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On propagation of exponential moments for the Landau kinetic equation

Abstract. The paper is devoted to study of asymptotic properties (for large values of energy) of radially symmetric solutions to the spatially homogeneous Landau equation for Coulomb forces. The main result of the paper is the proof of propagation in time of the exponential moment of the third order and some explicit time-dependent estimates of this moment. Roughly speaking, this means that the high energy tails of the form $\exp[-b(t)v^k]$ with some $k \geq 3$ are typical for solutions of the Landau equation with initial data having compact support. A comparison with related results for similar kinetic equations is briefly discussed.

Keywords. Landau kinetic equation, radial symmetry, distribution function, power moments, exponential moments.

Mathematics Subject Classification: 82D10, 82C70, 82C31.

The paper is dedicated to memory of Giampiero Spiga.

1 - Introduction

We investigate in this paper some asymptotic (for large values of energy) properties of solutions of the Landau kinetic equation for Coulomb forces. The words “kinetic” and “for Coulomb forces” are omitted below for brevity. This equation attracted a lot of attention in recent years, see, in particular, important papers [**10, 11**], [**6**]. A brief review of results and open problems in this area can be found in our previous paper [**1**] and in related references from it.

In the present paper we continue the line of [**1**] and try to extend and clarify one of results of that paper. We concentrate below on the proof of

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propagation in time of the exponential moment of the third order for radially symmetric solutions to the Landau equation. In our opinion, this property is very important because it shows very clearly a big difference between large energy asymptotic properties of solutions to the Landau equation and solutions to similar kinetic equations for hard forces.

We remind the known results in that area for the Boltzmann equation. The exponential moment $I_k(\lambda, t)$ of order $k > 0$ reads as

$$I_k(\lambda, t) = \frac{1}{4\pi} \int\limits_{\mathbb{R}^3} dv f(v,t) \exp(\lambda |v|^k), \quad \lambda > 0,$$

where $f(v,t)$ is a non-negative solution of the Boltzmann equation. Here $v \in \mathbb{R}^3$ and $t \geq 0$ are standard notations for velocity and time respectively. Usually we assume that $I_k(\lambda_0, 0) < \infty$ for some $\lambda_0 > 0$ and fixed $k > 0$. If $I_k(\lambda, t) < \infty$ for some $t > 0$ and sufficiently small $0 < \lambda < R(t)$, then we say that the exponential moment $I_k(\lambda, t)$ propagates in time. It is clear, that for fixed $t > 0$ the function $R(t)$ denotes the radius of convergence of the Taylor series in λ for analytic at $\lambda = 0$ function $I(\lambda)$.

The well-known lower estimate [**13**] for Boltzmann equation with hard forces and pseudo-Maxwell molecules states that $f(v,t) \geq a(t) \exp[-b(t)|v|^2]$ for any $t > 0$ and some non-explicitly known functions $a(t) > 0$ and $b(t) > 0$. The above lower pointwise estimate was also obtained in [**7**] for the Landau equation with hard potentials. Hence, the most interesting case for hard potentials is $k = 2$. The propagation of $I_2(\lambda, t)$ was proved for the Boltzmann equations in several publications listed below. Moreover, it was proved in papers [**4**] (hard spheres), [**9**] (hard potentials with cut-off), [**5**] (Maxwellian molecules with cut-off), [**8**] (hard potentials without cut-off) that $I_2(\lambda, t)$ remains bounded uniformly in time for all $t > 0$ for sufficiently small values of λ, say, $0 < \lambda < \lambda_*$, where the constant λ_* depends on initial condition $f(v, 0)$.

We shall see below that the properties of the Landau equation are quite different. In particular, we shall prove that the exponential moment of the third order $I_3(\lambda, t)$ propagates in time for radial solutions of the Landau equation. Here and below the words "$I_3(\lambda, t)$ propagates in time" are understood in the following sense: if $I_3(\lambda, 0) < \infty$ for some $\lambda > 0$, then for any $t > 0$ there exists $\lambda_1(t) > 0$ such that $I_3[\lambda_1(t), t] < \infty$.

The paper is organized as follows. In Section 2 we introduce the radially symmetric Landau equation, which will be the main subject of our study. Then we derive the set of equations for power moments of the solution. The exponential moments are introduced in Section 3. Their explicit connection with power moments is discussed. Some estimates for time-derivatives of power moments are proved in Section 4 (Lemma 1). Then the main result of the paper, the propagation and explicit estimates of the exponential moment of the third order, is formulated and proved in Theorem 1 from Section 5. The results of the paper are briefly discussed in Conclusions.

2 - The Landau equation for radial solutions

We consider the spatially homogeneous Landau equation for $f(v,t)$ in standard notations

$$\begin{aligned}
&\frac{\partial f}{\partial t} = \frac{\partial}{\partial v_i} \int_{\mathbb{R}^3} dw\; T_{ij}(u) \left(\frac{\partial}{\partial v_j} - \frac{\partial}{\partial w_j} \right) f(v) f(w), \\
&T_{ij}(u) = \frac{|u|^2 \delta_{ij} - u_i u_j}{|u|^3}, \quad u = v - w
\end{aligned} \tag{1}$$

with summation over repeating indices $i, j = 1, 2, 3$. A constant factor in front of the collision integral (see e.g. [**2**], [**3**]) is omitted without loss of generality. This is an original form of the Landau equation from [**12**]. This form is very convenient for the proof of standard conservation laws and H-theorem. However, for practical purposes we normally transform (1) by integration by parts to the form of nonlinear diffusion-type equation (or, equivalently, the Fokker–Planck equation [**14**])

$$\frac{\partial f}{\partial t} = \frac{\partial}{\partial v_i} \left[D_{ij}(v,t)\, \frac{\partial f}{\partial v_j} + F_i(v,t)\, f \right], \tag{2}$$

where

$$\begin{aligned}
&D_{ij}(v,t) = \int_{\mathbb{R}^3} dw f(w,t)\; T_{ij}(v-w), \\
&F_i(v,t) = -\frac{\partial D_{ij}}{\partial v_j} = -2\, \frac{\partial}{\partial v_i} \int_{\mathbb{R}^3} \frac{dw f(w,t)}{|v-w|}\,.
\end{aligned} \tag{3}$$

We consider below the radially symmetric form of the Landau equation (1) obtained by substitution of

$$f(v,t) = \tilde{f}(\tilde{v}, \tilde{t}), \quad \tilde{v} = |v|, \quad \tilde{t} = 8\pi t \tag{4}$$

into (1). After straightforward calculations we obtain the equation for $\tilde{f}(\tilde{v}, \tilde{t})$, where tildes are omitted

$$\frac{\partial f}{\partial t} = \frac{1}{v^2} \frac{\partial}{\partial v} \left[A(v) \frac{\partial f}{\partial v} + B(v) f \right], \quad v > 0, \tag{5}$$

$$A(v) = \frac{1}{3v}\left[\int_0^v dw\, w^4 f(w) + v^3 \int_v^\infty dw\, w\, f(w)\right], \quad B(v) = \int_0^v dw\, w^2 f(w). \tag{6}$$

Here and below the argument t is often omitted. The letters v, w, x, y, ... denote scalar non-negative variables (in contrast with Section 1 and Eqs. (1) – (3)). The Landau equation (5), (6) will be the main object of our study. It can be also written in the form similar to (1)

$$\begin{aligned} \frac{\partial f}{\partial t} &= \frac{1}{v^2}\frac{\partial}{\partial v}\int_0^\infty dw\, w\, K(v,w)\left(\frac{1}{v}\frac{\partial}{\partial v} - \frac{1}{w}\frac{\partial}{\partial w}\right) f(v)f(w), \\ K(v,w) &= \frac{1}{3}\min\left(v^3, w^3\right). \end{aligned} \tag{7}$$

We consider in this paper a class of initial data for Eq. (4) that includes, for example, such functions as

$$f_0(v) = f(v,0) \le C\exp(-b|v|^3), \quad v \ge 0, \tag{8}$$

for some positive numbers C and b. In fact we shall need below a weaker restriction for $f_0(v)$. To explain it we introduce notations for power moments

$$m_n(t) = \int_0^\infty dv f(v,t) v^{2+n}, \quad n = 0, 1, \dots \tag{9}$$

Then the weaker restrictions read

$$m_{3n}(0) = \int_0^\infty dv f(v,0) v^{2+3n} \le C \int_0^\infty dv e^{-bv^3} v^{2+3n} = \frac{C\, n!}{3\, b^{n+1}}, \quad n = 0, 1, \dots \tag{10}$$

Obviously these inequalities follow from Eq. (8) with the same values of C and b. On the other hand, there are many functions, which satisfy conditions (10) without satisfying conditions (8).

Let us consider a "nice" solution $f(v,t)$ of the Landau equation (5), (6) on the time-interval $[0,T]$ with some $T > 0$. In particular, we assume that all moments (9) of the solution are bounded for all $0 \le t \le T$. Then we obtain from equations (5), (6) the following equations for moments

$$
\begin{aligned}
\frac{dm_n(t)}{dt} &= \int_0^\infty dv v^n \frac{\partial}{\partial v}\left[A(v)\, f_v + B(v) f\right] \\
&= v^n \left[A(v)\, f_v + B(v) f\right]\Big|_0^\infty - n \int_0^\infty dv v^{n-1}\left[A(v)\, f_v + B(v) f\right] \\
&= \left\{v^n A(v)\, f_v + v^{n-1}\left[vB(v) - nA(v)\right] f\right\}\Big|_0^\infty \\
&\quad + n \int_0^\infty dv f(v) \left[\frac{\partial}{\partial v} v^{n-1} A(v) - B(v) v^{n-1}\right], \qquad n = 0, 1, \ldots
\end{aligned}
\tag{11}
$$

where the argument t of function $f(v,t)$ and its partial derivative $f_v(v,t)$ is omitted.

For simplicity of presentation it is assumed that $f(v,t) \geq 0$ and $f_v(v,t)$ are bounded continuous functions of $v \geq 0$ for any $0 \leq t \leq T$. We also assume that for any $n \geq 0$

$$
\lim_{v\to\infty} v^n \max[f(v,t),\, |f_v(v,t)|] = 0, \quad 0 \leq t \leq T.
$$

Under these assumptions Eqs. (11) reduce to

$$
\frac{dm_n(t)}{dt} = n \int_0^\infty dv f(v) \left[\frac{\partial}{\partial v} v^{n-1} A(v) - B(v) v^{n-1}\right], \quad n \geq 0,
$$

in the notation of Eq. (6). Then we obtain by differentiation

$$
\frac{\partial}{\partial v} v^{n-1} A(v) = \frac{n-2}{3}\left[v^{n-3}\int_0^v dw f(w) w^4 + 4v^n \int_v^\infty dw f(w) w\right].
$$

Hence,

$$
\frac{dm_n(t)}{dt} = I_n^{(1)}(t) + I_n^{(2)}(t) + I_n^{(3)}(t), \quad n \geq 0,
$$

where

$$
I_n^{(1)}(t) = \frac{n(n-2)}{3}\int_0^\infty dv f(v) v^{n-3} \int_0^v dw f(w) w^4,
$$

$$
I_n^{(2)}(t) = \frac{n(n+1)}{3}\int_0^\infty dv f(v) v^{n} \int_v^\infty dw f(w) w,
$$

$$I_n^{(3)}(t) = n\int_0^\infty dv f(v) v^{n-1} \int_0^v dw f(w) w^2 .$$

It is convenient to change the order of integration in the integral $I_n^{(2)}(t)$. Then we obtain by exchange of notations for v and w the following formula

$$I_n^{(2)}(t) = \frac{n(n+1)}{3}\int_0^\infty dv f(v) v \int_0^v dw f(w) w^n .$$

Hence, the equations for moments (9) of $f(v,t)$ have the following form

$$\frac{dm_0}{dt} = 0; \qquad \frac{dm_n}{dt} = \int_0^\infty dv f(v,t) \int_0^v dw f(w,t) Q_n(v,w), \tag{12}$$

where

$$\begin{gathered} Q_n(v,w) = \alpha_n v^{n-3} w^4 + \beta_n v w^n - \gamma_n v^{n-1} w^2, \\ \alpha_n = \frac{n(n-2)}{3}, \quad \beta_n = \frac{n(n+1)}{3}, \quad \gamma_n = n;\ n \geq 0. \end{gathered} \tag{13}$$

These equations were used in [**1**] without details of their derivation. Of course, they are valid also for any real $n \geq 0$. We consider below only such solutions $f(v,t)$ of the Landau equations (5) that have all moments satisfying Eqs. (12) at least in their integral form

$$m_n(t) = m_n(0) + \int_0^t d\tau \int_0^\infty dv f(v,\tau) \int_0^v dw f(w,\tau) Q_n(v,w) \tag{14}$$

in the notation of Eqs. (13).

3 - Exponential moments

If the distribution function $f(v,t)$ is such that

$$I_k(\lambda,t) = \int_0^\infty dv f(v,t) v^2 \exp(\lambda v^k) < \infty \tag{15}$$

for some $k > 0$ and $\lambda > 0$ for any $0 \leq t \leq T$, then we call the integral $I_k(\lambda,t)$ "the exponential moment of order $k > 0$". The variable λ can be considered as a free parameter. For example, the initial data satisfying conditions (8), have bounded exponential moment $I_3(\lambda, 0)$ for any $0 < \lambda < b$. For brevity we consider below the only case $k = 3$, i.e. exponential moments of the third order. Our goal is to prove their propagation in time.

There is an obvious connection between power moments $m_{3n}(t)$ and $I_3(\lambda, t)$. By using the Taylor series for the exponential function in (15) we obtain

$$I_3(\lambda, t) = \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} m_{3n}(t), \quad m_{3n}(t) = \int_0^{\infty} dv f(v, t) v^{2+3n}, \tag{16}$$

where $n = 0, 1, \dots$ The radius $R(t)$ of convergence of the series in (16) is given by the well-known formula

$$R(t) = \left\{ \lim_{n \to \infty} \sup \left[\frac{m_{3n}(t)}{n!} \right]^{1/n} \right\}^{-1},$$

but we shall not use this formula below. In spite of it we can use direct estimates of the integral $I_3(\lambda, t)$ in the form of series (16) provided that the initial moments $m_n(0)$ satisfy conditions (10). In particular, it follows from (10) that

$$I_3(\lambda, 0) \le \frac{C}{3b} \sum_{n=0}^{\infty} \left(\frac{\lambda}{b} \right)^n = \frac{C}{3b} \left(1 - \frac{\lambda}{b} \right)^{-1}, \qquad |\lambda| < b. \tag{17}$$

In order to obtain similar estimates of $I_3(\lambda, t)$ for $t > 0$ we shall need some estimates for moments obtained in the next section.

4 - Estimates of moments

Let us consider equations (12) for moments. The first consequence of these equations is that

$$m_0(t) = m_0(0) = const., \quad m_2(t) = m_2(0) = const.,$$

i.e. the conservation laws for mass and energy. The set of equations (12) is obviously unclosed because the right hand side of (12) is not expressed in terms of moments $\{m_n(t),\ n = 0, 1, \dots\}$. However, we can derive from (12) a closed set of inequalities for moments. It can be done in the following way. The moments $m_0 = m_0(0)$, $m_2 = m_2(0)$ are given by initial data. The first moment $m_1(t)$ can be estimated by standard inequality

$$\frac{m_2}{m_0} \ge \left(\frac{m_1}{m_0} \right)^2,$$

which means that the average of square is greater or equal to the square of average value. Hence, we obtain

$$m_1(t) \leq (m_0 m_2)^{1/2} = const.$$

Therefore it is sufficient to consider Eqs. (12) for $n \geq 3$. Note that the polynomial $Q_n(v,w)$ in the integral can be written as

$$Q_n(v,w) = v^{n-1}w^2\,[\alpha_n(w/v)^2 + \beta_n(w/v)^{n-2} - \gamma_n],$$
$$v > 0, \quad 0 \leq w \leq v, \quad n = 3,4,\dots$$

in the notation of Eqs. (13). Since $w/v \leq 1$, we obtain

$$Q_n(v,w) \leq g_n v^{n-1} w^2, \quad g_n = \alpha_n + \beta_n - \gamma_n = \frac{2}{3}n(n-2).$$

Hence, it follows from Eqs. (12) that

$$\frac{dm_n(t)}{dt} \leq g_n \int\limits_0^\infty dv f(v,t) v^{n-1} \int\limits_0^v dw f(w,t) w^2, \quad n \geq 3.$$

Finally we change the upper limit to infinity in the inner integral and obtain the following simple inequality for moments

$$\frac{dm_n(t)}{dt} \leq g_n m_{n-3}(t) m_0(t), \quad n \geq 3.$$

The result can be formulated as follows.

Lemma 1. *If the set* $\{m_n(t),\ n = 0,1,\dots\}$ *of integer moments* (9) *of non-negative function* $f(v,t)$ *satisfies equations* (12), (13) *on some interval* $0 \leq t \leq T$, *then*

(a)

(18) $$m_0(t) = m_0(0), \quad m_2(t) = m_2(0);$$

(b)

(19) $$m_1(t) \leq [m_0(0)\, m_2(0)]^{1/2};$$

(c) *the moments of orders* $n = 2,3,\dots$ *satisfy linear differential inequalities*

(20) $$\frac{dm_n(t)}{dt} \leq g_n m_0(0) m_{n-3}(t), \quad g_n = \frac{2}{3}n(n-2), \quad 0 \leq t \leq T.$$

In exactly the same way one can prove a more general (integral) version of Lemma 1, where it is assumed that the moments $\{m_n(t),\ n = 0, 1, \dots\}$ satisfy integral equations (14). In such version the inequalities (20) for $n \geq 3$ should be replaced by integral inequalities

$$m_n(t) \leq m_n(0) + g_n m_0(0) \int_0^\tau d\tau m_{n-3}(\tau) \tag{21}$$

in the notation of Eqs. (20). Perhaps the integral version of Lemma 1 can be applied to Villani's H -solutions [**16**] of the Landau equation (the propagation of moments for these solutions is proved in [**6**]).

In the next section we apply Lemma 1 to evaluation of series (16).

5 - Estimates of exponential moment $I_3(\lambda, t)$

Our goal in this section is to prove the following theorem

Theorem 1. *Let $f(v,t)$ be a non-negative solution of the Landau equation on the time-interval $0 \leq t \leq T$. It is assumed that*

(i) *power moments* $\{m_{3n}(0),\ n = 0, 1, \dots\}$ (9) *satisfy Eqs.* (12), (13) *(or integral equations* (14) *for $0 \leq t \leq T$)*

(ii) *the initial values* $\{m_{3n}(0),\ n = 0, 1, \dots\}$ *of these moments satisfy inequalities*

$$m_{3n}(0) \leq \frac{Cn!}{3b^{n+1}}, \quad n = 0, 1, \dots \tag{22}$$

for some constants $C > 0$ and $b > 0$. Then the following estimate of the exponential moment $I_3(\lambda, t)$ (16) *is valid for all $0 \leq t \leq T$:*

$$I_3(\lambda, t) \leq \frac{C}{3}\,[b_1(t) - \lambda]^{-1}, \quad b_1(t) = b\exp[-6m_0(0)\,bt], \quad 0 < \lambda < b_1(t). \tag{23}$$

Remark 1. *In fact the condition* (ii) *of Theorem 1 can be replaced by the equivalent condition* (ii'): *$I_3(\lambda_0, 0) < \infty$ for some $\lambda_0 > 0$. Such condition means that the series* (16) *for $I_3(\lambda, 0)$ has a positive radius of convergence $R > 0$. Then it follows from above formula for R that*

$$\frac{m_{3n}(0)}{n!} \leq m_0(0) b^{-n}, \quad n = 0, 1, \dots$$

for some $b > 0$. These are exactly inequalities (22) *with $C = 3m_0(0)b$.*

To prove the theorem we need the following lemma.

L e m m a 2. *Let functions* $\{x_n(t),\ n = 0, 1, \dots\}$ *satisfy for* $t \geq 0$ *inequalities*

$$\begin{aligned} &\frac{dx_n(t)}{dt} \leq n x_{n-1}, \quad n = 1, 2, \dots; \\ &x_0(t) = x_0(0); \\ &0 \leq x_n(0) \leq 1, \quad n = 0, 1, \dots \end{aligned} \tag{24}$$

Then

$$x_n(t) \leq \exp[(n+1)t], \quad n = 0, 1, \dots \tag{25}$$

P r o o f. We integrate inequalities (24) and obtain

$$x_n(t) \leq 1 + n \int\limits_0^t d\tau x_{n-1}(\tau), \quad n \geq 1.$$

The estimate (24) is obviously correct for $n = 0$, since $x_0(0) \leq 1$. Then we use induction and get for any $n \geq 1$

$$x_n(t) \leq 1 + n \int\limits_0^t d\tau \exp(n\tau) = \exp(nt) \leq \exp[(n+1)t], \quad t \geq 0.$$

This completes the proof. □

Then we pass to the proof of Theorem 1.

P r o o f. We note that the exponential moments $I_3(\lambda, t)$ depends only on power moments $m_{3n}(t)$, $n \geq 0$, as it is seen from the series (16). We also remind to the reader that accordingly to (10), the assumption (ii) of Theorem 1 means simply that the moments $m_{3n}(0)$ are controlled by similar moments of the function $C \exp(-bv^3)$ with arbitrary parameters $C > 0$ and $b > 0$. It was already shown that this assumption leads to a simple estimate (17) of $I_3(\lambda, 0)$. In fact, Theorem 1 gives an extension of that estimate to positive values of time t.

To prove this extension we first simplify inequalities (20) in obvious way and obtain

$$\frac{dm_n(t)}{dt} \leq \frac{2n^2}{3} m_0(0) m_{n-3}(t), \quad n \geq 3.$$

Then we consider these inequalities for a subset of indices $n = 3\tilde{n}$, $\tilde{n} \geq 1$. Omitting tildes, we obtain

$$\frac{dm_{3n}(t)}{dt} \leq 6n^2 m_0(0) m_{3(n-1)}(t), \quad n = 1, 2, \ldots; \quad m_0(t) = m_0(0).$$

After that we change variables by transformation

$$m_{3n}(t) = \frac{C\, n!}{3\, b^{n+1}} x_n(\tau), \quad \tau = 6\, m_0(0)\, b\, t\,, \quad n \geq 0.$$

Then we obtain

$$\frac{dx_n(\tau)}{d\tau} \leq n\, x_{n-1}(\tau), \quad x_n(0) \leq 1, \quad n \geq 1; \quad x_0(\tau) = x_0(0) \leq 1.$$

It remains to apply Lemma 2 to this set of inequalities. Coming back to initial variables, we get the following estimates for moments

$$m_{3n}(t) \leq \frac{Cn!}{3b^{n+1}} \exp[6\, m_0(0)\, b\, (n+1)\, t], \quad n = 0, 1, \ldots$$

Finally we substitute these estimates into the series (16) and obtain the resulting inequality (23).

This completes the proof of Theorem 1. □

6 - Conclusions

We have considered in this paper some properties of radially symmetric solutions $f(v,t)$ of the Landau equation, where $v \geq 0$ denotes the absolute value of the velocity. The main result is the proof of propagation in time $t > 0$ of the exponential moment

$$I_3(\lambda, t) = \int_0^\infty dv f(v,t) v^2 \exp(\lambda v^3), \quad \lambda > 0,$$

of the third order. The result is formulated in Theorem 1 and Remark 1 (Section 5). It simplifies the proof of similar result from [**1**] and contains some new explicit estimates of the integral $I_3(\lambda, t)$.

We expect that $f(v,t)$ converges to a Maxwellian $M = Ae^{-bv^2}$, as $t \to \infty$. Then formally the integral $I_3(\lambda, t)$ tends to infinity for any fixed $\lambda > 0$. This does not contradict to Theorem 1. In fact we have proved that, for any $t > 0$,

$I_3(\lambda, t)$ can be represented by its Taylor series in λ, but the radius $R(t)$ of convergence of the series tends to zero exponentially in time t, as $t \to \infty$.

The propagation of $I_3(\lambda, t)$ for the original Landau equation (1) from [**12**] considered in the present paper shows a big difference in asymptotic properties with the Boltzmann equation for hard forces. It was already discussed in Introduction that the propagation of the exponential moment $I_2(\lambda, t)$ of the second order is proved and studied in detail for the Boltzmann equation. The moment $I_3(\lambda, t)$ does not exist in that case because of the Maxwellian lower bound [**13**] mentioned in Introduction. It is clear that similar lower bound cannot exist for solutions of the Landau equation (1). On the other hand, the Maxwellian lower bound is also valid for the Landau equation with hard potentials [**7**]. Of course, such difference in properties is not related to grazing collisions limit, but rather to the difference in collision frequency as the function of relative velocity. It would be more reasonable to compare solutions of equation (1) with solutions of the Boltzmann equation for soft potentials, but we do not know much about exponential moments for that case. It is interesting that the asymptotic solutions of the Landau equation constructed in [**3**] are in complete agreement with Theorem 1 applied to initial data $f(v, 0)$ with compact support. Roughly speaking, it follows from Theorem 1 that the solution $f(v, t)$ for such initial data decays, as $v \to \infty$, like $\exp(-bv^3)$ or faster. Of course, it does not contradict to relaxation of $f(v, t)$ to some Maxwellian for large values of time.

Data Availability. Data sharing is not applicable to this article as no datasets were generated or analysed during the current study.

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