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The only solitary wave that matters: the sech^2-type

Abstract. In this note we show that on introducing an appropriate point transformation, any solitary wave solution of a 1+1 partial differential equation (PDE) in a single unknown is equivalent to the sech^2 soliton of the Korteweg-de Vries (KdV) equation.

Keywords. Solitary waves, exact solutions, qualitative analysis *à la* Weierstrass.

Mathematics Subject Classification: 35Q53, 35L05, 74J35.

1 - Introduction

Solitary waves are a special and very important class of solutions of some partial differential equations (PDEs). From a physical point of view, they are originated by the cancellation of the nonlinearities due to dispersive and/or dissipative effects. From a mathematical perspective, they are non trivial solutions in a fixed coordinate frame that we may choose in view of the invariance of the PDE with respect to translations of the independent variables.

One of the most celebrated solitary waves is the *soliton* of the KdV equation

$$u_t + uu_x + u_{xxx} = 0. \tag{1.1}$$

As it is well known, this equation includes nonlinear terms due to transport phenomena and dispersion. To find a solitary wave solution of (1.1), we introduce the ansatz

$$u(x,t) = f(\xi), \quad \xi = x - ct, \tag{1.2}$$

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with c being a positive parameter and f such that

$$\lim_{\xi\to\pm\infty}\frac{\mathrm{d}^n f}{\mathrm{d}\xi^n}=0 \quad \text{for all} \quad n\in\mathbb{N}_0, \tag{1.3}$$

into (1.1) and reduce it to the ordinary differential equation (ODE)

$$-cf'+ff'+f'''=0, \tag{1.4}$$

where the prime denotes differentiation with respect to ξ. Integrating once (1.4) and taking into account the conditions at infinity (1.3) lead to the ODE

$$f''=cf-\frac{f^2}{2}. \tag{1.5}$$

Next, by multiplying both sides of (1.5) by f', integrating and, again, taking into account (1.3) we obtain the energy integral

$$f'^2=f^2\left(c-\frac{f}{3}\right). \tag{1.6}$$

Finally, limiting our search to bright solitary waves, i.e. to solutions in the form (1.2) with f positive, from (1.6) we deduce that the amplitude of the solitary wave is equal to $3c$ and

$$f'=\begin{cases} -f\sqrt{c-\dfrac{f}{3}} & \text{if } \xi\geq 0,\\ f\sqrt{c-\dfrac{f}{3}} & \text{if } \xi<0. \end{cases} \tag{1.7}$$

Integrating (1.7) yields the well-known soliton solution

$$f(\xi)=3c\,\mathrm{sech}^2\left[\frac{\sqrt{c}}{2}(\xi-\xi_0)\right]. \tag{1.8}$$

The integration constant ξ_0 in (1.8) gives the location at which the wave form attains its maximum and does not affect the shape of the solitary wave. Such an integration constant is then unessential and, for simplicity, it can be set equal to zero.

The method illustrated above to determine the bright solitary wave (1.8) of the KdV equation can be used also for many other PDEs of interest in physics and mechanics such as the generalised KdV, Gardner, Boussinesq and improved Boussinesq equations as well as some classes of second order semi-linear wave equations. In all these cases, on introducing the ansatz (1.2) and the conditions at infinity (1.3), and by following similar arguments as for the KdV equation one arrives at the the second order autonomous equation

$$f'' = F(f), \tag{1.9}$$

which admits the first integral

$$f'^2 = V(f), \tag{1.10}$$

where V is twice the antiderivative of F that vanishes at $f = 0$. An analysis *à la* Weierstrass of (1.10) reveals that a bright solitary wave is admissible if and only if the potential V admits a positive zero (say $f^\star$) with multiplicity less than two, $f = 0$ is a zero with multiplicity greater than or equal to two, and is strictly positive in $]0, f^\star[$. In other words, V is in the form

$$V(f) = \mathcal{V}(f) f^m (f^\star - f)^p \tag{1.11}$$

with $m \geq 2$, $0 < p < 2$ and $\mathcal{V}(f) > 0$ for all $f \in]0, f^\star[$. For illustration, the potential in the energy integral (1.6) of the KdV equation is in the form (1.11) with $m = 2$, $p = 1$, $f^\star = 3c$ and $\mathcal{V}(f) = 1/3$. Equation (1.10) can be integrated by quadrature. Only in particular cases (such as for the KdV equation) (1.10) can be integrated analytically.

Due its simplicity, equation (1.6) can be regarded as an archetype for determining bright solitary wave solutions. It is then natural to investigate whether a bright solitary wave of a general PDE which, after introducing the traveling wave ansatz (1.2), reduces to (1.9) with V as in (1.11) can be obtained via a point transformation of the soliton (1.8) of the KdV equation. This represent the main goal of the present manuscript.

It is well known that the KdV equation may be obtained in the weakly nonlinear asymptotic limit from a very large class of nonlinear evolution equations, and, in view of this, it is considered a universal nonlinear evolution PDE [**2**]. Here, by focusing on only a special class of solutions, the bright solitary waves, we extend exactly (and not asymptotically) such a universality to other types of nonlinear evolution PDEs.

This short note is dedicated to the memory of a profound and insightful scholar who was also a great gentleman and, above all, a sincere and gentle friend: Giampiero Spiga.

2 - An equivalence transformation

The energy integral (1.6) plays a fundamental role in determining the soliton of the KdV equation. In order to determine the class of energy integrals that admit bright solitary wave solutions and can be determined via an equivalence transformation from (1.6), we introduce the change of variables

$$\xi = \phi(s,g), \quad f = \psi(s,g), \qquad \text{with} \quad \phi_s\psi_g - \phi_g\psi_s \neq 0. \tag{2.1}$$

In the new variables s and $g = g(s)$ defined by (2.1), equation (1.6) takes the form

$$\left[\psi_g^2 - \psi^2\left(c - \frac{\psi}{3}\right)\phi_g^2\right]\left(\frac{\mathrm{d}g}{\mathrm{d}s}\right)^2 + 2\left[\psi_s\psi_g - \psi^2\left(c - \frac{\psi}{3}\right)\phi_s\phi_g\right]\frac{\mathrm{d}g}{\mathrm{d}s} + \left[\psi_s^2 - \psi^2\left(c - \frac{\psi}{3}\right)\phi_s^2\right] = 0, \tag{2.2}$$

which represents an energy integral providing that

$$\psi_s\psi_g - \psi^2\left(c - \frac{\psi}{3}\right)\phi_s\phi_g = 0 \tag{2.3a}$$

and

$$V(g) = -\frac{\psi_s^2(s,g) - \psi^2(s,g)\left[c - \dfrac{\psi(s,g)}{3}\right]\phi_s^2(s,g)}{\psi_g^2(s,g) - \psi^2(s,g)\left[c - \dfrac{\psi(s,g)}{3}\right]\phi_g^2(s,g)}. \tag{2.3b}$$

A special solution of the nonlinear equations (2.3) is given by

$$\xi = \phi(s,g) = s, \qquad f = \psi(s,g) = \Psi(g), \tag{2.4}$$

with $\Psi_g \neq 0$, in which case (2.2) reduces to

$$g'^2 = \frac{\Psi^2(g)}{\Psi_g^2(g)}\left[c - \frac{\Psi(g)}{3}\right] = V(g). \tag{2.5}$$

In view of the discussion in the previous section, equation (2.5) admits bright solitary wave solutions if and only if Ψ is a monotonically increasing function such that

$$\lim_{g\to 0^+} \Psi(g) = 0. \tag{2.6}$$

Indeed, if this is the case, the potential V on the right-hand side of (2.5) admits a simple zero at $g = g^\star = \Psi^{-1}(3c)$, a zero with multiplicity greater than or equal to two at $g = 0$, and is strictly positive in the interval $]0, g^\star[$. In other words, the potential is in the form (1.11), with $m \geq 2$ and $p = 1$, which implies that the necessary and sufficient conditions for the existence of a bright solitary wave solution are satisfied.

We may then conclude that, given an arbitrary function Ψ that satisfies (2.6) and is such that $\Psi_g > 0$, one can map the energy integral (1.6) of the KdV equation into the first integral (2.5) which admits a bright solitary wave solution. In particular, since Ψ is fixed, the potential V that allows the existence of this particular traveling wave is determined by Ψ and its first derivative according to the expression in the right-hand side of (2.5).

Vice versa, given a potential $V(g) = \mathcal{V}(g)g^2(g^\star - g)$, with $\mathcal{V}(g) > 0$ for all $g \in]0, g^\star[$, one can determine the map $f = \Psi(g)$ which defines the equivalence transformation (2.4). More precisely, by requiring that $\Psi(g^\star) = 3c$ we find that the equivalence transformation is given by

$$f = \Psi(g) = 3c\,\mathrm{sech}^2\left[\frac{\sqrt{c}}{2}\int_g^{g^\star}\frac{\mathrm{d}\upsilon}{\sqrt{V(\upsilon)}}\right]. \tag{2.7}$$

From the inverse Ψ^{-1} of this transformation and the soliton (1.8) of the KdV equation, the bright solitary waves deducible from (2.5) is found to be

$$g(\xi) = \Psi^{-1}\left(3c\,\mathrm{sech}^2\left(\frac{\sqrt{c}}{2}\xi\right)\right). \tag{2.8}$$

3 - Some examples

In this section we report some examples of equivalence transformations thanks to which one can easily determine the bright solitary waves of some nonlinear evolution equations of particular interest in mechanics.

3.1 - *The generalised Korteweg-de Vries equation*

The generalised Kortewev-de Vries equation is given by [**8**]

$$u_t + u^k u_x + u_{xxx} = 0, \tag{3.1}$$

with k being a positive parameter. In looking for bright solitary waves of (3.1), one arrives at the energy integral

$$g'^2 = g^2\left[c - \frac{2g^k}{(k+1)(k+2)}\right], \tag{3.2}$$

which, in view of (2.7), can be deduced from the energy integral (1.6) of the classical KdV equation by means of the transformation

$$f = \Psi(g) = 3c\,\mathrm{sech}^2\left[\frac{1}{k}\mathrm{arccosh}\sqrt{\frac{c(k+1)(k+2)}{2g^k}}\right]. \tag{3.3}$$

Next, by inverting (3.3) one obtains the bright solitary wave solution of the generalised KdV equation

$$g(\xi) = \left[\frac{c(k+1)(k+2)}{2}\mathrm{sech}^2\left(\frac{k\sqrt{c}}{2}\xi\right)\right]^{\frac{1}{k}}. \tag{3.4}$$

3.2 - *Shear strain waves in nonlinear dispersive solids*

The dimensionless equation governing the nonlinear waves in an incompressible hyperelastic dispersive material reads [**3**]

$$u_{tt} = [(\alpha_T^2 + \alpha_{nl}^2 u^2)u]_{xx} + \nu^2 u_{ttxx} \tag{3.5}$$

where α_T, α_{nl} and ν are positive constants. It is easy to prove that the energy integral associated with (3.5) is given by

$$g'^2 = \frac{g^2}{\nu^2 c^2}\left(c^2 - \alpha_T^2 - \frac{\alpha_{nl}^2}{2}g^2\right), \tag{3.6}$$

which, on using (2.7), can be deduced from (1.6) by means of the transformation

$$f = 3c\,\mathrm{sech}^2\left[\frac{\nu c\sqrt{c}}{2\sqrt{c^2-\alpha_T^2}}\cosh\left(\frac{\sqrt{2(c^2-\alpha_T^2)}}{\alpha_{nl}}g\right)\right]. \tag{3.7}$$

From the inverse of this transformation and the soliton of the KdV equation the bright solitary wave solution of (3.6) is found to be

$$g(\xi) = \frac{\sqrt{2(c^2-\alpha_T^2)}}{\alpha_{nl}}\mathrm{sech}\left(\frac{\sqrt{c^2-\alpha_T^2}}{\nu c}\xi\right). \tag{3.8}$$

3.3 - *Semilinear Klein-Gordon equations*

The propagation of transverse waves in an incompressible hyperelastic solid supported by an elastic foundation is given by the semi-linear Klein-Gordon equation [**6**]

$$u_{tt} - c_T^2 u_{xx} + \gamma(u^2)u = 0, \tag{3.9}$$

where c_T is a positive constant and γ is the restored force corresponding to the substrate potential $\Gamma(u^2)/2$. The energy integral associated with (3.9) is given by

$$g'^2 = \frac{\Gamma(g^2)}{c_T^2 - c^2}. \tag{3.10}$$

In what follows we assume that $c < c_T$, and the potential Ψ is polynomial in g and of the form

$$\Gamma(g) = a_0 g^2 + a_1 g^3 + a_2 g^4, \tag{3.11}$$

where a_0, a_1 and a_2 are constants. We now consider two cases of particular interest.

3.3.1 - Algebraic solitary wave

If $a_0 = 0$, $a_1 > 0$ and $a_2 < 0$ in (3.11), then, with the aid of (2.7), one deduces that (3.10) can be obtained from (1.6) via the transformation

$$f = 3c\,\mathrm{sech}^2\left[\frac{\sqrt{c(c_T^2 - c^2)}}{a_1}\sqrt{\frac{a_1}{g} + a_2}\right], \tag{3.12}$$

which, once inverted and combined with (1.8), yields the algebraic bright solitary wave solution

$$f(\xi) = \frac{a_1}{\dfrac{a_1^2\xi^2}{4(c_T^2 - c^2)} - a_2}. \tag{3.13}$$

3.3.2 - Flat-top solitary waves

If $a_0 > 0$, $a_1 < 0$, $a_2 > 0$ and $a_1^2 < 4a_0a_2$ in (3.11), from the analysis in the previous section we deduce that (3.10) can be obtained from (1.6) by means of the transformation

$$f = 3c\,\mathrm{sech}^2\left[\frac{1}{2}\sqrt{\frac{c(c_T^2 - c^2)}{a_0}}\mathrm{arccosh}\left(\frac{2a_0 + a_1 g}{g\sqrt{a_1^2 - 4a_0a_2}}\right)\right], \tag{3.14}$$

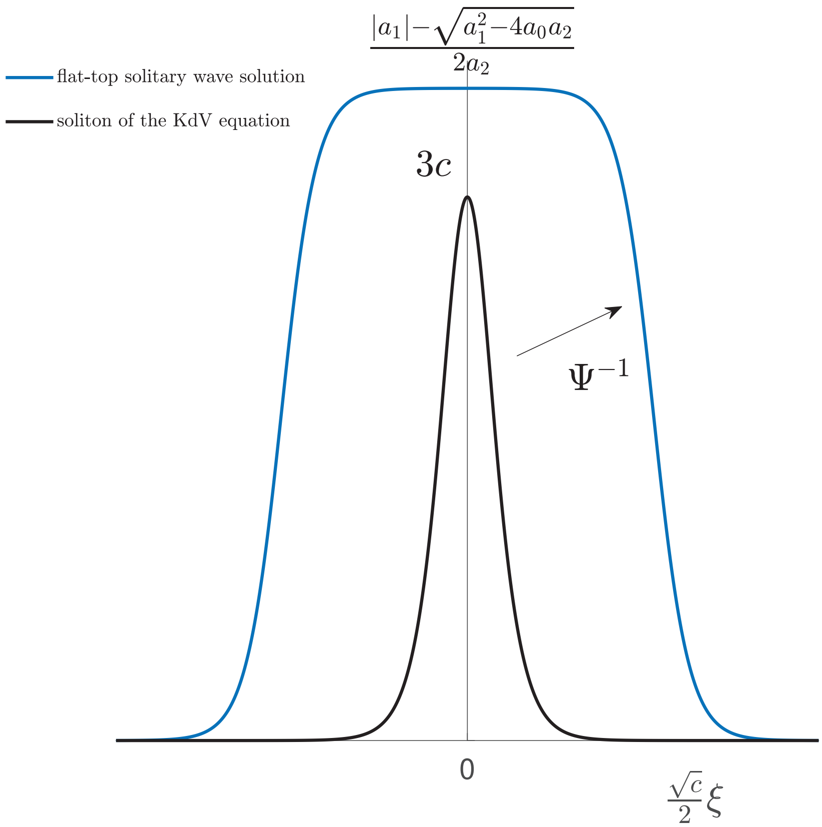

Fig. 1. Flat-top solitary wave of the semi-linear Klein-Gordon equation (3.9) as a transformation of the soliton pf the KdV equation under the map Ψ^{-1}, with Ψ as in the right-hand side of (3.15).

whose inverse, once combined with (1.8), gives the bright solitary wave

$$f(\xi) = \frac{2a_0}{|a_1| + \sqrt{a_1^2 - 4a_0a_2}\cosh\left(\sqrt{\frac{a_0}{c_T^2 - c^2}}\xi\right)}. \tag{3.15}$$

It is worth pointing out that if $a_1^2 - 4a_0a_2 \ll 1$, then the solitary wave (3.15) becomes almost flat at its top (see Figure 1). This is due to the fact that the bright solitary wave (3.15) dissolves into a kink when $a_1^2 = 4a_0a_2$ [**6**, **7**].

4 - Concluding Remarks

In this short note we highlight the centrality and universality of the soliton of the KdV equation. In particular, we prove that any bright solitary wave solution of a large class of models can be related to the celebrated sech^2 soliton. This sort of universal property is interesting and not at all surprising as the KdV equation itself is, in some sense, universal. In mechanics there are many situations in which systems of PDEs or ODEs represent a sort of hyphen between various theories. Just think of the small oscillations around a stable equilibrium configuration of any discrete mechanical system. This problem is always investigated by means of a system of harmonic oscillators which, except for the values of the entries of the mass and stiffness matrices, are identical.

The equivalence transformation that we have used to show the centrality of the soliton solution of the KdV equation is not the usual one as the appropriate change of variables to be introduced has to satisfy additional conditions necessary to preserve the characteristics of the solitary wave solutions.

The results in this note can be generalised in several directions. For instance, by using the geometrical methods developed in [**5**], one might find more general solutions of (2.3) and determine other transformations of mathematical and physical interest.

Other aspects worth to be investigated are the generalisation of our method to other classes of nonlinear evolution PDEs and/or to other classes of solutions such as kinks and cnoidal waves. The analytical approach developed here may be used to investigate the universality of these wave solutions starting from the identification of archetypal equations.

Finally, a possible generalisation of the present work could involve more general transformations such as the Sundman transformation [**4**].

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