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Sebastian Geisler

Mathematical modelling with experiments: students sense of validation and its relevance

Abstract. Mathematical modelling is important yet challenging. Especially the validation of ones model seems to be a hurdle for many students. Thus, fostering students validation competence is an important aim in mathematics lessons. In this paper, I follow an idea to foster model validation that has been discussed in the literature before: the combination of modelling with scientific experiments. An analysis of the validation ideas from 111 students working on two different modelling tasks with experiments shows that a sufficient validation is not self-evident. In contrast, dealing with measurement errors that occur during experimentation might even hinder validation. It seems that some students put more trust in their mathematical models than in their experimental data and misinterpret the relation between data and model. Practical implications of these results are discussed.

Keywords. Mathematical modelling, experiments, validation.

Mathematics Subject Classification: 97M10, 97C70.

1 - Introduction

Mathematical modelling is a key competence with great relevance for modern societies [**18**]. Likewise, several national standards documents [**14**] and the PISA framework [**19**] name the development of modelling competencies as an aim of mathematics lessons. Unfortunately, modelling and especially model validation is challenging for students as well as teachers [**1**]. Furthermore, some students do not see validation as a necessary step in modelling processes [**12**].

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One approach to support students to develop validation competence and a sense of the relevance of validation, is to use tasks that combine mathematical modelling with experiments. In these kind of tasks, students perform experiments to gather data that can be used for the subsequent modelling. Following this approach, students from six German mathematics classrooms worked on either a modelling task with experiment concerning the decay of beer froth or with an experiment concerning the cooling of coffee. In this contribution I present extended results concerning students ideas for model validation with respect to their experimental data and their opinions on the relevance of validation derived from data that I have reported on before [**8**], [**10**].

2 - Theoretical background

2.1 - *Mathematical modelling*

Most theories describe the modelling process as circular. The well known conceptualisation by Blum and Leiss [**1**] involves seven steps (see Figure 1). Niss [**18**] states that validation is the "single most important point related to

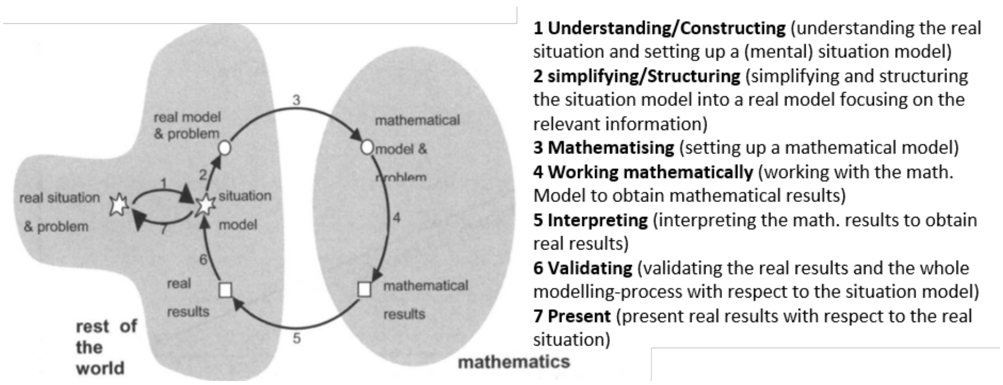

Fig. 1. modelling cycle following Blum and Leiss [**1**].

mathematical modelling". According to Hankeln and Greefrath [**13**] the validation step involves that learners check their solution and reflect on it, or revise parts of their models, if it becomes apparent that the solutions or the model itself are not suitable for the real situation. Furthermore, reflection if other models or solutions are possible is also part of a thorough validation. However, especially the validation step seems to be a hurdle for many student. Blum and Leiss [**1**] report that many students do not validate their models and that the validation is therefore a recurring shortcoming in students' modelling processes. Likewise, within a short intervention study Borromeo Ferri, Grünewald and Kaiser [**3**] found that students validation competence was the weakest modelling sub-competence. Moreover, some students seem to think that a validation of there models is not even relevant [**12**]. But even if students validate their models, they sometimes do so by relying on rather intuitive feelings whether their model could be correct or inappropriate [**2**].

2.2 - *Modelling with experiments*

One approach to foster students validation competence is to use tasks that combine modelling and (scientific) experiments. In this paper I follow the conceptualisation of Ganter [**7**] who proposed a cyclical experimentation model. The core of this conceptualisation is that experiments are theory-driven and aim at answering questions for which a hypothesis already exist. Furthermore, experiments are planned and controlled and therefore go far beyond pure "trial and error". Thus, the first step for an experiment is the formulation of a concrete hypothesis [**7**]. In the next step, an experiment is planed in order to confirm or reject the hypothesis. After conducting the experiment (step 3), the derived data is analysed (step 4). If the hypothesis has to be rejected, a new hypothesis can be the starting point for a second experimental cycle.
Ludwig and Oldenburg [**15**] have argued that combining modelling and experiments is worthwhile because experiments tie the whole modelling process to students' experiences within the real world. They especially see benefits regarding model validation as students can validate their model using their own experimental data. Zell and Beckman [**22**] argue that experiments are valuable with regard to validation because experimental data are always affected by measurement errors and alike and therefore the data does never fit perfectly to an intended model, offering occasions for validation and to discuss about the limitations of a chosen model. Engel [**6**] argues for the use of real data as well, criticizing that many tasks in schoolbooks use unrealistic or even smoothed data that fits well to the intended model. In his opinion, using such unrealistic data does not lead to realistic validations and hinders students to understand why validation is important [**6**]. Halverscheid [**11**] sees a natural link between modelling and experiments, too:

> Experiments related to mathematics find their natural place in the framework of mathematical modelling because they represent the "rest of the world" for which mathematical models are built (p. 225).

Thus, experiments can foster a deeper understanding of the real situation [**4**]. As most experiments make use of idealizations during the planing of the experiment, the experiment itself can be seen as a real model in the sense of Blum and Leiss. Following this assumption, I have proposed an integrated model for modelling with experiments [**9**]. This model is based on the modelling cycle by Blum and Leiss and the conceptualisation of experiments by Ganter, tying together both cyclical processes (Figure 2).

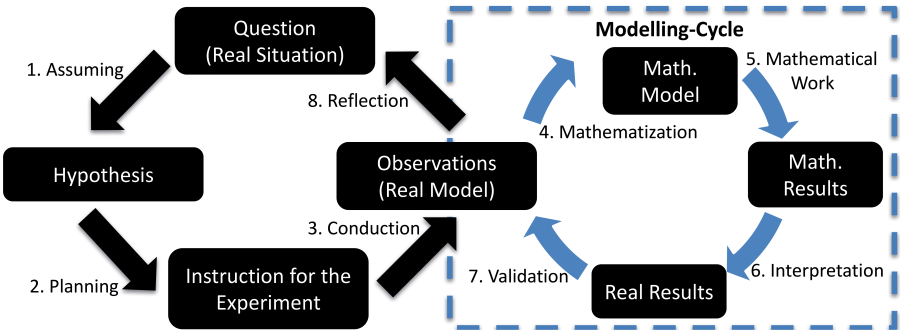

Fig. 2. Integrated model for modelling with experiments [**9**], p. 205.

Despite the theoretical potential of modelling tasks with experiments, research on those tasks is scarce and as a consequence there is not much empirical evidence for the assumed potentials. Zell and Beckmann used physics experiments to foster secondary students' modelling competencies [**22**]. They conclude that students were able to validate their models once they had accepted that their models are not perfect and that measurement errors can occur. In contrast, Maull and Berry [**16**] report from a study with undergraduate students, in which most students did not validate their models derived from experimental data. Moreover, trying to validate ones model does not guarantee an appropriate validation. In a study by Carrejo and Marshall [**5**], undergraduate students worked on a modelling task with experiment. The students used models that were based on an inappropriate assumption and had systematics shortcomings. Despite that they noticed deviations between their models and their experimental data, students did not decide to adjust their models because they attributed the deviations to measurement errors instead of their underlying assumptions when setting up the model. It seems that in this case, using experimental data even hindered a useful validation due to a more complex real situation (compared to a modelling task without experiment). Summarizing, only a few studies exist that deal with the benefits and constraints of modelling tasks with experiments and more research is needed regarding students validation processes when solving modelling tasks with experiments.

3 - The current study

The purpose of the current study is to reduce the above discussed research gap concerning students validation in modelling tasks with experiments. Furthermore, I want to shed light on how experiments can help students to recognize the relevance of model validation. This leads to the following questions:

1. How do students validate their models and do they rely on their experimental data and their experiences with the experiment for the validation?
2. Which ideas for improvements of their models do students develop?
3. In which way do students reason the relevance of model validation?

I reported on these questions before [**8**], [**10**]. For this paper, a larger sample and a reanalysis was used. In the following I describe the sample in detail and give an overview of the modelling tasks used, before discussing the methodology.

3.1 - *Sample*

The sample consists of 111 students from six German 10th grade classes. The students were between 15 and 17 years old and 51 % were female. None of the classes had word on modelling tasks with experiments before. However, the students had conducted experiments before in nature science lessons. The tasks used in the study were located in the topic of exponential functions. All classes had covered exponential functions and their characteristics before. In particular, students already new how to set up an exponential function based on given values in a table. However, exponential functions were not the current topic in class at the time of the study. Therefore, a certain timespan lay between covering the topic exponential functions and the study.

3.2 - *Modelling tasks "Stale Beer" and "Cold Coffee"*

Two modelling tasks with related experiments have been developed for the study. Their design basically followed the integrated model presented in Section 2 (Figure 2). Thus, formulating an initial hypothesis was part of both tasks. To save time, students did not have to plan the experiments on their own.

The concrete modelling tasks have been described in detail before [**8**], [**10**]. That is why I restict myself to describe the overall structure here. Both tasks start with a short explanation of the context and stating the overall problem of the task. In the case of “Stale Beer” it is stated that the speed of froth decay is a criterion for the quality of beer. Students are asked to model the decay of froth and to judge the quality of the beer accordingly. In the case of the task “Cold Coffee” the context is that people prefer to drink coffee at different temperatures and that it takes some time after brewing until the coffee can be drunken delightfully. Students were asked to model the cooling of the coffee and to identify a time from which on the coffee can be drunken in their opinion.

In both tasks, students were asked to state a hypothesis how the process will be like and to conduct an experiment to check their hypothesis. For the “Stale Beer” task, students measured the height of froth from freshly poured beer every 30 seconds. The experiment in the “Cold Coffee” task involved measuring the temperature of freshly brewed coffee every minute. In both tasks, students captured their data in a table and visualized it either in a diagram by hand or via GeoGebra. Afterwards they were asked to find a function that describes their data. No hints were given, which kind of function could be appropriate.

Both processes can approximately be described by exponential decay. However, due to measurement errors and the physically complex nature of the processes they are not perfectly exponential [**21**], thus offering occasions for validation and reflection upon limitations of chosen models. A first attempt for modelling the decay of beer froth can be to use an exponential function of the form $f : \mathbb{R} \to \mathbb{R}$, $f(x) = c \cdot a^x, c \in \mathbb{R}^+, a \in (0; 1)$. The height of the beer froth directly after pouring it can be used as an estimate for c. a can be estimated as the quotient of the height in intervals of 1 minute (e.g., $a = \frac{h_{1\mathrm{min}}}{h_{0\mathrm{min}}}$). This estimation is becoming more accurate if several intervals are considered. A similar approach can be used to model the cooling of coffee [**10**]. The subtask “Compare your function to your measured data. Does your function describe the data accurately enough? (How) could your model be improved?” served as validation prompt in both tasks. Furthermore, students were asked why it was relevant to ask these questions and in which way it was important for judging the beer quality respectively identifying a good drinking temperature. This last subtask served as a reflection of the relevance of model validation.

3.3 - *Methods*

The students in all participating classes were randomly distributed to the both modelling tasks and worked on them in pairs of two during a 90 minutes lesson. Students answers to the validation prompt and the subsequent reflection task were used as data for the analysis to answer the research questions. Moreover, six students have been interviewed concerning their validation and ideas for model-improvement after working on the task. Students’ solutions served as stimulus for these interviews (stimulated recall). Within an qualitative content analysis [**17**] inductive categories have been derived from the data and formed a coding guide comprising of a coding rule and an coding example. Parts of the data haven been anlysed before [**8**], [**10**] but for the current contribution a reanalysis has been done leading to an improved coding guide. Following the research questions, three main categories have been formulated: 1) *Validation* (students' judgement whether their function described the measured values appropriately); 2) *Ideas for Model Improvement*; 3) *Relevance of the Validation.* The whole data was coded independently by two coders. Afterwards the inter-rater reliability was calculated resulting in Cohens $\kappa = .66$ indicating a good reliability of the coding guide.

4 - Results

Given the similar structure of both used modelling tasks, I present the results not divided by the tasks. Nearly all students were able to set up an exponential function as a model for the decay of beer froth or for the cooling of the coffee. It is worth noticing, that most students did not use all of their experimental data to set up a model. The vast majority used only the first two measured values with the result that most functions deviated clearly from the other measured values (see Figure 3 for some examples).

4.1 - *Validation*

96 students explicitly stated ideas concerning the validation of their model. 35 students wrote that their function adequately describes their data and thus no relevant deviations between function and data occurred. For example:

> The function describes our measured values quite exact. There are hardly any bigger deviations.

In contrast 42 students identified relevant deviations between their data and the function as becomes apparent in the following statement:

> No, the function leads to little deviations from the original data and is therefore not appropriate (often 1 or 2 °C deviation).

Comparing the two presented statements, it is obvious that students judgement of the models is quite subjective. In both solutions, students wrote about little or not to big deviations. However, the conclusions they draw are completely different. This is a phenomenon that can be recurrently found in the data:

Students judge models that have similar shortcomings very differently. This can also be seen in the solutions in Figure 3. Whereas the students who produced the left diagram stated that they see no relevant deviations between data and function, the students producing the right diagram stated that their function deviates considerably from their measured data.

Besides these rather general judgements, 19 students wrote more elaborated statements in which they described that the function was a suitable model only in a limited area. Often, the function described the data at the beginning of the process well but deviated considerably at the end of the process or vice versa:

> Whereas the first 4 measured values fit well to the function, the following values are strongly deviating.

First ideas concerning the limits of ones own model become apparent here.

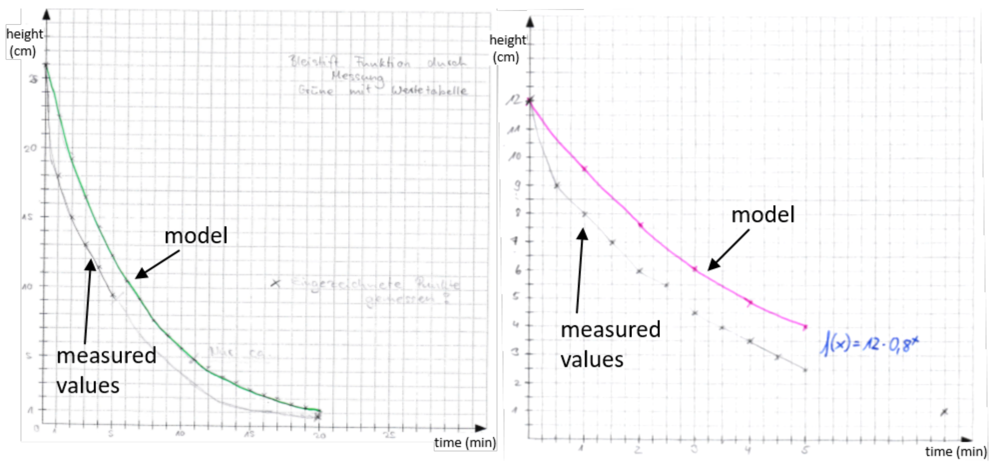

Fig. 3. Students diagrams visualizing their measured data and set up functions concerning the decay of beer froth.

4.2 - *Ideas for Model-improvement*

Most students wrote down ideas for model-improvement - independently from whether they had judged their model to be appropriate with respect to the data or not. 10 students explicitly stated that they have no ideas. Only 6 students argued that one should take into account concrete characteristics of the context as explicit assumptions when setting up a model. In the case of the "Stale Beer" task this involved for example characteristics of the froth:

> One could consider that the froth is higher at the edge of the glass than in the middle.

Students working on the “Cold Coffee” task mentioned the room temperature as a characteristic that should be taken into account:

> ... this was because the influence of the room temperature was not considered. This could further improve the function.

Only 17 students wrote down ideas on how to improve the function based on the measured values they already had. These ideas involved to find a better function through testing different parameters. Some also suggested to use more measured values to estimate the parameters of the function:

> Moreover, it would be helpful to calculate a for all measurements and to calculate the arithmetic mean afterwards.

However, most ideas that students mentioned were related to improving the experiment and the quality of the measured values instead of working with the function itself ($N = 64$). This mainly involved simple improvements like measuring more often or more precisely:

> Shorter intervals for the measurement, more precise measurement.

It seems that many students put more trust in their model than in their measured data and thus suggest to improve the experiment instead of the model. These impressions are supported by students’ statements in the interviews.

From the six students interviewed, only two offered concrete ideas to work on the function while all other suggested to improve the experimental process. Those two students were named *Student A* and *Student B*. In her written solution, *Student A* wrote that she sees only little deviations between function and measured values and that the function is appropriate. Furthermore, she suggested to measure more precisely during the experiment. However, in the stimulated recall interview she offered another idea:

Interviewer:	So if you already have the data and you have to work with this data. Do you have an idea what you maybe could do then?
Student A:	We have noticed that beginning from a certain value it is constant. So maybe that one splits it into two functions. One for the first time and another where it continues constantly.

Student A offers an idea of a partly defined function that would account for the fact that the function they already found was only suitable for the first few measured values. Thus, a first step towards the idea of limitations of a model and how to deal with such limitations is already touched by *Student A*'s answer. *Student B* is following a different idea on how to work on the model:

Interviewer:	You have thought about how one can improve your function and you wrote “one could calculate a for every single value and take the mean then”. What do you mean by every single value?
Student B:	We have values for every time point and we could calculate an a for every time point and calculate the mean of these as.
Interviewer:	And what would happen if you do so?
Student B:	Then one value would lie exactly on the function and it would be consistent what lies over and under the graph.

The answers show that she is aware that all measured values should be used to set up the function in order to have an overall sufficient fit between function and data. Moreover, she is formulating a criterion (without explicitly naming it) for a function that fits well: the amount of data points above and beneath the function should be similar. This shows that *Student B* has established a sense of what it means to have a good fit between function and data.

All other interviewed students had no ideas for working on the function to improve the fit to the data. *Student C* and *Student D* only stated ideas concerning the experiment because for them the function relies mainly on the data. From their statements, it is clear that they are aware that improving the experiment will not only result in different data but also in a different function:

Student C:	So I think the analysis is really dependent on the experiment and so it is more relevant what values we gather from the experiment to have the right analysis.
Interviewer:	Ok, so you say it depends on the data. What would you say would happen to your function if you gather other data, if you measure more often or more precisely?
Student C:	I think nothing. It would be the same. It would just be more detailed and one could better see how the function...
Interviewer:	So if you would have measured more precisely it would be the same function but everything would fit better?
Student C:	So maybe there would be a different number [for the parameter] but basically it would be the same.

Student D argues in a similar way:

Student D:	I think it wouldn't change a lot. So you had the function in GeoGebra and the points were not exactly on it, so a little away. And I think, this could lead to that the points are moving closer to the function. But not a lot else.
Interviewer:	Ok, what about the function, what would happen with it?
Student D:	Actually I think yes, because the values are different if you set up the calculation differently you get other values. But it would be precisely then.

While these two students are aware that using new measured values will lead to some differences in the function as well, *Student E* and *Student F* seem not to understand how data and function are related:

Interviewer:	Do you have also ideas how to improve your function if you already have the values and you cannot change them? [...]
Student E:	I have no idea!
Interviewer:	And if you would have a better measuring cylinder and you measured the values more precisely, do you think at the end you would have a function that fits better to the measured values?
Student E:	I think we would have the same function then, but eventually the measured values would fit better to it.

Student F argued in the same way. It is worth noticing that these two students are aware that they would get new measured values if they improve the experiment but they believe that their function would still be the same. This is remarkable because they used their measured values to find the function. Thus, their function is not independent from the data gathered in the experiment.

4.3 - *Relevance of the validation*

14 students explicitly wrote that they see no sense in validating their models. However, the majority stated that comparing function and measured values was relevant. 32 students did not give a specific reason and just wrote that it was important in order to judge the beer quality respectively the drinking temperature of the coffee. A typical answer in this category is the following:

> Yes, because this way we could analyse the beer quality.

Six answers involved the idea that one can improve ones model:

> If the model had errors, we could have recognized these errors through this task and furthermore one had to think about ideas for improvement. Thus, in the future one could maybe work better.

These arguments are clearly in line with the sense of validation [**13**]. In 16 answers, the argument that comparing model and data is important with respect to the initial real problem can be found. These students point out, that the validation was for example necessary to judge the beer quality:

> It was important because so we could see that - in contrast to our function - the froth decreased much faster, showing that the quality was not good.

This statement points out consequences if a validation is absent: one could solve the initial problem based on an inappropriate model. This argumentation was also found amongst students who worked on the "Cold Coffee" task:

> It was important to see if the function is correct. If it deviates a lot from reality, it could happen that one burns the tongue.

However, students also mentioned arguments that are not really related to validation. For example, 10 students wrote that comparing function and data enabled them to judge the experiment:

> It is relevant in order to see whether the experiment worked well.

5 - Discussion and implications

Theoretical assumptions suggest that combining modelling with experiments can be valuable to foster students validation competence [**15**], [**22**]. However, the results of the study presented here, show that a sensible validation is not self-evident even when using experiments. Students' judgements on their models are very subjective. Thus students seem to follow different norms or criteria for validating their models. Some students present elaborated ideas and also mention limits of their functions. In the interview *Student A* even speaks about ideas to deal with this limits. *Student B* also presented ideas to work on the function in order to overcome systematic shortcomings that occur if only the first measured values are used for the modelling process. However, several students did not recognize systematic shortcomings of their models (Figure 3).

Furthermore, sometimes real data from experiments might even hinder an appropriate validation. Systematic shortcomings of ones model - like systematically overestimating the data - were frequently attributed to measurement errors which is in line with the results of prior studies [**5**]. It is therefore not surprising that most students suggest to improve the experiments and thus their measured values instead of working on the function. The stimulated recall interviews revealed that beneath lack of concrete ideas for working on the function, some students seem to have misconceptions of the relation between function and data. *Student C* explicitly stated that the function depends on the data and that good data is important to set up a good model. Indeed, as the function is set up using the data, the function heavily depends on ones data. Likewise, *Student C* is aware that new data would also lead to a slightly different function. In contrast, *Student E* and *Student F* seem not to understand how function and data are related. Both believe that measuring new values in an improved experiment will result in the same function but that these new values will fit better to the function. This misconception could be rooted in the believe to have already found the "correct" function. Obviously both students put more trust in their model than in their data. It is an open question, why they trust the model more. One reason could be that they are used to mathematics tasks with only one correct answer. According to Schoenfeld many students believe that mathematics tasks have always only one correct answer [**20**]. Another reason could lie in students experiences from science lessons where experiments are often done to rediscover a model that is already known.

Most students state that a model validation is relevant for them. Unfortunately, only a few students grap the real intention of validation and point to possible consequences for the whole modelling process when skipping the validation, like answering the initial real problem based on an inappropriate model. Misconceptions concerning the relation of experiment and model can be seen in students argumentations here again. Some students think that the validation should be used to check whether the experiment worked. This is a delicate issue. Indeed the function is dependent from the data but beneath the actual data, students decisions when setting up the function are relevant. These decisions involve, among others, the class of function that is used and the amount of data that is taken into account. These decisions can lead to substantial deviations between data and function. Therefore, it is not possible to derive measuring errors only because deviations between data and function have been observed.

What implications for teaching modelling to secondary students can be derived from these results? One could argue that using experiments even increases the complexity of modelling and is therefore no good idea. In my opinion this conclusion would be premature. Engel argues that using real data is necessary to learn validation and to understand its relevance [**6**]. Only if students experience that setting up an initial model does not always lead to an appropriate model and that models can have flaws and limits, they understand why validating ones models is so important. This experience can not occur if unrealistic or even smoothened data is used in modelling tasks [**6**]. However, the results show that students can misinterpret the relation between data and model as well as the influence of measuring errors. In particular, students have to differentiate between deviations of model and data that depend on measuring errors and are thus inevitable to some extant and those deviations that are the result of systematic shortcomings of the model and point to unfavourable decisions made during the modelling process. This delicate distinction can only be learned when working with real data. Therefore, using modelling tasks with experiments can be the starting point for fruitful discussions in the classroom and offer the possibility to talk about several validation related topics. These topics include (but are not limited to) decisions and assumptions made when setting up a model, the extant of deviations that are acceptable for a model, limits of a model because a certain function describes only one part of a process. If these topics are intensively discussed among students, modelling tasks with experiments can have great potential for fostering students' validation competence.

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Sebastian Geisler
University of Potsdam
Karl-Liebknecht-Str. 24-25
Potsdam, 14476, Germany
e-mail: sebastian.geisler@uni-potsdam.de