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Regularity of solutions of Fokker-Planck equations with rough coefficients

Abstract. The purpose of this article is to review recent progress on the regularity theory for kinetic models of Fokker-Planck type. Such equations are known to be hypoelliptic, and our aim is to explain how De Giorgi-Nash-Moser iterations can be used on such problems. Most of this note is based on our recent joint work with C. Imbert, C. Mouhot and A. Vasseur [Ann. Sc. Norm. Super. Pisa Cl. Sci. (5), Vol. XIX (2019), 253–295].

Keywords. Fokker-Planck equations, Hypoelliptic equations, De Giorgi-Nash-Moser iteration method.

Mathematics Subject Classification: 35H10, 35B65, 35Q84.

In memory of our friend and colleague Prof. Giampiero Spiga

1 - Local Regularity for Fokker-Planck Equations

We are concerned with the local regularity of weak solutions of Fokker-Planck equations of the form

$$(\partial_t+v\cdot\nabla_x)f(t,x,v)=\mathrm{div}_v(A(t,x,v)\nabla_v f(t,x,v))+g(t,x,v)\,,\quad x,v\in\mathbf{R}^d\,,\tag{1}$$

or more generally

$$(\partial_t+b(v)\cdot\nabla_x)f(t,x,v)=\mathrm{div}_v(A(t,x,v)\nabla_v f(t,x,v))+g(t,x,v)\,,\quad x,v\in\mathbf{R}^d\,.\tag{2}$$

In these equations, the unknown is the real-valued function $f\equiv f(t,x,v)$. In some cases f is the (velocity) distribution function as in the kinetic theory of

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gases, i.e. $f(t,x,v)$ is the density at time t of particles located at the position x and moving at velocity v, in which case $f(t,x,v) \geq 0$. But f could also designate more general objects, such as fluctuations of the distribution function about some equilibrium state, in which case f has no definite sign. Likewise, v designates the particles' velocity in (1), but could designate other physical quantities as in (2), where the particles' velocity is $b(v)$ instead of v. (One typical example in the kinetic theory of relativistic particles of mass m at rest is $b(v) := \nabla_v \sqrt{m^2c^4 + c^2|v|^2}$, with c being the speed of light in vacuum and v the momentum variable.)

The diffusion coefficient $(t,x,v) \mapsto A(t,x,v) \in \mathbf{R}^{N\times N}$ is a given, measurable matrix-valued map s.t.

$$\frac{1}{\Lambda} I \leq A(t,x,v) = A(t,x,v)^T \leq \Lambda I\,, \qquad \text{for some real constant } \Lambda > 1\,.$$

Finally, the function $g \equiv g(t,x,v)$ on the r.h.s. of (1) or (2) is a given source term.

Although the techniques presented in this paper work for (2) as well as for (1), we shall mostly focus on (1), and provide the reader interested in (2) with the relevant references.

Let us briefly discuss the notion of weak solution of (1) to be considered below.

Multiplying both sides of (1) by $2f$ and integrating by parts in $x, v \in \mathbf{R}^d$, assuming that f decays fast enough as $|x| + |v| \to \infty$, one (formally) obtains the inequality[1]

$$\begin{aligned}
&\|f(t)\|^2_{L^2_{x,v}} + \frac{2}{\Lambda}\int_{t_0}^t \|\nabla_v f(s)\|^2_{L^2_{x,v}} ds \\
&\leq \|f(t)\|^2_{L^2_{x,v}} + 2\int_{t_0}^t \iint_{\mathbf{R}^{2d}} \nabla_v f(s,x,v)\cdot A(s,x,v)\nabla_v f(s,x,v)dxdvds \\
&= \|f(t_0)\|^2_{L^2_{x,v}} + 2\int_{t_0}^t \iint_{\mathbf{R}^{2d}} f(s,x,v)g(s,x,v)dxdvds \\
&\leq \|f(t_0)\|^2_{L^2_{x,v}} + 2\int_{t_0}^t \|g(s)\|_{L^2_{x,v}}\|f(s)\|_{L^2_{x,v}} ds\,,
\end{aligned}$$

so that

$$\begin{aligned}
&-T < t_0 < t \implies \|f(t)\|^2_{L^2_{x,v}} + \frac{2}{\Lambda}\int_{t_0}^t \|\nabla_v f(s)\|^2_{L^2_{x,v}} ds \\
&\leq \left(\|f(t_0)\|^2_{L^2_{x,v}} + \int_{t_0}^t \|g(s)\|^2_{L^2_{x,v}} ds\right) + \int_{t_0}^t \|f(s)\|^2_{L^2_{x,v}} ds\,.
\end{aligned} \tag{3}$$

[1] In the sequel, $L^2_{x,v}$ stands for $L^2(\mathbf{R}^d \times \mathbf{R}^d)$.

This is the "energy" inequality for weak solutions of (1) (or (2)).

Assuming $g \in L^2_{loc}(-T, \infty; L^2_{x,v})$, we can apply the Gronwall inequality to find that

$$-T < t_0 < t \Rightarrow \|f(t)\|^2_{L^2_{x,v}} + \frac{2}{\Lambda}\int_{t_0}^{t} \|\nabla_v f(s)\|^2_{L^2_{x,v}} ds \leq (\|f(t_0)\|^2_{L^2_{x,v}} + \|g\|^2_{L^2([t_0,t];L^2_{x,v})})e^{T+t}\,.$$

This justifies considering the following notion of weak solution of (1).

Definition 1.1. *A weak solution of* (2) *on* $(-T, +\infty) \times \mathbf{R}^d \times \mathbf{R}^d$ *is a measurable function*

$$f \in C(-T, +\infty; L^2_{x,v})\,, \qquad \textit{s.t. } \nabla_v f \in L^2_{loc}(-T, +\infty; L^2_{x,v})$$

satisfying the following "renormalized" form of (1)*: for all* $\chi \in C^2(\mathbf{R})$ *such that* $z \mapsto \chi(z)/(1+z^2)$ *is bounded on* $\mathbf{R}$*, it holds*[2]

$$(\partial_t + b(v)\cdot\nabla_x)\chi(f) = \operatorname{div}_v(A\nabla_v\chi(f)) - \chi''(f)A : (\nabla_v f)^{\otimes 2} + \chi'(f)g \tag{4}$$

in the sense of distributions on $(-T, +\infty) \times \mathbf{R}^d \times \mathbf{R}^d$*.*

In the sequel, it will be convenient to use the following notations. Let $\mathcal{O}$ be a domain of $\mathbf{R} \times \mathbf{R}^d \times \mathbf{R}^d$. For each $\Lambda > 1$ we set

$$S[\mathcal{O}, \Lambda] := \{\mathcal{O} \ni (t,x,v) \mapsto A(t,x,v) = A^T(t,x,v) \in \mathbf{R}^{d\times d} \text{ measurable s.t. } \Lambda^{-1}I \leq A(t,x,v) \leq \Lambda I \text{ a.e. on } \mathcal{O}\}\,.$$

For each domain $\mathcal{O}$ of the form $I \times \Omega$ where $I \subset \mathbf{R}$ is an open interval and Ω a domain of $\mathbf{R}^d \times \mathbf{R}^d$, each $\Lambda > 1$, each diffusion matrix $A \in S[\mathcal{O}, \Lambda]$ and each source term $g \in L^2_{loc}(I; L^2(\Omega))$, we set

$$\begin{aligned} FP[A, g, \mathcal{O}] := \{f \in C(I; L^2(\Omega)) \text{ s.t. } \nabla_v f \in L^2_{loc}(I; L^2(\Omega)) \text{ and} \\ (\partial_t + v\cdot\nabla_x)\chi(f) = \operatorname{div}_v(A\nabla_v\chi(f)) - \chi''(f)A : (\nabla_v f)^{\otimes 2} + \chi'(f)g \text{ in } \mathcal{D}'(\Omega) \\ \text{for all } \chi \in C^2(\mathbf{R}) \text{ s.t. } z \mapsto \tfrac{\chi(z)}{1+z^2} \in L^\infty(\mathbf{R})\}\,. \end{aligned}$$

[2]If $u \in \mathbf{R}^d$, the notation $u^{\otimes 2}$ designates the second order tensor $u \otimes u$, identified with the matrix with entries $u_i u_j$ for $1 \leq i, j \leq d$, where $u_1, \ldots, u_d$ are the components of the vector u. The notation $A : u^{\otimes 2}$ designates the contraction of the second order tensors A and $u \otimes u$, i.e.

$$A : u^{\otimes 2} := \sum_{i,j=1}^{d} A_{ij} u_i u_j\,.$$

Finally, for each $r>0$, we set

$$Q[r]:=(-r,0]\times B(0,r)\times B(0,r)\,,\qquad \hat{Q}:=(-\tfrac{3}{2},-1]\times B(0,1)\times B(0,1)\,.$$

(Notice the difference with parabolic cylinders of the form $(-r^2,0]\times B(0,r)$ used in the context of space-homogeneous solutions $f\equiv f(t,v)$ of (1), i.e. of solutions of (1) that are independent of the space-variable x, in which case the resulting equation is a variant of the heat equation.)

Our first main result in this paper is the following theorem (Theorem 1.4 in [**10**]).

Theorem 1.2. *Let Ω be a domain of $\mathbf{R}^d\times\mathbf{R}^d$, and let I be an open interval of $\mathbf{R}$. Let $A\in S[I\times\Omega;\Lambda]$ where $\Lambda>1$, and let $g\in L^\infty(I\times\Omega)$. Then, there exists a Hölder exponent $\sigma\in(0,1)$ such that*

$$f\in FP[A,g,I\times\Omega]\implies f\big|_K\in C^{0,\sigma}(K)$$

for each compact $K\subset I\times\Omega$.

The proof of Theorem 1.2 is based on using De Giorgi-Nash-Moser iterations, in a way that differs from the combined works of [**26**] and [**29, 30**]. In particular, our approach does not use the fundamental solution of the Kolmogorov operator

$$\partial_t+v\cdot\nabla_x-\Delta_v$$

at variance with [**26, 29, 30**]. One advantage of our approach is that it applies without major modifications to (2): see [**31**].

De Giorgi-Nash-Moser iterations appeared for the first time in connection with Hilbert's 19th problem: let $L\in C^\omega(\mathbf{R}^d)$ such that $\Lambda^{-1}I\le\nabla^2L\le\Lambda I$ on $\mathbf{R}^d$, and let O be an open set of $\mathbf{R}^d$. Do extremals of the functional

$$\int_O L(\nabla u(x))dx$$

belong to $C^\omega(O)$? After various contributions by Bernstein, Petrovski, Hopf..., the problem was solved by Morrey [**23**] in the late 1930s in the case $d=2$, and by De Giorgi [**6**] and Nash [**25**] independently in any space dimension. Moser proposed a slightly different approach later in [**24**], aimed at symplifying Nash's argument.

While De Giorgi-Nash-Moser iterations have been known for a long time in the context of elliptic or parabolic equations, their applications in the context of kinetic models is more recent — see for instance [**9, 11, 16, 19, 20**] to quote only a few references. The present paper is meant to be a "gentle" (i.e. pedagogical) introduction to this circle of ideas in the context of kinetic equations. We shall mostly follow [**10**], suggesting alternative approaches whenever appropriate. In order to keep this introduction as concise as possible, we shall however not discuss the Harnack inequality stated as Theorem 1.6 in [**10**], although it is intimately related to the control of oscillations of weak solutions of (1) (and therefore to their Hölder modulus of continuity). The reader interested in the Harnack inequality is advised to read [**2**, **15**, **16**], in addition to Theorem 1.6 in [**10**].

Of course, there exist various presentations of the De Giorgi argument in the recent literature: see for instance [**5**,**17**,**28**] to quote only a few. Moser's original paper [**24**] is very short and nicely written. Most results on second order elliptic PDE's with variable coefficients postulate either that the diffusion matrix is close to some constant multiple of the identity (as in the Cordes-Nirenberg theory), of that the diffusion matrix is continuous (as in the Schauder or the Calderón-Zygmund theories): see the introduction of the book [**4**]. The idea is that, when zooming in near some point, the oscillations in the coefficients converge to zero, so that, in the limit, the equation to be studied is expected to behave as an equation with constant coefficients. For instance, in the case of a second order elliptic equation, solutions are expected to retain some features of harmonic functions. Now, in the case of bounded but discontinuous coefficients, for instance with a jump discontinuity in one of the variables, zooming in near a point of discontinuity will not does not change anything as this jump discontinuity persists. De Giorgi iterations are precisely aimed at exploring the regularity properties of solutions of elliptic (or parabolic) equations when the method of frozen coefficients does not apply, i.e. in the case of bounded but possibly discontinuous coefficients.

While the first applications of the De Giorgi-Nash-Moser iterations were in the field of elliptic or parabolic equations, one should not think of it as a purely elliptic or parabolic method. Very recently, striking applications of the De Giorgi-Nash-Moser method have been proposed in the field of hyperbolic conservation laws [**27**].

In the present paper as in [**10**], we discuss the case of hypoelliptic equations. Of course, the prototype of hypoelliptic operators is Kolmogorov's operator

$$\partial_t + v \cdot \nabla_x - \Delta_v$$

for which a fundamental solution can be computed explicitly in terms of the Fourier transform of a Gaussian [**21**]. Generalizations of this operator have been considered by Hörmander [**18**], and take the form

$$X_0 + \sum_{j=1}^{n} X_j^2$$

where X_j for $j = 0, \ldots, n$ are vector fields on $\mathbf{R}^d$ such that the rank of the Lie algebra generated by $X_0, \ldots, X_n$ is d at each point of $\mathbf{R}^d$. Of course, the Kolmogorov operator falls in this class, as can be seen by taking

$$X_0 := \partial_t + v \cdot \nabla_x \,, \quad X_j = \partial_{v_j} \,, \quad j = 1, \ldots, d \,.$$

Indeed

$$\partial_{x_j} = [X_j, X_0] \,, \quad j = 1, \ldots, d \,,$$

so that the tangent space at each point of $\mathbf{R}^d$ is spanned by

$$X_0, X_1, \ldots, X_d, [X_1, X_0], \ldots, [X_d, X_0] \,.$$

Of course, Hörmander considers only vector fields with C^∞ coefficients. The purpose of [**10**] and of the present paper is to explain how this assumption can be alleviated in the case of the Fokker-Planck equation (1).

Giampiero Spiga has been a distinguished leader in the field of kinetic models. His untimely passing away is a great shock to our community. This modest contribution is dedicated to his memory.

2 - From the Energy Class to L^∞

We have split the proof of Theorem 1.2 into two parts: in the first part, we prove that weak solutions of (1), which are only known to satisfy the bounds deduced from the energy estimate, and therefore are only square-integrable in x a priori, are in fact locally bounded. Hölder regularity will be proved in the next section. It is often the case that this first step immediately implies local regularity for specific examples of Fokker-Planck equations (1) enjoying additional properties. The space homogeneous Landau equation (with Coulomb potential) is known to fall in this case. Thus, the method used in this section is already of independent interest for applications to some specific kinetic models.

Our aim is to prove the following statement, which is Theorem 3.1 in [**10**].

Theorem 2.1. *For each $\Lambda > 1$, each $\gamma > 0$ and each Lebesgue exponent $q > 12d + 6$, there exists $\kappa[d, \Lambda, \gamma, q] \in (0, 1)$ satisfying the following property: for each diffusion matrix $A \in S[Q[\frac{3}{2}], \Lambda]$, each $g \in L^q(Q[\frac{3}{2}])$ satisfying the bound $\|g\|_{L^q(Q[\frac{3}{2}])} \leq \gamma$ and each $f \in C(-\frac{3}{2}, 0; L^2(B(0, \frac{3}{2})^2)) \cap FP[A, g, Q[\frac{3}{2}]]$, it holds*

$$\int_{Q[\frac{3}{2}]} f(t, x, v)_+^2 dt dx dv < \kappa \implies f \leq \tfrac{1}{2} \ \text{a.e. on } Q[\tfrac{1}{2}] \,.$$

In the sequel, we sketch the proof of this theorem.

All the proofs involve some preliminary step in which L^2 integrability on some cylinder is improved into L^p integrability with $p > 2$ on a smaller cylinder. The gain in Lebesgue exponent is independent of the reduction in the size of the cylinder, but the embedding constant is obviously not. In the sequel, we study in detail this improvement of integrability — see sections 2.1 to 2.3. Once this is done, we shall present two different iteration methods (the De Giorgi and the Moser iteration methods) leading to the inequality in Theorem 2.1.

2.1 - *Local Energy Estimate*

Write the weak formulation of the renormalized variant of equation (2) with normalizing nonlinearity $\chi(f) := \frac{1}{2}(f-c)_+^2$, and pick the test function in the form

$$\psi(\tau, x, v) := \mathbf{1}_{s<\tau<t}\phi(x, v)\,, \qquad \phi(x, v) = \eta(x)\eta(v)^2$$

where $\eta \in C_c^\infty(\mathbf{R}^d)$. (Obviously, we should replace $f \mapsto \chi(f) := \frac{1}{2}(f-c)_+^2$ with a C^2 approximation, and $\mathbf{1}_{s<\tau<t}$ with a C^∞ approximation $\tau \mapsto \theta(\tau)$, but we shall skip both steps since they do not involve any technical difficulty.)

Since $\chi''(f) = \mathbf{1}_{f>c}$, one has

$$\chi''(f)(\nabla_v f)^{\otimes 2} = (\nabla_v (f-c)_+)^{\otimes 2}\,,$$

and hence

$$\begin{aligned}&2\eta(v)A : \nabla_v \chi(f) \otimes \nabla\eta(v) + \eta(v)^2\chi''(f)A : (\nabla_v f)^{\otimes 2}\\ &= A : (\nabla_v(\eta(v)(f-c)_+))^{\otimes 2} - (f-c)_+^2 A : (\nabla_v \eta)^{\otimes 2}\,.\end{aligned}$$

(For the reader already familiar with the De Giorgi iteration method in the case of elliptic equations, we observe that this identity is reminiscent of the (proof of the) Caccioppoli inequality, which is itself reminiscent of Cauchy's inequality for holomorphic functions, and is a first step in the De Giorgi method: see inequality (20) in [**6**].)

Substituting the r.h.s. of the preceding identity in the dissipation term of the "energy" equality, one finds that

$$
\begin{aligned}
&\tfrac{1}{2}\iint_{\mathbf{R}^{2d}}\eta(x)\eta(v)^2(f(t,x,v)-c)_+^2dxdv\\
&\quad+\frac{1}{\Lambda}\int_s^t\iint_{\mathbf{R}^{2d}}\eta(x)|\nabla_v(\eta(v)(f(\tau,x,v)-c)_+)|^2dxdvd\tau\\
&\le\tfrac{1}{2}\iint_{\mathbf{R}^{2d}}\eta(x)\eta(v)^2(f(s,x,v)-c)_+^2dxdv\\
&\quad+\Lambda\int_s^t\iint_{\mathbf{R}^{2d}}\eta(x)(f(\tau,x,v)-c)_+^2|\nabla\eta(v)|^2dxdvd\tau\\
&\quad+\int_s^t\iint_{\mathbf{R}^{2d}}\eta(v)^2\tfrac{1}{2}(f(\tau,x,v)-c)_+^2b(v)\cdot\nabla\eta(x)dxdvd\tau\\
&\quad+\int_s^t\iint_{\mathbf{R}^{2d}}\eta(v)^2g(\tau,x,v)(f(\tau,x,v)-c)_+\eta(x)dxdvd\tau\,.
\end{aligned}\tag{5}
$$

This is the local variant of the energy inequality (3) used to define the notion of weak solution of (1) or (2). Observing that

$$g(f-c)_+=g\mathbf{1}_{f>c}(f-c)_+\le\tfrac{1}{2}g^2\mathbf{1}_{f>c}+\tfrac{1}{2}(f-c)_+^2\,,$$

we pick $0<r<R<+\infty$ and assume that

$$\mathbf{1}_{B(0,r)}\le\eta\le\mathbf{1}_{B(0,R)}\qquad\text{and}\qquad\|\nabla\eta\|_{L^\infty}\le\tfrac{2}{R-r}\,.$$

Then we average both sides of (5) in $s\in(-R,r)$ to arrive at the following statement.

Lemma 2.2 (Local energy bound). *Let $\Lambda>2$, and $R>r>0$. Assume that $A\in S[Q[R],\Lambda]$, and let f be a weak solution, or a nonnegative subsolution of* (2) *on $Q[R]$. Then*

$$
\begin{aligned}
&\sup_{-r<t<0}\int_{B(0,r)^2}(f(t,x,v)-c)_+^2dxdv+\int_{Q[r]}|\nabla_v(f(\tau,x,v)-c)_+|^2dxdvd\tau\\
&\le\frac{\Lambda}{2}\left(1+\frac{1+2\|b\|_{L^\infty(B(0,R))}}{R-r}+\frac{8\Lambda}{(R-r)^2}\right)\int_{Q[R]}(f(\tau,x,v)-c)_+^2dxdvd\tau\\
&\quad+\frac{\Lambda}{2}\int_{Q[R]}g(\tau,x,v)^2\mathbf{1}_{f(\tau,x,v)>c}dxdvd\tau\,.
\end{aligned}\tag{6}
$$

2.2 - *A Local Barrier Function*

Consider a positive subsolution of (2), i.e.

$$(\partial_t+b(v)\cdot\nabla_x)f-\operatorname{div}_v(A(t,x,v)\nabla_vf)\le g\,,\qquad\text{and}\qquad f\ge0\quad\text{a.e. on }Q[R]\tag{7}$$

for some $R > 0$.

Pick $0 < r < R$, and set $Q_{ext} := Q[R]$, while $Q_{int} := Q[r]$ and $Q_{mid} := Q[\frac{R+r}{2}]$. Choose $\chi, X \in C^\infty(\mathbf{R} \times \mathbf{R}^d \times \mathbf{R}^d)$ such that

$$\mathbf{1}_{Q_{int}} \le \chi \le \mathbf{1}_{Q_{mid}} \le X \le \mathbf{1}_{Q_{ext}}\,, \qquad \text{with } |\nabla\chi|, |\nabla X| \le \tfrac{4}{R-r}\,. \tag{8}$$

Multiplying both sides of (7) by χ, we find that

$$(\partial_t + b(v)\cdot\nabla_x)(\chi f) - \operatorname{div}_v(A(t,x,v)\nabla_v(\chi f)) \le H_0 + \operatorname{div}_v H_1 \quad \text{ on } \mathbf{R}\times\mathbf{R}^d\times\mathbf{R}^d\,,$$

where

$$\begin{cases} H_0 := \chi g \mathbf{1}_{f>0} + Xf(\partial_t + b(v)\cdot\nabla_x)\chi - \nabla_v\chi\cdot A\nabla_v(Xf)\,, \\ H_1 := -\,XfA\nabla_v\chi\,. \end{cases} \tag{9}$$

Let $F \equiv F(t,x,v)$ be the solution of

$$\begin{cases} (\partial_t + b(v)\cdot\nabla_x)F - \operatorname{div}_v(A(t,x,v)\nabla_v F) = H_0 + \operatorname{div}_v H_1\,, \quad x, v \in \mathbf{R}^d\,, \\ F\big|_{t=-R} = 0\,. \end{cases} \tag{10}$$

Equivalently

$$\begin{cases} (\partial_t + b(v)\cdot\nabla_x)F - \operatorname{div}_v(A(t,x,v)\nabla_v F) = H_0 + \operatorname{div}_v H_1\,, \quad x, v \in \mathbf{R}^d,\ t \in \mathbf{R}\,, \\ \operatorname{supp}(F) \subset (-R, +\infty)\times\mathbf{R}^d\times\mathbf{R}^d\,. \end{cases}$$

Then, we claim that

$$0 \le \chi f \le F \quad \text{ a.e. on } \mathbf{R}\times\mathbf{R}^d\times\mathbf{R}^d\,, \qquad \text{so that } 0 \le f \le F \text{ a.e. on } Q[r]\,. \tag{11}$$

Indeed, by linearity of (2), the difference $h = \chi f - F$ satisfies

$$\begin{cases} (\partial_t + b(v)\cdot\nabla_x)h \le \operatorname{div}_v(A(t,x,v)\nabla_v h)\,, \quad x, v \in \mathbf{R}^d\,, \\ h\big|_{t=-R} = 0\,. \end{cases}$$

Now, arguing as in (3), we conclude that h_+, which is also a subsolution of (2) with $g = 0$, by convexity of $h \mapsto h_+$ and (4), must be identically 0, and this implies the desired conclusion.

2.3 - *Gain of Local Integrability*

In this section, we shall prove a gain of local regularity on the local barrier function F, but not on the positive subsolution f itself. However, this is enough to imply a gain of local integrability on f, which is all we need in the sequel.

First, the energy inequality for F is

$$\tfrac{1}{2}\iint_{\mathbf{R}^d\times\mathbf{R}^d} F(t,x,v)^2 dxdv + \frac{1}{\Lambda}\int_{-R}^{t}\iint_{\mathbf{R}^d\times\mathbf{R}^d}|\nabla_v F(s,x,v)|^2 dxdvds$$
$$\le \int_{-R}^{t}\iint_{\mathbf{R}^d\times\mathbf{R}^d} H_0 F(s,x,v)dxdvds - \int_{-R}^{t}\iint_{\mathbf{R}^d\times\mathbf{R}^d} H_1\cdot\nabla_v F(s,x,v)dxdvds\,,$$

which implies that

$$\tfrac{1}{2}\iint_{\mathbf{R}^d\times\mathbf{R}^d} F(t,x,v)^2 dxdv + \frac{1}{2\Lambda}\int_{-R}^{t}\iint_{\mathbf{R}^d\times\mathbf{R}^d}|\nabla_v F(s,x,v)|^2 dxdvds$$
$$\le \tfrac{1}{2}\int_{-R}^{0}\iint_{\mathbf{R}^d\times\mathbf{R}^d} H_0(s,x,v)^2 dxdvds + \tfrac{\Lambda}{2}\int_{-R}^{0}\iint_{\mathbf{R}^d\times\mathbf{R}^d} H_1(s,x,v)^2 dxdvds$$
$$+ \tfrac{1}{2}\int_{-R}^{t}\iint_{\mathbf{R}^d\times\mathbf{R}^d} F(s,x,v)^2 dxdvds\,,$$

for all $t \ge -R$, since $H_0 = H_1 = 0$ for $t > 0$. By Gronwall's inequality

$$\tfrac{1}{2}\iint_{\mathbf{R}^d\times\mathbf{R}^d} F(t,x,v)^2 dxdv + \frac{1}{2\Lambda}\int_{-R}^{t}\iint_{\mathbf{R}^d\times\mathbf{R}^d}|\nabla_v F(s,x,v)|^2 dxdvds$$
$$\le e^{t+R}\int_{-R}^{0}\iint_{\mathbf{R}^d\times\mathbf{R}^d} \tfrac{1}{2}(H_0(s,x,v)^2 + \Lambda H_1(s,x,v)^2)dxdvds\,.$$

Pick $\Theta \equiv \Theta(t)$ in $C^\infty(\mathbf{R})$ such that

$$\mathbf{1}_{[-(3R+r)/4,0]} \le \Theta \le \mathbf{1}_{[-R,(R-r)/4]}\,, \qquad \text{and } |\Theta'| \le \tfrac{8}{R-r}\,.$$

Then

$$\int_{\mathbf{R}\times\mathbf{R}^d\times\mathbf{R}^d} (\Theta(t)F(t,x,v))^2 dxdvdt + \int_{\mathbf{R}\times\mathbf{R}^d\times\mathbf{R}^d} |\nabla_v(\Theta(t)F(t,x,v))|^2 dxdvdt$$
$$\le 4\Lambda R e^{2R}(\|H_0\|_{L^2}^2 + \|H_1\|_{L^2}^2)\,.$$

At this point, one has the choice of two strategies.

2.3.1 - Bouchut's Hypoellipticity Lemma

The first method applies to the special case $b(v) = v$. Pick $\eta \in C^\infty(\mathbf{R}^d)$ such that $\mathbf{1}_{B(0,2R)} \le \eta \le \mathbf{1}_{B(0,3R)}$. Observe that

$$(\partial_t + v\cdot\nabla_x)(\Theta(t)\eta(v)F(t,x,v)) = \Theta(t)\eta(v)H_0(t,x,v)$$
$$+ \operatorname{div}_v(\Theta(t)\eta(v)(H_1(t,x,v) + A(t,x,v)\nabla_v F(t,x,v))$$
$$- \Theta(t)\nabla\eta(v)\cdot(H_1(t,x,v) + A(t,x,v)\nabla_v F(t,x,v))$$
$$+ \Theta'(t)\eta(v)F(t,x,v)\,.$$

Applying Theorem 1.3 of [**3**] with $p = 2$, $r = 0$ and $m = 1$ shows that

$$
\begin{aligned}
&\|\nabla_v(\Theta(t)\eta(v)F(t,x,v))\|_{L^2}^2 \\
&\quad + \|D_t^{1/3}(\Theta(t)\eta(v)F(t,x,v))\|_{L^2}^2 + \|D_x^{1/3}(\Theta(t)\eta(v)F(t,x,v))\|_{L^2}^2 \\
&\le C[d,\Lambda]^2(1+R^2)^2\left(1+\tfrac{8}{R-r}\right)^2 e^{2R}(\|H_0\|_{L^2}^2 + \|H_1\|_{L^2}^2)\,.
\end{aligned} \tag{12}
$$

2.3.2 - Velocity Averaging

The second method is not specific to (1), and applies to the more general case (2). We only sketch the argument here, and refer the interested reader to the relevant literature.

The idea is to use velocity averaging combined with the dissipation rate coming from the energy inequality (3). Velocity averaging refers to a method for obtaining a regularizing effect from the free transport operator. Of course, this seems to be a desperate endeavour, since the free transport operator is hyperbolic, and therefore propagates singularities. Velocity averaging refers to a regularizing effect on the macroscopic densities obtained from the velocity distribution function by averaging in the velocity variable v, observed for the first time in [**1**, **13**] and studied more systematically in [**12**]. However, for the purpose of proving Theorem 2.1, we need a regularizing effect on the velocity distribution function itself, and not only on its averages in the velocity variable.

The idea is to start from the equation satisfied by $\Theta(t)\eta(v)F(t,x,v)$, i.e.

$$(\partial_t + b(v)\cdot\nabla_x)(\Theta(t)\eta(v)F(t,x,v)) = K_0(t,x,v) + \operatorname{div}_v K_1(t,x,v)$$

with[3] $K_0, K_1 \in L^2_{comp}(\mathbf{R}\times\mathbf{R}^d\times\mathbf{R}^d)$, and to apply a poor man's version of the DiPerna-Lions variant of velocity averaging [**8**] used to construct global renormalized solutions to the Vlasov-Maxwell system.

Now DiPerna and Lions treated the case $b(v) = v$, but the more general case of interest here can be found in Proposition 3.2 of [**14**]. The final remark of [**14**] makes it very clear that Proposition 3.2 of of [**14**] cannot give the optimal regularity exponent — however, this is by no means important in the present work. The idea is to use the proof of Proposition 3.2 of of [**14**] to obtain a bound on velocity averages of the form

$$\left\|\int_{\mathbf{R}^d}\Theta(t)\eta(w)F(t,x,w)\zeta_\epsilon(v-w)dw\right\|_{H^s_{t,x}}$$

[3]The notation $L^2_{comp}(\mathbf{R}^n)$ designates the subspace of elements of $L^2(\mathbf{R}^n)$ vanishing a.e. in the complement of some compact subset of $\mathbf{R}^n$.

for some $s > 0$ which depends on the exponent $\alpha \in (0, 1)$ such that

$$\sup_{\omega^2+|k|^2=1} |\{v \in \mathbf{R}^d \text{ s.t. } |\omega + b(v) \cdot k| \le \delta\}| \le C\delta^\alpha .$$

In the expression above, ζ_ϵ is a mollifier on $\mathbf{R}^d$, and the idea is to consider the decomposition

$$\begin{aligned}\Theta(t)\eta(v)F(t, x, v) = &\int_{\mathbf{R}^d} \Theta(t)\eta(w)F(t, x, w)\zeta_\epsilon(v - w)dw \\ &+ \Theta(t)\eta(v)F(t, x, v) - \int_{\mathbf{R}^d} \Theta(t)\eta(w)F(t, x, w)\zeta_\epsilon(v - w)dw .\end{aligned}$$

The bound on the first term coming from velocity averaging involves the $W^{1,\infty}$ norm of ζ_ϵ, which is of order $1/\epsilon^{d+1}$. The second term can be recast as

$$\int_{\mathbf{R}^d} \Theta(t)(\eta(v)F(t, x, v) - \eta(v + z)F(t, x, v + z))\zeta_\epsilon(z)dz$$

and is vanishingly small in $L^2(\mathbf{R}\times\mathbf{R}^d\times\mathbf{R}^d)$ as $\epsilon \to 0^+$, because of the estimate on $\|\nabla_v(\Theta(t)F)\|_{L^2}$ coming from the local energy bound.

This strategy for obtaining a regularizing effect on the velocity distribution itself, and not only on its macroscopic averages, comes from [**7**], where it used on a *model* of the Boltzmann equation without angular cutoff. A precursor of this idea has been used by [**22**] to prove the strong L^1_{loc} compactness of sequences of solutions of the Landau equation.

We refer the interested reader to Lemmas 2.1 and 2.4 of [**31**], where these (rather technical) estimates are worked out in detail.

2.3.3 - Conclusion

At this point, we specialize our discussion to (1), i.e. $b(v) = v$, and combine the inequality (11) with (12). The Sobolev embedding and (12) imply that

$$\begin{aligned}&\left(\int_{\mathbf{R}\times\mathbf{R}^d\times\mathbf{R}^d} |\Theta(t)\eta(v)F(t, x, v)|^p dtdxdv\right)^{1/p} \\ &\qquad\le C_S(d)C[d, \Lambda](1 + R^2)\left(1 + \tfrac{8}{R-r}\right)e^R(\|H_0\|_{L^2} + \|H_1\|_{L^2})\end{aligned}$$

with $\frac{1}{p} = \frac{1}{2} - \frac{1}{3(2d+1)}$, i.e. $p := \frac{12d+6}{6d+1} > 2$. Since $\Theta(t)\eta(v)$ and χ are identically equal to 1 on $Q[r]$, this inequality and (11) imply that

$$\|f\|_{L^p(Q[r])} \le C_S(d)C[d, \Lambda](1 + R^2)\left(1 + \tfrac{8}{R-r}\right)e^R(\|H_0\|_{L^2} + \|H_1\|_{L^2}) .$$

It remains to bound the right-hand side. By construction

$$\|H_1\|_{L^2} \leq \tfrac{4\Lambda}{R-r}\|f\|_{L^2(Q[R])}\,,$$

while

$$\|H_0\|_{L^2} \leq \|g\mathbf{1}_{f>0}\|_{L^2(Q[R])} + \tfrac{4(1+R)}{R-r}\|f\|_{L^2(Q[R])} + \tfrac{4\Lambda}{R-r}\|\nabla_v f\|_{L^2(Q[(R+r)/2])}\,.$$

This last term is controlled by (6): assuming without loss of generality that $\Lambda > 2$, one has

$$\|\nabla_v f\|^2_{L^2(Q[(R+r)/2])} \leq \tfrac{\Lambda}{2}\|g\mathbf{1}_{f>0}\|^2_{L^2(Q[R])} + \tfrac{\Lambda}{2}(1 + \tfrac{2(1+2R)}{R-r} + \tfrac{32\Lambda}{(R-r)^2})\|f\|^2_{L^2(Q[R])}\,,$$

so that

$$\|\nabla_v f\|_{L^2(Q[(R+r)/2])} \leq \tfrac{\Lambda}{2}\|g\mathbf{1}_{f>0}\|_{L^2(Q[R])} + \tfrac{\Lambda}{2}\left(1 + 2R + \tfrac{4\Lambda}{R-r}\right)\|f\|_{L^2(Q[R])}\,,$$

and hence

$$\|H_0\|_{L^2} \leq (1 + \tfrac{2\Lambda^2}{R-r})\|g\mathbf{1}_{f>0}\|_{L^2(Q[R])} + \left(\tfrac{4(1+R)(1+\Lambda^2)}{R-r} + \tfrac{8\Lambda^3}{(R-r)^2}\right)\|f\|_{L^2(Q[R])}\,.$$

Summarizing, we have proved the following estimate.

Lemma 2.3. *Let $\Lambda > 2$, and $R > r > 0$. Assume that $A \in S[Q[R], \Lambda]$, and let f be a nonnegative weak subsolution of* (2) *on $Q[R]$. Then*

$$\|f\|_{L^p(Q[r])} \leq C[d, R, \Lambda]\left((1 + \tfrac{1}{(R-r)^3})\|f\|_{L^2(Q[R])} + (1 + \tfrac{1}{R-r})\|g\mathbf{1}_{f>0}\|_{L^2(Q[R])}\right)$$

for some constant $C[d, R, \Lambda] > 0$ which is increasing in $R > 0$, and for the Lebesgue exponent $p := (12p + 6)/(6p + 1) > 2$.

2.4 - *De Giorgi's Iterations*

First, we set up a system of dyadic truncations as follows. The idea is to pick level sets corresponding to values of the Fokker-Planck solution increasing from 0 to 1/2, while, at the same time, this Fokker-Planck solution is restricted to a decreasing sequence of nested cylinders. In other words, we are restricting simultaneously the *domain* on which the solution is observed and the *set of values* taken by that solution.

Specifically, for each integer $k \geq 0$, we set

$$R_k := \tfrac{1}{2}(1 + 2^{-k}) =: -T_k\,, \qquad B_k := B(0, R_k)\,, \qquad Q_k := Q[R_k]\,,$$

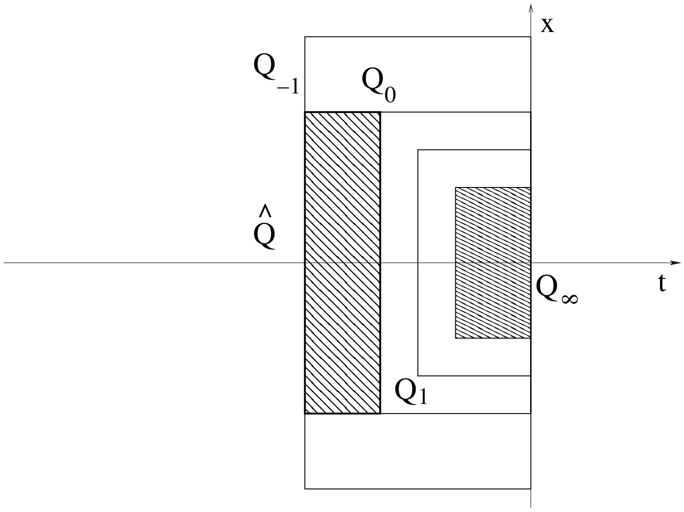

Fig. 1. The nested system of space-time cylinders Q_k in the past centered at the origin, and the space-time cylinder $\hat{Q}$

and we pick $\eta \in C^\infty(\mathbf{R}^d)$ such that

$$\mathbf{1}_{B_k} \leq \eta_k \leq \mathbf{1}_{B_{k-1}}\,, \quad \|\nabla \eta_k\|_{L^\infty} \leq 2^{k+2}\,.$$

The nested cylinders mentioned above are the Q_k's, and restricting the solution to Q_k is achieved by means of the blob function η_k.

On the other hand, we set

$$C_k := \tfrac{1}{2}(1-2^{-k})\,, \qquad \text{and} \qquad f_k := (f - C_k)_+\,.$$

In other words, the decreasing sequence of level sets of the solution f is defined by the sequence of inequalities $f \geq C_k$.

With the definitions above, we set

$$\begin{aligned} U_k := &\sup_{T_k \leq t \leq 0} \tfrac{1}{2} \iint_{\mathbf{R}^{2d}} \eta_k(x)\eta_k(v)^2 f_k(t,x,v)^2 dxdv \\ &+ \frac{1}{\Lambda}\int_{T_k}^0 \iint_{\mathbf{R}^{2d}} \eta_k(x)|\nabla_v(\eta_k(v) f_k(\tau,x,v))|^2 dxdvd\tau\,. \end{aligned}$$

Observe that, by construction,

$$0 \leq \ldots \leq U_k \leq U_{k-1} \leq \ldots \leq U_1 \leq U_0 < +\infty\,.$$

At this point, we

(a) write the local energy inequality (5) with $\eta = \eta_k$, with $c = C_k$ and for all $s \in (T_{k-1}, T_k)$, and
(b) average both sides of the resulting inequality over $s \in (T_{k-1}, T_k)$.

After elementary computations left to the reader, we arrive at the inequality

$$U_k \leq 2^{2k+3}(1+2\Lambda)\int_{Q_{k-1}} (f_k(\tau,x,v) + |g(\tau,x,v)|)f_k(\tau,x,v)dxdvd\tau \tag{13}$$

for all $k \geq 0$.

2.4.1 - The Nonlinearization Procedure

Now comes the first truly original argument in the De Giorgi iteration method. In the preceding inequality, U_k is a quadratic quantity in f_k, while the r.h.s. is quadratic in $(f_k, |g|)$. In other words, both sides of this inequality have the same homogeneity, which is only natural since (1) is a linear equation with source term g.

But the definition of f_k suggests involving the level set defined by the inequality $f_k > 0$, and this will be used to change the homogeneity of the r.h.s.. Specifically, let $q > 12d+6$, and recall that $p := \frac{12d+6}{6d+1}$; hence

$$\frac{1}{p} + \frac{2}{q} < \frac{6d+1}{12d+6} + \frac{2}{12d+6} = \frac{6d+3}{12d+6} = \frac{1}{2}\,. \tag{14}$$

By Hölder's inequality[4]

$$\begin{aligned}\int_{Q_{k-1}} (|g|+f_k)f_k dxdvd\tau &= \int_{Q_{k-1}} (|g|+f_k)f_k \mathbf{1}_{f_k>0} dxdvd\tau \\ &\leq \|f_k\|_{L^p(Q_{k-1})}\|g\|_{L^q(Q_{k-1})}|\{f_k>0\}\cap Q_{k-1}|^{1-\frac{1}{p}-\frac{1}{q}} \\ &\quad + \|f_k\|^2_{L^p(Q_{k-1})}|\{f_k>0\}\cap Q_{k-1}|^{1-\frac{2}{p}}\,.\end{aligned}$$

Then

$$|\{f_k>0\}| = |\{f > C_k\}| = |\{f_{k-1} > C_k - C_{k-1} = 2^{-k-1}\}|\,,$$

and, applying the Bienaymé-Chebyshev inequality shows that

$$|\{f_k>0\}| \leq 2^{2k+2}\|f_{k-1}\|^2_{L^2(Q_{k-1})} \leq 2^{2k+2}|T_{k-1}|U_{k-1} \leq 3\cdot 2^{2k+1}U_{k-1}\,. \tag{15}$$

[4]For each measurable $A \subset R^d$, the notation $|A|$ designates the Lebesgue measure of A.

Injecting this information in the r.h.s. of the inequality (13), we conclude that

$$
\begin{aligned}
U_k \leq & 3(1+2\Lambda)\cdot 2^{4k+4} U_{k-1}^{1-\frac{2}{p}} \|f_k\|^2_{L^p(Q_{k-1})} \\
& + 3\gamma \cdot 2^{2k+1} U_{k-1}^{1-\frac{1}{p}-\frac{1}{q}} \|f_k\|_{L^p(Q_{k-1})} \,,
\end{aligned}
$$

since $\|g\|_{L^q} \leq \gamma$.

2.4.2 - Using the Gain in Integrability

By Lemma 2.3 and (15), it holds

$$
\begin{aligned}
\|f_k\|_{L^p(Q_{k-1})} \leq & C[d, \tfrac{3}{2}, \Lambda]((1+2^{3k})\|f\|_{L^2(Q_{k-2})} + (1+2^k)\gamma|\{f_k > 0\}|^{1/2-1/q}) \\
\leq & C[d, \tfrac{3}{2}, \Lambda](2^{3k+1}\|f\|_{L^2(Q_{k-2})} + 2^{2k+3}\gamma U_{k-1}^{1/2-1/q}) \\
\leq & C[d, \tfrac{3}{2}, \Lambda]\cdot 2^{3(k+1)} \cdot \left(|T_{k-2}|^{1/2} U_{k-2}^{1/2} + \gamma U_{k-1}^{1/2-1/q}\right) \\
\leq & C[d, \tfrac{3}{2}, \Lambda]\cdot 2^{3(k+1)} \cdot (U_0^{1/q} + \gamma) U_{k-2}^{1/2-1/q} \,.
\end{aligned}
$$

Hence

$$
\begin{aligned}
U_k \leq & 3(1+2\Lambda) C[d, \tfrac{3}{2}, \Lambda]^2 (U_0^{1/q} + \gamma)^2 \cdot 2^{10k+10} U_{k-2}^{2-\frac{2}{p}-\frac{2}{q}} \\
& + 3\gamma \cdot C[d, \tfrac{3}{2}, \Lambda] \cdot (U_0^{1/q} + \gamma) \cdot 2^{5k+5} U_{k-2}^{\frac{3}{2}-\frac{1}{p}-\frac{2}{q}} \,,
\end{aligned}
$$

and since

$$
\left(2 - \frac{2}{p} - \frac{2}{q}\right) - \left(\frac{3}{2} - \frac{1}{p} - \frac{2}{q}\right) = \frac{1}{2} - \frac{1}{p} > 0 \,,
$$

we conclude that

$$
U_k \leq M \cdot 2^{10k} U_{k-2}^{\alpha} \,,
$$

with

$$
\begin{aligned}
M := & 3 \cdot 2^{10} (1+2\Lambda) C[d, \tfrac{3}{2}, \Lambda]^2 (U_0^{1/q} + \gamma)^2 U_0^{\frac{1}{2}-\frac{1}{p}} 2^{10} \\
& + 3 \cdot 2^5 \gamma \cdot C[d, \tfrac{3}{2}, \Lambda] \cdot (U_0^{1/q} + \gamma) \,, \\
\alpha := & \frac{3}{2} - \frac{1}{p} - \frac{2}{q} \,.
\end{aligned}
$$

2.4.3 - Conclusion

The idea is then to use the nonlinearity in U_{k-2} on the r.h.s. to fight the exponential growth in 2^{10k}. Setting

$$
\rho := 2^{10}(1+M) \,, \qquad V_k := U_{2k} \,,
$$

one has

$$V_{k+1} \leq \rho^k V_k^\alpha \,,$$

and an easy induction shows that

$$V_k \leq \rho^{k+\alpha(k-1)+\alpha^2(k-2)+\ldots+\alpha^{k-1}} V_0^{\alpha^k} \,.$$

Since

$$k+\alpha(k-1)+\alpha^2(k-2)+\ldots+\alpha^{k-1} = k\frac{\alpha^k-1}{\alpha-1} - \alpha\left(\frac{\alpha^k-1}{\alpha-1}\right)' \leq \frac{\alpha^{k+1}}{(\alpha-1)^2} \,,$$

we conclude that

$$V_k \leq \left(\rho^{\frac{\alpha}{(\alpha-1)^2}} U_0\right)^{\alpha^k} \,,$$

and the choice

$$U_0 < \rho^{-\frac{\alpha}{(\alpha-1)^2}}$$

implies that

$$U_{2k} \to 0 \quad \text{ as } k \to \infty \,.$$

Since

$$\int_{Q_k} (f - C_k)_+^2 dtdxdv \leq T_k U_k \leq U_k \,,$$

applying Fatou's lemma shows that

$$f - \tfrac{1}{2} \leq 0 \text{ a.e. on } \bigcap_{k \geq 0} Q_k = \overline{Q[\tfrac{1}{2}]} \,.$$

On the other hand, (13) shows that

$$U_0 \leq 8(1+2\Lambda)\|f_0\|_{L^2(Q[3/2])}\left(\|f_0\|_{L^2(Q[3/2])} + \gamma |Q[3/2]|^{\frac{1}{2}-\frac{1}{q}}\right) \,,$$

so that $U_0 < \rho^{-\frac{\alpha}{(\alpha-1)^2}}$ can be realized by taking $\|f_0\|_{L^2(Q[3/2])}$ smaller than some threshold depending only on $\rho, \Lambda, \alpha, \gamma, d$. This concludes the proof of Theorem 2.1.

2.5 - *Moser's Iterations*

In this section, we briefly sketch Moser's iteration argument in the simplest possible setting, so that the interested reader can compare it with De Giorgi's. In the interest of simplicity, we shall assume that the source term $g = 0$.

2.5.1 - Step 1: constructing positive subsolutions

Assuming that $A \in S[Q[\frac{3}{2}], \Lambda]$, consider

$$f \in C(-\tfrac{3}{2}, 0; L^2(B(0, \tfrac{3}{2})^2)) \cap FP[A, 0, Q[\tfrac{3}{2}]].$$

Pick $\chi_\epsilon \in C^2(\mathbf{R})$ so that $\chi_\epsilon'' \geq 0$ while $\chi_\epsilon(z) = 0$ for all $z \geq 0$, and $\chi_\epsilon(z) \to z_+^\beta$ with $1 < \beta < 2$ as $\epsilon \to 0^+$.

We deduce from (4) that f_+^β is a nonnegative subsolution of (1), i.e.

$$(\partial_t + v \cdot \nabla_x) f^\beta \leq \operatorname{div}_v(A \nabla_v f^\beta) \quad \text{ on } Q[\tfrac{3}{2}]\,.$$

2.5.2 - Step 2: using the gain in integrability

Using Lemma 2.3 shows that

$$\left(\int_{Q[1]} f_+(t,x,v)^{\beta p} dt dx dv \right)^{\frac{2}{p}} \\ \leq C[d, \tfrac{3}{2}, \Lambda]^2 \left(1 + \tfrac{1}{(\frac{3}{2}-1)^3} \right)^2 \int_{Q[\frac{3}{2}]} f_+(t,x,v)^{2\beta} dt dx dv\,,$$

where $p = (12d + 6)/(6d + 1) > 2$. In particular, we can apply to f_+^β the procedure described in Step 1 and iterate.

2.5.3 - Step 3: designing the iterations

Define $\beta_n := (p/2)^n$ for all integer $n \geq 0$, together with

$$1 = r_0 > r_1 > r_2 > \ldots > r_n > r_{n+1} > \ldots \text{ s.t. } r_{n-1} - r_n = \frac{1}{\sigma n^2}\,,$$

so that

$$r_n = r_0 - \tfrac{1}{\sigma} \sum_{k=0}^{n} \frac{1}{k^2} \to 1 - \tfrac{\pi^2}{6\zeta} > \tfrac{1}{2}$$

if $\zeta > \frac{\pi^2}{12}$. For instance, one can choose $\zeta = \pi^2$. Set $Q_n := Q[r_n]$, and

$$A_n := \left(\int_{Q_n} f_+(t,x,v)^{2\beta_n} dt dx dv \right)^{1/2\beta_n}, \qquad n \geq 0\,.$$

We deduce from Step 2 applied to $f_+^{2\beta_n}$ the inequality

$$A_{n+1} \le C[d, \tfrac{3}{2}, \Lambda]^{1/\beta_n}(1+\zeta^3(n+1)^6)^{1/\beta_n} A_n\,, \quad n \ge 0\,.$$

Hence

$$A_{n+1} \le A_0 C[d, \tfrac{3}{2}, \Lambda]^{\sum_{0\le j\le n}\frac{1}{\beta_j}} \prod_{j=0}^{n}(1+\zeta^3(j+1)^6)^{1/\beta_j}\,, \quad n \ge 0\,.$$

2.5.4 - Step 4: Passing to the limit as $n \to \infty$

Observe that

$$\sum_{j\ge 0}\frac{1}{\beta_j} = \sum_{j\ge 0}\left(\frac{2}{p}\right)^j = \frac{p}{p-2} < \infty\,,$$

while

$$\begin{aligned}\ln\prod_{j=0}^{n}(1+\zeta^3(j+1)^6)^{1/\beta_j} =& \sum_{j=0}^{n}\left(\frac{2}{p}\right)^j \ln(1+\zeta^3(j+1)^6)\\ \le& \zeta^3\sum_{j\ge 0}^{n}(j+1)^6\left(\frac{2}{p}\right)^j \in [\zeta^3, +\infty)\,.\end{aligned}$$

Hence the infinite product

$$\prod_{j\ge 0}(1+\zeta^3(j+1)^6)^{1/\beta_j}$$

is absolutely convergent, and

$$\|f_+\|_{L^\infty(Q[\frac{1}{2}])} \le \varlimsup_{n\ge 0} A_n \le A_0 C[d, \tfrac{3}{2}, \Lambda]^{\frac{p}{p-2}} \prod_{j\ge 0}(1+\zeta^3(j+1)^6)^{1/\beta_j}\,.$$

This concludes the proof of Theorem 2.1 by Moser's iterations.

3 - From L^∞ to $C^{0,\sigma}$

In this section, we shall prove Theorem 1.2. The proof is based on the following ingredients

(a) the action of a special one-parameter group of scaling transforms on the Fokker-Planck equation,

(b) an "intermediate values" theorem showing that a Fokker-Planck solution needs "enough room" to jump from 0 to 1, and

(c) a "reduction of oscillations" argument resulting from (b).

Zooming in on any point $(t_0, x_0, v_0) \in I \times \Omega$ results in a modulus of Hölder continuity near (t_0, x_0, v_0). Since (t_0, x_0, v_0) is arbitrary, the desired result follows.

3.1 - *Zooming and the Fokker-Planck equation*

Consider the zooming transformation defined as follows

$$\mathcal{T}_\epsilon[t_0, x_0, v_0]F(s, y, \xi) := F(t_0 + \epsilon^2 s, x_0 + \epsilon^3 y + \epsilon^2 s v_0, v_0 + \epsilon\xi)$$

for each $\epsilon > 0$, each $x_0, v_0 \in \mathbf{R}^d$, and each $t_0 \in \mathbf{R}$.

One easily checks that the zooming transformation acts on the Fokker-Planck equation in the following manner:

$$\begin{aligned}
&(\partial_s + \xi \cdot \nabla_y)\mathcal{T}_\epsilon[t_0, x_0, v_0]F(s, y, \xi) \\
&= \epsilon^2(\partial_t + (v_0 + \epsilon\xi) \cdot \nabla_x)F(t_0 + \epsilon^2 s, x_0 + \epsilon^3 y + \epsilon^2 s v_0, v_0 + \epsilon\xi) \\
&= \epsilon^2 \operatorname{div}_v(A\nabla_v F(t_0 + \epsilon^2 s, x_0 + \epsilon^3 y + \epsilon^2 s v_0, v_0 + \epsilon\xi)) \\
&\quad + \epsilon^2 G(t_0 + \epsilon^2 s, x_0 + \epsilon^3 y + \epsilon^2 s v_0, v_0 + \epsilon\xi) \\
&= \operatorname{div}_\xi(\mathcal{T}_\epsilon[t_0, x_0, v_0]A(s, y, \xi)\nabla_\xi \mathcal{T}_\epsilon[t_0, x_0, v_0]F(s, y, \xi)) \\
&\quad + \epsilon^2 \mathcal{T}_\epsilon[t_0, x_0, v_0]G(s, y, \xi)\,.
\end{aligned} \tag{16}$$

The Hölder continuity of the solution of the Fokker-Planck equation is obtained by combining the zooming procedure above with the next two (fundamental) lemmas.

3.2 - *The De Giorgi Intermediate Values Lemma*

Lemma 3.1. *Let $\Lambda > 1$, $\eta > 0$ and $\omega \in (0, 1 - 2^{-d})$. Then there exist $\theta \in (0, \frac{1}{2})$ and $\alpha > 0$ satisfying the following property.*

For all $A \in S[\hat{Q} \cup Q[1], \Lambda]$, and for all f, g such that $f \in FP[A, g, \hat{Q} \cup Q[1]]$, satisfying

$$f, |g| \le 1 \text{ a.e. on } \hat{Q} \cup Q[1] \quad \text{ and } |\{f \le 0\} \cap \hat{Q}| \ge \tfrac{1}{2}|\hat{Q}|$$

where $\hat{Q} := (-\frac{3}{2}, 1] \times B(0,1) \times B(0,1)$, the following conclusion holds:

(a) $|\{f \geq 1-\theta\} \cap Q[\frac{\omega}{4}]| < \eta$, *or*

(b) $|\{0 < f < 1-\theta\} \cap (\hat{Q} \cup Q[1])| \geq \alpha$.

In the original, elliptic setting, the analogous property is a consequence of the De Giorgi isoperimetric inequality (Lemma II in [**6**]): let $w \in H^1(B(0,1))$ (where $B(0,1)$ is the open unit ball of $\mathbf{R}^d$), and set

$$A := \{w = 0\} \cap B(0, \tfrac{1}{2})\,,\ C := \{w = 1\} \cap B(0, \tfrac{1}{2})\,,\ D := \{0 < w < 1\} \cap B(0, \tfrac{1}{2})\,.$$

Then

$$|A| \cdot |C|^{1-1/d} \leq |D|^{1/2} \|\nabla w\|_{L^2(D)}\,.$$

In particular, a H^1 function in $\mathbf{R}^d$ cannot "jump" from 0 to 1, but "needs some room" to vary from 0 to 1. How much "room" is needed is made precise by the inequality above.

The proof of Lemma 3.1 given below is quite different from the elliptic case recalled above; for want of an isoperimetric inequality adapted to our setting, we shall instead argue by contradiction, by means of a compactness argument. For a quantitative proof of this result, see [**16**].

P r o o f o f L e m m a 3.1. If this conclusion were wrong, there would exist sequences $A_n \in S[\hat{Q} \cup Q[1], \Lambda]$ and f_n, g_n s.t. $f_n \in FP[A_n, g_n, \hat{Q} \cup Q[1]]$ for all integer $n \geq 0$, with

$$\begin{cases} f_n \leq 1\,, \quad |g_n| \leq 1\,, \quad |\{f_n \leq 0\} \cap \hat{Q}| > \frac{1}{2}|\hat{Q}|\,, \\ |\{0 < f_n < 1 - 2^{-n}\} \cap (\hat{Q} \cup Q[1])| < 2^{-n}\,, \\ |\{f_n \geq 1 - 2^{-n}\} \cap Q[\frac{\omega}{4}]| > \eta\,. \end{cases}$$

This would imply in particular that, possibly after extracting a subsequence of f_n and A_n, it holds

$$f_n^+ \to \mathbf{1}_P(t,x) \quad \text{in} \quad L^p(\hat{Q} \cup Q[1])$$

for $1 \leq p < \infty$, and that

$$A_n \nabla_v f_n^+ \rightharpoonup h \quad \text{in} \quad L^2_{loc}(\hat{Q} \cup Q[1])$$

as $n \to \infty$, with

$$(\partial_t + v \cdot \nabla_x)\mathbf{1}_P(t,x) \leq \operatorname{div}_v h + 1 \qquad \text{in} \quad \mathcal{D}'(\hat{Q} \cup Q[1])\,.$$

(That the limiting distribution function is a.e. equal to an indicator function comes from the fourth assumption on f_n. That the set where this indicator function is positive is of the form $P \times \mathbf{R}_v^d$ up to a Lebesgue negligible set, in other words that this indicator function is a.e. equal to a function independent of v comes from the L^2 bound on $\nabla_v f_n$ implied by the local energy estimate (6). At this point, we have used the same argument as De Giorgi's, i.e. the fact that an indicator function which belongs to the Sobolev space H^1 must be a.e. constant.)

Pick $v_0 \in B(0, \frac{1}{3})$ and let ζ_ϵ be a radial mollifier at the origin. Multiplying both sides of the inequality above by $\zeta_\epsilon(v - v_0)$ and averaging in v leads to

$$(\partial_t + v \cdot \nabla_x)\mathbf{1}_P(t,x) \leq 1 + \|\nabla\zeta_\epsilon\|_{L^2}\|h(t,x,\cdot)\|_{L^2(B(0,1))} \in L^2((-\tfrac{3}{2},0] \times B(0,1))\,.$$

Since the function $s \mapsto \mathbf{1}_P(t_0 + s, x_0 + sv_0)$ has only jump discontinuities for a.e. $(t_0, x_0) \in (-\frac{3}{2}, 0) \times B(0,1)$, the inequality above implies that

$$(\partial_t + v_0 \cdot \nabla_x)\mathbf{1}_P(t,x) \leq 0 \qquad \text{in } \mathcal{D}'((-\tfrac{3}{2},0) \times B(0,1))\,.$$

Thus, the condition $|\{f_n \leq 0\} \cap \hat{Q}| > \frac{1}{2}|\hat{Q}|$ implies that $|P^c \cap \hat{Q}| > \frac{1}{2}|\hat{Q}| > 0$, and hence, by propagation (see Figure 2) it holds $|(P \times B(0,1)) \cap Q[\frac{\omega}{4}]| = 0$.

On the other hand, the condition $|\{f_n \geq 1 - 2^{-n}\} \cap Q[\frac{\omega}{4}]| > \eta$ satisfied for each integer $n \geq 0$ and the fact that $f_n \to \mathbf{1}_P(t,x)$ in $L^p(\hat{Q} \cup Q[1])$ implies that $|(P \times B(0,1)) \cap Q[\frac{\omega}{4}]| \geq \eta > 0$, which leads to a contradiction. $\square$

3.3 - *Reduction of Oscillations*

First we recall the notion of oscillation of a real-valued function defined on a set.

Definition 3.2. *Let X be a set, and let $f : X \to \mathbf{R}$. The oscillation of f on X is*

$$\mathrm{osc}_X f := \sup_{x \in X} f(x) - \inf_{x \in X} f(x)\,.$$

For (weak) solutions of the Fokker-Planck equation, zooming in on a phase space point leads to a reduction of oscillations. This is the core of the proof of Hölder regularity for such solutions.

Lemma 3.3. *Let $\Lambda > 1$ and $A \in S[\hat{Q} \cup Q[1], \Lambda]$. For all $\omega \in (0, 1 - 2^{-d})$, there exist $\beta, \mu \in (0,1)$ satisfying the following property. For all f, g such that $f \in FP[A, g, \hat{Q} \cup Q[1]]$ and*

$$|f| \leq 1 \quad \textit{while} \quad |g| \leq \beta \qquad \textit{a.e. on } \hat{Q} \cup Q[1]\,,$$

it holds

$$\mathrm{osc}_{Q[\frac{\omega^3}{54}]} f \leq 2\mu\,.$$

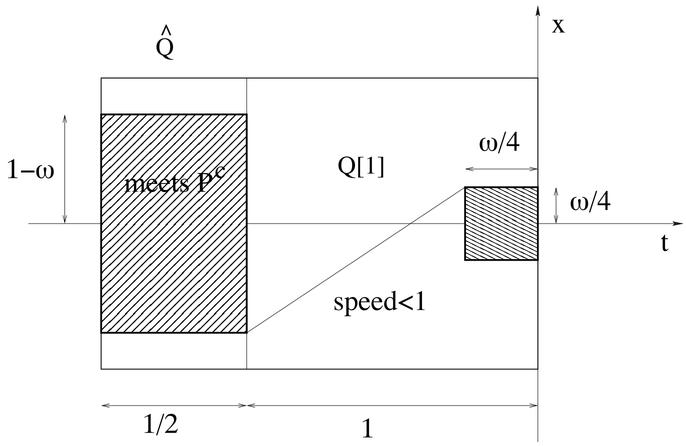

Fig. 2. Any point in the left shaded region traveling with speed < 1 can reach the right shaded region. Since P^c meets the left shaded region and $\mathbf{1}_P$ is nonincreasing along characteristics, the right shaded region meets P in a set of measure 0. Because the maximum speed in this phase space domain is 1, it is absolutely essential that the domains $\hat{Q}$ and $Q[\frac{\omega}{4}]$ be away from one another of a distance $1-\frac{\omega}{4}$ in the time variable (see Remark 4 in [**16**] on p. 1161). Since the maximal spatial distance of a trajectory joining the two shaded regions is at most $1-\omega+\frac{\omega}{4} < 1-\frac{\omega}{4}$, this distance can be traveled by a point moving at speed 1 in time less than $1-\frac{\omega}{4}$.

Sketch of the proof. Given $\omega \in (0, 1-2^{-d})$, set

$$\eta := \left(\tfrac{\omega}{3}\right)^{4d+2} \kappa[d, \Lambda, \tfrac{\omega^2}{9}, \infty],$$

where κ is the constant in Theorem 2.1. With these data, Lemma 3.1 provides us with two positive constants $\theta \in (0, \frac{1}{2})$ and $\alpha > 0$. Pick then $0 < \beta \ll 1$ small enough so that

$$\ln \frac{1}{\beta} \geq \left(\frac{\frac{1}{2}|\hat{Q}| + |Q[1]|}{\alpha} + 2 \right) \ln \frac{1}{\theta} \, .$$

Assume that $|\{f \leq 0\} \cap \hat{Q}| \geq \frac{1}{2}|\hat{Q}|$, and define

$$f_k := \tfrac{1}{\theta}(f_{k-1} - 1) + 1 \, , \qquad f_0 := f \, .$$

Thus

$$1 - f_k = \frac{1}{\theta}(1 - f_{k-1}) = \ldots = \frac{1}{\theta^k}(1 - f_0) \geq 0 \, .$$

In particular

$$1 - f_k \geq (1 - f_{k-1})$$

since $1/\theta > 1$ (in fact $1/\theta > 2$), so that

$$\ldots \leq f_k \leq f_{k-1} \leq \ldots \leq f_0 = f \leq 1 \, .$$

Besides
$$f_k \in FP[A, \theta^{-k} g, \hat{Q} \cup Q[1]]$$
since the Fokker-Planck operator
$$f \mapsto (\partial_t + v \cdot \nabla_x) f - \mathrm{div}_v (A \nabla_v f)$$
is linear. Now observe that
$$\{f \le 0\} \subset \{f_1 \le 0\} \subset \ldots \subset \{f_{k-1} \le 0\} \subset \{f_k \le 0\} \subset \ldots$$
so that
$$\tfrac{1}{2}|\hat{Q}| \le |\hat{Q} \cap \{f \le 0\}| \le \ldots \le |\hat{Q} \cap \{f_k \le 0\}|\,.$$
Besides
$$\{f_k \le 0\} = \{f_{k-1} \le 0\} \cup \{0 < f_{k-1} \le 1 - \theta\}\,.$$
Consider then
$$m_k := |\{f_k \le 0\} \cap (\hat{Q} \cup Q[1])| \quad \text{for all } k \ge 0\,;$$
this sequence satisfies
$$m_k = m_0 + \sum_{j=1}^{k} |\{0 < f_{k-1} \le 1 - \theta\} \cap (\hat{Q} \cup Q[1])|\,, \quad \text{and} \quad m_0 > \tfrac{1}{2}|\hat{Q}|\,.$$
Define then
$$\hat{k} = \left[\frac{\frac{1}{2}|\hat{Q}| + |Q[1]|}{\alpha}\right] + 1 \le \left[\frac{\ln(1/\beta)}{\ln(1/\theta)}\right]\,.$$
It is impossible that
$$|\{0 < f_{k-1} \le 1 - \theta\} \cap (\hat{Q} \cup Q[1])| \ge \alpha \quad \text{for } j = 1, \ldots, \hat{k}\,,$$
since this would imply that
$$\tfrac{1}{2}|\hat{Q}| + \hat{k}\alpha \le m_0 + \hat{k}\alpha \le m_{\hat{k}} \le |\hat{Q}| + |Q[1]|\,.$$
Then
$$m_0 > \tfrac{1}{2}|\hat{Q}| \implies \hat{k}\alpha \le \tfrac{1}{2}|\hat{Q}| + |Q[1]|\,,$$
which is incompatible with our choice of $\hat{k}$. This rules out case (b) in Lemma 3.1.

Notice that our choice of β implies that
$$\theta^{-\hat{k}} \beta \le 1\,, \qquad \text{so that } \theta^{-k}|g| \le 1 \text{ on } \hat{Q} \cup Q[1] \text{ for } k = 0, \ldots, \hat{k}\,.$$

Hence conclusion (a) of Lemma 3.1 must hold for some $\tilde{k} \in \{0, \ldots, \hat{k} - 1\}$, i.e.

$$|\{f_{\tilde{k}} \geq 1 - \theta\} \cap Q[\omega/2]| < \eta\,.$$

Therefore

$$\begin{aligned}\int_{Q[\omega/2]} (f_{\tilde{k}+1})_+^2 dtdxdv &= \int_{Q[\omega/2]} (f_{\tilde{k}+1})_+^2 \mathbf{1}_{f_{\tilde{k}} \geq 1-\theta} dtdxdv \\ &\leq \int_{Q[\omega/2]} \mathbf{1}_{f_{\tilde{k}} \geq 1-\theta} dtdxdv < \eta\,.\end{aligned}$$

(The penultimate inequality follows from the fact that

$$1 \geq f_{\tilde{k}+1} = 1 + \tfrac{1}{\theta}(f_{\tilde{k}} - 1) \geq 0$$

if and only if $f_{\tilde{k}} \geq 1 - \theta$.)

Now

$$\begin{aligned}\eta &> \int_{Q[\omega/2]} (f_{\tilde{k}+1})_+^2 dtdxdv \\ &= \left(\tfrac{\omega}{3}\right)^{4d+2} \int_{(-\frac{9}{2\omega},0]\times B(0,\frac{27}{2\omega^2})\times B(0,\frac{3}{2})} (\mathcal{T}_{\omega/3}[0,0,0] f_{\tilde{k}+1}(s,y,\xi))_+^2 dsdyd\xi \\ &\geq \left(\tfrac{\omega}{3}\right)^{4d+2} \int_{Q[\frac{3}{2}]} (\mathcal{T}_{\omega/3}[0,0,0] f_{\tilde{k}+1}(s,y,\xi))_+^2 dsdyd\xi\end{aligned}$$

implies that $\mathcal{T}_{\omega/3}[0,0,0] f_{\tilde{k}+1}$ satisfies the conditions of De Giorgi's first lemma (Theorem 2.1), so that

$$\mathcal{T}_{\omega/3}[0,0,0] f_{\tilde{k}+1} \leq \tfrac{1}{2} \text{ a.e. on } Q[\tfrac{1}{2}]\,,$$

which implies in turn that

$$f_{\tilde{k}+1} \leq \tfrac{1}{2} < 1 - \theta \text{ a.e. on } Q[\tfrac{\omega^3}{54}]\,.$$

Since $1 - f = \theta^{\tilde{k}+1}(1 - f_{\tilde{k}+1})$, we conclude that

$$f < 1 - \theta^{\tilde{k}+2} \text{ a.e. on } Q[\tfrac{\omega^3}{54}]\,.$$

Hence

$$\operatorname{osc}_{Q[\frac{\omega^3}{54}]} f \leq 1 - \theta^{\tilde{k}+2} - (-1) = 2 - \theta^{\tilde{k}+2} \leq 2\mu$$

with

$$\mu =: 1 - \theta^{\hat{k}+2} \geq 1 - \tfrac{1}{2}\theta^{\tilde{k}+2}\,,$$

since $0 \leq \tilde{k} < \hat{k}$ and $\theta \in (0, \frac{1}{2})$.

This concludes the proof of the reduction of oscillations lemma. $\square$

Notice that

$$0 < \theta < \tfrac{1}{2} \implies \tfrac{1}{2} < 1 - (\tfrac{1}{2})^3 \leq 1 - \theta^{\hat{k}+2} = \mu < 1$$

so that $2\mu < 2$. Since one assumes that $|f| \leq 1$ on $\hat{Q} \cup Q[1]$, then $\mathrm{osc}_{\hat{Q} \cup Q[1]} f$ could be as large as 2, so that Lemma 3.3 corresponds indeed to a reduction of the maximum oscillation of f from 2 to $2\mu < 2$ when reducing the domain of definition of f from $\hat{Q} \cup Q[1]$ to $Q[\frac{\omega^3}{54}]$.

3.4 - *Hölder Continuity*

Observe that

$$\mathrm{osc}_{Q[\epsilon^3 r]} f \leq \mathrm{osc}_{Q[r]} \mathcal{T}_\epsilon[0,0,0] f = \mathrm{osc}_{\tilde{Q}_\epsilon[r]} f \leq \mathrm{osc}_{Q[\epsilon r]} f \,, \tag{17}$$

for all $\epsilon \in (0,1)$, where

$$\tilde{Q}_\epsilon[r] := (-\epsilon^2 r, 0] \times B(0, \epsilon^3 r) \times B(0, \epsilon r) \subset Q[\epsilon r] \,.$$

If f and g satisfy the assumptions of Lemma 3.3, i.e. if $f \in FP[A, g, \hat{Q} \cup Q[1])$ with $|f| \leq 1$ while $|g| \leq \beta$ a.e. on $\hat{Q} \cup Q[1]$, it holds

$$\mathrm{osc}_{Q[\frac{\omega^3}{54}]} f \leq 2\mu \,.$$

Setting $\epsilon = \omega^3/81$ and $r = 3/2$, we deduce from the second inequality in (17) that

$$\mathrm{osc}_{\hat{Q} \cup Q[1]} \frac{1}{\mu} \mathcal{T}_{\omega^3/81} f \leq \mathrm{osc}_{Q[\frac{3}{2}]} \frac{1}{\mu} \mathcal{T}_{\omega^3/81} f \leq \mathrm{osc}_{Q[\frac{\omega^3}{54}]} \frac{1}{\mu} f \leq 2 \,.$$

Because of (16)

$$\frac{1}{\mu} \mathcal{T}_{\omega^3/81}[0,0,0] f \in \mathcal{FP}\left[\mathcal{T}_{\omega^3/81}[0,0,0] A, \frac{\omega^3}{81\mu} \mathcal{T}_{\omega^3/81}[0,0,0] g, Q[\tfrac{3}{2}]\right]$$

and

$$\left\| \frac{\omega^3}{81\mu} \mathcal{T}_{\omega^3/81}[0,0,0] g \right\|_{L^\infty(Q[3/2])} \leq \frac{\omega^3}{81\mu} \|g\|_{L^\infty(Q[\omega^3/54])} \leq \beta \,,$$

since $\omega^3 \leq 1 < 81/2 \leq 81\mu$ and $\|g\|_{L^\infty(Q[\omega^3/54])} \leq \|g\|_{L^\infty(\hat{Q} \cup Q[1])} \leq \beta$, while $\mathcal{T}_{\omega^3/81}[0,0,0] A \in S[Q[\frac{3}{2}], \Lambda]$. Therefore

$$\mathrm{osc}_{Q[\frac{\omega}{54}]} \frac{1}{\mu} \mathcal{T}_{\omega^3/81}[0,0,0] f \leq 2\mu$$

by Lemma 3.3 (where the function f is replaced with

$$f - \tfrac{1}{2}\left(\sup_{\hat{Q}\cup Q[1]} f + \inf_{\hat{Q}\cup Q[1]} f\right)$$

if needed). Iterating this argument $n-1$ times shows that

$$\mathrm{osc}_{Q[\frac{\omega}{54}]} \frac{1}{\mu^{n-1}} \mathcal{T}^{n-1}_{\omega^3/81}[0,0,0]f \le 2\mu\,,$$

so that, by the first inequality in (17),

$$\mathrm{osc}_{Q[\frac{3}{2}\frac{\omega^{9n-8}}{81^{3n-2}}]} f \le \mathrm{osc}_{Q[\frac{\omega}{54}]} \mathcal{T}^{n-1}_{\omega^3/81}[0,0,0]f \le 2\mu^n\,.$$

Thus

$$-\tfrac{3}{2}\tfrac{\omega^{9n-8}}{81^{3n-2}} < s \le 0 \text{ and } |y|,|\xi| \le \tfrac{3}{2}\tfrac{\omega^{9n-8}}{81^{3n-2}} \implies |f(s,y,\xi) - f(0,0,0)| \le 2\mu^n\,,$$

so that

$$0 \le -s, |y|, |\xi| \le \tfrac{\omega}{54} \implies |f(s,y,\xi) - f(0,0,0)| \le C\max(|s|,|y|,|\xi|)^\sigma$$

with

$$\sigma := \ln\mu/3\ln(\omega^3/3^4) \quad \text{and} \quad C := 2\left(2\cdot 3^7/\omega^6\right)^\sigma\,.$$

This holds for $f \in FP[A,g,\hat{Q}\cup Q[1]]$ satisfying both $\|f\|_{L^\infty(\hat{Q}\cup Q[1])} \le 1$ and $\|g\|_{L^\infty(\hat{Q}\cup Q[1])} \le \beta$. Therefore, one can remove the restrictions on the sizes of f and g by changing C into $C(1+\|f\|_{L^\infty(\hat{Q}\cup Q[1])})(1+\frac{1}{\beta}\|g\|_{L^\infty(\hat{Q}\cup Q[1])})$.

Finally, let $f \in FP[A,g,I\times\Omega]$ with $f,g \in L^\infty(I\times\Omega)$, and let $(t_0,x_0,v_0) \in I\times\Omega$. Pick $\epsilon > 0$ small enough so that

$$\mathcal{T}_\epsilon[t_0,x_0,v_0]Q[\tfrac{3}{2}] \subset I\times\Omega\,.$$

Thus $F = \mathcal{T}_\epsilon[t_0,x_0,v_0]f$ is a solution of (1) with diffusion matrix $\mathcal{T}_\epsilon[t_0,x_0,v_0]A$ and source $\epsilon^2\mathcal{T}_\epsilon[t_0,x_0,v_0]g$. Arguing as above with F and setting

$$s = t - t_0\,, \quad \xi = v - v_0\,, \quad y = x - x_0 - sv_0\,,$$

we conclude that

$$|f(t,x,v) - f(t_0,x_0,v_0)| \le C((1+|v_0|)|t-t_0| + |x-x_0| + |v-v_0|)^\sigma$$

provided that

$$0 < t_0 - t < \frac{\epsilon^3\omega^3}{54(1+|v_0|)} \text{ and } \max(|x-x_0|,|v-v_0|) < \frac{\epsilon^3\omega^3}{54(1+|v_0|)}\,,$$

which is the desired continuity estimate.

4 - Conclusion

This note proposes a reading guide for the somewhat more technical references [**10**, **31**]. An alternate point of view on the same result for (1) can be found in [**26**, **29**, **30**].

A natural sequel to the discussion presented above is the Harnack inequality for (1): see Theorem 1.6 in [**10**]. Other approaches to the Harnack inequality can be found in [**2**, **15**, **16**].

Let us conclude with a physically realistic kinetic model to which the mathematical tools described in this note can be applied.

Consider the Landau equation (for charged particles with velocity distribution function $f \equiv f(t,x,v) \geq 0$ a.e. interacting via a Coulomb potential):

$$(\partial_t + v\cdot\nabla_x)f(t,x,v) = \operatorname{div}_v \int_{\mathbf{R}^3} a(v-w)(\nabla_v - \nabla_w)(f(t,x,v)f(t,x,w))dw\,, \tag{18}$$

where $a(z) := \nabla^2|z|$. Set

$$\rho_f(t,x) := \int_{\mathbf{R}^3} f(t,x,v)dv\,.$$

The Landau equation can be put in the form

$$(\partial_t + v\cdot\nabla_x)f = \operatorname{div}_v((f \star_v a)\nabla_v f) - \operatorname{div}_v(f(f\star_v \nabla a))$$

where $\star_v$ designates the convolution in the v variable. Except for the first order differential operator $f \mapsto \operatorname{div}_v(f(f\star_v \nabla a))$, this is an example of Fokker-Planck equation analogous to (1), except that the diffusion matrix $f\star_v a$ depends on the solution f itself.

Theorem 4.1. *Assume that*

$$f \in L^\infty(Q[1]) \cap L^2((-1,0)\times B(0,1)_x; H^1(B(0,1)_v))$$

while

$$(\partial_t + v\cdot\nabla_x)f \in L^2((-1,0)\times B(0,1)_x; H^{-1}(B(0,1)_v))\,,$$

and that f is a distributional solution of (18), *such that*

$$M := \underset{(-1,0)\times B(0,1)}{\text{ess-sup}} \left(\rho_f(t,x) + \frac{1}{\rho_f(t,x)} + \int_{\mathbf{R}^3} f(t,x,v)(\tfrac{1}{2}|v|^2 + \ln f(t,x,v))dv\right) < \infty.$$

Then $f \in C^\alpha(Q[\frac{1}{2}])$, where $\alpha \equiv \alpha[M] \in (0,1)$.

(This is Theorem 1.1 in [**10**]). At the time of this writing, the conditions required on the distribution function f in this theorem are not known to be verified. Put in other words, the "natural" bounds on a weak solution f of (18) lead to much worse Lebesgue or Sobolev exponents. In fact, even in the space homogeneous case, whether there is global existence of a classical solution or finite time blow-up of solutions to the Cauchy problem for (18) remains an outstanding open problem in the mathematical analysis of kinetic models. Only very partial information is known on this problem: see for instance [**9**,**11**] and the references therein. After the present paper was accepted, Guillen and Silvestre have claimed a proof of regularity for space-homogeneous solutions of the Landau equation: see [**32**]. (Note and reference added in proof.)

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