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# Mixed anisotropic and nonlocal Dirichlet $(p, q)$-eigenvalue problem

**Abstract.** In this article, we consider an anisotropic and a combination of anisotropic and nonlocal Dirichlet $(p, q)$-eigenvalue problems. We establish existence and regularity of eigenfunctions in a bounded domain $\Omega \subset \mathbb{R}^N$ under the assumption that $1 < p < \infty$ and $1 < q < p^*$ where $p^* = \frac{Np}{N-p}$ if $1 < p < N$ and $p^* = \infty$ if $p \geq N$.

**Keywords.** Mixed local and nonlocal $p$-Laplacian, Anisotropic $p$-Laplace operator, Dirichlet eigenvalue problem, existence, regularity.

**Mathematics Subject Classification:** 35M12, 35J92, 35R11, 35P30, 35A01.

## 1 - Introduction

In this article, we study the following $(p, q)$-eigenvalue problems

$$-H_p u + \alpha(-\Delta_p)^s u = \lambda \|u\|^{p-q}_{L^q(\Omega)} |u|^{q-2} u \text{ in } \Omega, \tag{1.1}$$

where $\alpha = 0$ or $1$ and $\Omega \subset \mathbb{R}^N$ is a bounded domain. For $\alpha = 0$, it will be understood that (1.1) holds under the Dirichlet boundary condition $u = 0$ on $\partial\Omega$ and for $\alpha = 1$, the boundary condition $u = 0$ in $\mathbb{R}^N \setminus \Omega$ holds. Throughout the rest of the paper, we assume that $0 < s < 1$, $1 < p < \infty$, $1 < q < p^*$, where $p^* = \frac{Np}{N-p}$ if $1 < p < N$ and $p^* = \infty$ if $p \geq N$ unless otherwise stated. For $0 < s < 1$,

$$(-\Delta_p)^s u = \text{P.V.} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))}{|x - y|^{N+ps}} \, dy,$$

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is the fractional $p$-Laplace operator, where P.V. denotes the principal value, see **[13]** for more details. Further $H_p u = \operatorname{div}(H(\nabla u)^{p-1}\nabla H(\nabla u))$ is the anisotropic $p$-Laplace operator, where $H : \mathbb{R}^N \to [0,\infty)$ is a Finsler-Minkowski norm, that is $H$ is nonnegative in $\mathbb{R}^N$, which is $C^1(\mathbb{R}^N \setminus \{0\})$ and strictly convex such that $H(x) = 0$ iff $x = 0$ and

$$c_1|x| \le H(x) \le c_2|x|, \quad \forall x \in \mathbb{R}^N, \tag{1.2}$$

for some positive constants $c_1, c_2$ and $H$ is even, positively homogeneous of degree 1, so that

$$H(tx) = |t|H(x), \text{ for every } x \in \mathbb{R}^N \text{ and } t \in \mathbb{R}. \tag{1.3}$$

From the proof of [**27**, Lemma 5.9], it follows that

$$\langle H(x)^{p-1}\nabla H(x) - H(y)^{p-1}\nabla H(y), x - y\rangle > 0, \tag{1.4}$$

for every $x, y \in \mathbb{R}^N$ such that $x \neq y$.

The dual $H_0 : \mathbb{R}^N \to [0,\infty)$ of $H$ is defined by

$$H_0(\xi) := \sup_{x\in\mathbb{R}^N\setminus\{0\}} \frac{\langle x, \xi\rangle}{H(x)}. \tag{1.5}$$

A typical example of $H$ includes the $l^r$-norm defined by

$$H(\zeta) = \Big(\sum_{i=1}^{N} |\zeta_i|^r\Big)^{\frac{1}{r}}, \quad r > 1, \tag{1.6}$$

where $\zeta = (\zeta_1, \zeta_2, \ldots, \zeta_N)$. When $H$ is the $l^r$-norm as in (1.6), we have

$$H_p u = \sum_{i=1}^{N} \frac{\partial}{\partial x_i}\bigg(\Big(\sum_{k=1}^{N}\Big|\frac{\partial u}{\partial x_k}\Big|^r\Big)^{\frac{p-r}{r}}\Big|\frac{\partial u}{\partial x_i}\Big|^{r-2}\frac{\partial u}{\partial x_i}\bigg). \tag{1.7}$$

For $r = 2$ in (1.7), $H_p$ becomes the usual $p$-Laplace operator $\Delta_p$. Moreover, for $r = p$ in (1.7), the operator $H_p$ reduces to the pseudo $p$-Laplace operator. Therefore, equation (1.1) covers a wide range of mixed local and nonlocal problems and in particular, extends the following mixed eigenvalue problem

(1.8)
$$\sum_{i=1}^{N} \frac{\partial}{\partial x_i}\bigg(\Big(\sum_{k=1}^{N}\Big|\frac{\partial u}{\partial x_k}\Big|^r\Big)^{\frac{p-r}{r}}\Big|\frac{\partial u}{\partial x_i}\Big|^{q-2}\frac{\partial u}{\partial x_i}\bigg) + \alpha(-\Delta_p)^s u = \lambda\|u\|^{p-q}_{L^q(\Omega)}|u|^{q-2}u \text{ in } \Omega.$$

In the local case ($\alpha = 0$), the $p$-Laplace eigenvalue problem

$$-\Delta_p u = \lambda \|u\|_{L^q(\Omega)}^{p-q} |u|^{q-2} u \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega,$$

has been studied widely. In this concern, for $p = q$, we refer to the works in [**25**, **32**, **34**] and the references therein. When $p \neq q$, we again refer to [**25**] including [**15**, **16**, **17**, **18**, **19**, **28**, **30**, **31**, **37**, **38**, **39**] and the references therein.

The pseudo Laplace $(p, q)$-eigenvalue problem

$$-\sum_{i=1}^{N} \frac{\partial}{\partial x_i}\left(\left|\frac{\partial u}{\partial x_i}\right|^{p-2} \frac{\partial u}{\partial x_i}\right) = \lambda \|u\|_{L^q(\Omega)}^{p-q} |u|^{q-2} u \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega \tag{1.9}$$

is studied for $p = q$ in [**3**]. For $p \neq q$, refer to [**36**] and the references therein. Further equation (1.9) is extended to the anisotropic $(p, q)$-eigenvalue problem

$$-H_p u = \lambda \|u\|_{L^q(\Omega)}^{p-q} |u|^{q-2} u \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega \tag{1.10}$$

in [**21**] for $q = 2$. For $p = q$, see [**1**, **2**, **11**, **12**] and the references therein.

In the nonlocal case, following fractional $p$-Laplace eigenvalue problem

$$(-\Delta_p)^s u = \lambda \|u\|_{L^q(\Omega)}^{p-q} |u|^{q-2} u \text{ in } \Omega, \quad u = 0 \text{ in } \mathbb{R}^N \setminus \Omega$$

is studied for $p = q$ in [**5**, **9**, **20**, **33**] and the references therein. For $p \neq q$, see [**18**].

When $H(x) = |x|$, in the mixed local and nonlocal case, the following eigenvalue problem

$$-\Delta_p u - \int_{\mathbb{R}^N} \mathcal{J}(x-y)|u(x) - u(y)|^{p-2}(u(y) - u(x))dy = \lambda |u|^{p-2} u \text{ in } \Omega,$$

$$u = 0 \text{ in } \mathbb{R}^N \setminus \Omega,$$

has been studied by [**10**, **26**], where $\mathcal{J} : \mathbb{R}^N \to \mathbb{R}^+$ is a nonsingular, radially symmetric, nonnegative and compactly supported kernel. Further, [**6**, **8**] studied the limiting problem for mixed local and nonlocal problems. Recently, the mixed local and nonlocal $(p, q)$-eigenvalue problem

$$-\Delta_p u + (-\Delta_p)^s u = \lambda \|u\|_{L^q(\Omega)}^{p-q} |u|^{q-2} u \text{ in } \Omega, \quad u = 0 \text{ in } \mathbb{R}^N \setminus \Omega \tag{1.11}$$

has been studied in [**24**].

In the work [**14**] the authors considered mixed local and nonlocal linear eigenvalue problems with Neumann boundary condition. For regularity results in the mixed case, see [**4**] and the references therein.

To the best of our knowledge, mixed anisotropic and nonlocal $p$-Laplace equation is very less understood. We refer to the recent works [**22**, **23**]. As far as we are aware, for such general class of the Finsler-Minkowski norm $H$ considered in this paper, mixed anisotropic and nonlocal $p$-Laplace eigenvalue problem (1.1) is not studied before even for $p = 2$. Our main purpose in this article is to investigate the existence and regularity of the eigenvalue problem (1.1) by considering both the purely anisotropic case ($\alpha = 0$) and mixed case ($\alpha = 1$) in (1.1). To this end, we follow the approach introduced in Ercole [**18**]. Further, we employ the recent regularity results from [**22**].

The organization of the paper is as follows: In Section 2, we present the functional setting, state some auxiliary results and the main results of this article. In Section 3, some preliminary results are proved. Finally, in Section 4, we prove the main results.

## 2 - Basic definitions and main results

In this section, we present some known results for the Sobolev spaces, see [**13**] for more details. Let $\Omega \subset \mathbb{R}^N$, $N \geq 2$ be a bounded domain. The fractional Sobolev space $W^{s,p}(\Omega)$, $0 < s < 1 < p < \infty$, is defined by

$$W^{s,p}(\Omega) = \Big\{ u \in L^p(\Omega) : \frac{|u(x) - u(y)|}{|x - y|^{\frac{N}{p}+s}} \in L^p(\Omega \times \Omega) \Big\}$$

and endowed with the norm

$$\|u\|_{W^{s,p}(\Omega)} = \Big( \int_\Omega |u(x)|^p \, dx + \int_\Omega \int_\Omega \frac{|u(x) - u(y)|^p}{|x - y|^{N+ps}} \, dx \, dy \Big)^{\frac{1}{p}}.$$

The Sobolev space $W^{1,p}(\Omega)$, $1 < p < \infty$, is defined as the Banach space of locally integrable weakly differentiable functions $u : \Omega \to \mathbb{R}$ equipped with the norm

$$\|u\|_{W^{1,p}(\Omega)} = \|u\|_{L^p(\Omega)} + \|\nabla u\|_{L^p(\Omega)}.$$

The Sobolev space $X_0 := W_0^{1,p}(\Omega)$ is defined as the closure of $C_c^\infty(\Omega)$ with respect to the norm $\|u\| = \|H(\nabla u)\|_{L^p(\Omega)}$.

To study the mixed problem, we consider the space $X_1$ defined as

$$X_1 = \{ u \in W^{1,p}(\mathbb{R}^N) : u|_\Omega \in X_0, u = 0 \text{ a.e. in } \mathbb{R}^N \setminus \Omega \}$$

under the norm

$$\|u\| = \|H(\nabla u)\|_{L^p(\Omega)} + \left\| \frac{u(x) - u(y)}{|x - y|^{\frac{N+ps}{p}}} \right\|_{L^p(\mathbb{R}^N \times \mathbb{R}^N)}.$$

Next, we have the following result from [**6**, Lemma 2.1].

Lemma 2.1. *There exists a constant* $c = c(N, p, s, \Omega)$ *such that*

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}\,dx\,dy \le c\int_\Omega |\nabla u|^p\,dx \tag{2.1}$$

*for every* $u \in X_1$.

For the following result, see [**40**].

Lemma 2.2. *Let* $\alpha = 0$ *and* 1. *Then the spaces* $X_\alpha$ *are real separable and uniformly convex Banach space.*

From [**29**, Lemma 2.1] we have the following result.

Lemma 2.3. *Let* $1 < p < \infty$ *and* $H : \mathbb{R}^N \to [0, \infty)$ *be a Finsler-Minkowski norm. If*

$$H(x)^p + (p-1)H(y)^p - p\langle x, H(y)^{p-1}\nabla H(y)\rangle = 0, \tag{2.2}$$

*for some* $x, y \in \mathbb{R}^N$, $y \neq 0$ *and* $H(x) = H(y)$. *Then* $x = y$.

Moreover, for the following properties, refer to [**29**, Pages 539-540].

Lemma 2.4. *Let* $1 < p < \infty$ *and* $H : \mathbb{R}^N \to [0, \infty)$ *be a Finsler-Minkowski norm. Then*

(*a*) $H_0(\nabla H(x)) = 1$ *for every* $x \in \mathbb{R}^N \setminus \{0\}$.

(*b*) *For every* $x, \xi \in \mathbb{R}^N$, *we have*

$$\langle x, \xi\rangle \le H_0(\xi)H(x), \tag{2.3}$$

*where the equality in* (2.3) *holds iff*

$$x = H(\xi)\nabla H(\xi) \text{ or, equivalently, } H_0(\xi) = H(x). \tag{2.4}$$

Next, we state the following result, which follows from [**7**, Theorem 9.14].

Theorem 2.5. *Let* $V$ *be a real separable reflexive Banach space and* $V^*$ *be the dual of* $V$. *Assume that* $A : V \to V^*$ *is a bounded, continuous, coercive and monotone operator. Then* $A$ *is surjective, i.e., given any* $f \in V^*$, *there exists* $u \in V$ *such that* $A(u) = f$. *If* $A$ *is strictly monotone, then* $A$ *is also injective.*

The next result follows from [**35**, Corollary 1.57].

Lemma 2.6. *Let $\Omega \subset \mathbb{R}^N$ be such that $|\Omega| < \infty$ and $1 < p < \infty$, $1 < r < p^*$. Then for every $u \in W_0^{1,p}(\Omega)$, there exists a positive constant $C = C(r, p, N)$ such that*

$$\left(\int_\Omega |u|^r\, dx\right)^{\frac{1}{r}} \leq C|\Omega|^{\frac{1}{r}-\frac{1}{p}+\frac{1}{N}} \left(\int_\Omega |\nabla u|^p\, dx\right)^{\frac{1}{p}}. \tag{2.5}$$

Next we define the notion of solution of the problem (1.1).

Definition 2.7. Let $\alpha = 0$ or 1. We say that $(\lambda, u) \in \mathbb{R} \times X_\alpha \setminus \{0\}$ is an eigenpair of (1.1) if for every $\phi \in X_\alpha$, we have

$$\begin{aligned}\int_\Omega & H(\nabla u)^{p-1} \nabla H(\nabla u) \nabla \phi\, dx \\ &+ \alpha \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))(\phi(x) - \phi(y))}{|x - y|^{N+ps}}\, dxdy \\ &= \lambda \|u\|_{L^q(\Omega)}^{p-q} \int_\Omega |u|^{q-2} u\phi\, dx.\end{aligned} \tag{2.6}$$

We observe that Lemma 2.1 ensures the above Definition in (2.6) is well stated. We refer to $\lambda$ as an eigenvalue and $u$ as an eigenfunction of (1.1) corresponding to the eigenvalue $\lambda$.

### **2.1 -** *Main results*

Our main results in this article reads as follows:

Theorem 2.8. *Let $\alpha = 0$ or 1. Suppose $0 < s < 1$, $1 < p < \infty$ and $1 < q < p^*$. Then the following properties hold:*

(a) *There exists a sequence $\{w_n\}_{n\in\mathbb{N}} \subset X_\alpha \cap L^q(\Omega)$ such that $\|w_n\|_{L^q(\Omega)} = 1$ and for every $v \in X_\alpha$, we have*

$$\begin{aligned}\int_\Omega & H(\nabla w_n)^{p-1} \nabla H(\nabla w_n) \nabla v\, dx \\ &+ \alpha \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|w_n(x) - w_n(y)|^{p-2}(w_n(x) - w_n(y))(v(x) - v(y))}{|x - y|^{N+ps}}\, dxdy \\ &= \mu_n \int_\Omega |w_n|^{q-2} w_n v\, dx,\end{aligned} \tag{2.7}$$

*where*

$$\mu_n \geq \lambda := \inf_{\{u\in X_\alpha\cap L^q(\Omega),\ \|u\|_{L^q(\Omega)}=1\}} \left\{ \int_\Omega H(\nabla u)^p\,dx + \alpha \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}\,dxdy \right\}.$$

(*b*) *Moreover, the sequences* $\{\mu_n\}_{n\in\mathbb{N}}$ *and* $\{\|w_{n+1}\|^p_{X_\alpha}\}_{n\in\mathbb{N}}$ *given by* (2.7) *are nonincreasing and converge to the same limit* $\mu$, *which is bounded below by* $\lambda$. *Further, there exists a subsequence* $\{n_j\}_{j\in\mathbb{N}}$ *such that both* $\{w_{n_j}\}_{j\in\mathbb{N}}$ *and* $\{w_{n_{j+1}}\}_{j\in\mathbb{N}}$ *converges in* $X_\alpha$ *to the same limit* $w \in X_\alpha \cap L^q(\Omega)$ *with* $\|w\|_{L^q(\Omega)} = 1$ *and* $(\mu, w)$ *is an eigenpair of* (1.1).

Theorem 2.9. *Let* $\alpha = 0$ *or* 1. *Assume that* $0 < s < 1$, $1 < p < \infty$ *and* $1 < q < p^*$. *Suppose* $\{u_n\}_{n\in\mathbb{N}} \subset X_\alpha \cap L^q(\Omega)$ *such that* $\|u_n\|_{L^q(\Omega)} = 1$ *and* $\lim_{n\to\infty} \|u_n\|^p_{X_\alpha} = \lambda$.

*Then there exists a subsequence* $\{u_{n_j}\}_{j\in\mathbb{N}}$ *which converges weakly in* $X_\alpha$ *to* $u \in X_\alpha \cap L^q(\Omega)$ *with* $\|u\|_{L^q(\Omega)} = 1$ *such that*

$$\lambda = \int_\Omega H(\nabla u)^p\,dx + \alpha \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}\,dxdy.$$

*Moreover,* $(\lambda, u)$ *is an eigenpair of* (1.1) *and any associated eigenfunction of* $\lambda$ *are precisely the scalar multiple of those vectors at which* $\lambda$ *is reached.*

Our final main result concerns the following qualitative properties of the eigenfunctions of (1.1).

Theorem 2.10. *Let* $\alpha = 0$ *or* 1. *Assume that* $0 < s < 1$, $1 < p < \infty$ *and* $1 < q < p^*$. *Suppose* $\lambda > 0$ *is an eigenvalue of the problem* (1.1) *and* $u \in X_\alpha \setminus \{0\}$ *is a corresponding eigenfunction. Then* (*a*) $u \in L^\infty(\Omega)$. (*b*) *Moreover, if* $u$ *is nonnegative in* $\Omega$, *then* $u > 0$ *in* $\Omega$. *Further, for every* $\omega \Subset \Omega$ *there exists a positive constant* $c$ *depending on* $\omega$ *such that* $u \geq c > 0$ *in* $\omega$.

## 3 - Coercive operators

In this section, we establish some preliminary results that are crucial to prove our main results. To this end, Let $\alpha = 0$ or 1. For $v \in X_\alpha$, we define the operators $A_\alpha : X_\alpha \to {X_\alpha}^*$ by

$$\langle A_\alpha(v), w\rangle = \int_\Omega H(\nabla v)^{p-2}\nabla H(\nabla v)\nabla w\,dx \tag{3.1}$$

$$+\alpha\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|v(x)-v(y)|^{p-2}(v(x)-v(y))(w(x)-w(y))}{|x-y|^{N+ps}}\,dxdy,\quad \forall w\in X_\alpha$$

and $B: L^q(\Omega)\to(L^q(\Omega))^*$ by

$$\langle B(v),w\rangle=\int_\Omega |v|^{q-2}vw\,dx,\quad \forall w\in X_\alpha. \tag{3.2}$$

The symbols ${X_\alpha}^*$ and $(L^q(\Omega))^*$ denotes the dual of $X_\alpha$ and $L^q(\Omega)$ respectively. First, we have the following result.

Lemma 3.1. *Let $\alpha=0$ or $1$. Then (i) The operators $A_\alpha$ defined by* (3.1) *and $B$ defined by* (3.2) *are continuous. (ii) Moreover, $A$ is bounded, coercive and monotone.*

Proof. (*i*) **Continuity:** Suppose $v_n\in X_\alpha$ such that $v_n\to v$ in the norm of $X_\alpha$. Thus, up to a subsequence $\nabla v_n(x)\to\nabla v(x)$ for almost every $x\in\Omega$. We observe that

$$\|H(\nabla v_n)^{p-1}\nabla H(\nabla v_n)\|_{L^{\frac{p}{p-1}}(\Omega)}\leq c\|H(\nabla v_n)\|^{p-1}_{L^p(\Omega)}\leq c, \tag{3.3}$$

for some constant $c>0$, which is independent of $n$. Thus, up to a subsequence, we have

$$H(\nabla v_n)|^{p-1}\nabla H(\nabla v_n)\rightharpoonup H(\nabla v)^{p-1}\nabla H(\nabla v)\text{ weakly in }L^{p'}(\Omega), \tag{3.4}$$

where $p'=\frac{p}{p-1}$. Moreover, for $\alpha=1$, by Lemma 2.1, up to a subsequence, we have

$$\frac{|v_n(x)-v_n(y)|^{p-2}(v_n(x)-v_n(y))}{|x-y|^{\frac{N+ps}{p'}}}\rightharpoonup\frac{|v(x)-v(y)|^{p-2}(v(x)-v(y))}{|x-y|^{\frac{N+ps}{p'}}} \tag{3.5}$$

weakly in $L^{p'}(\mathbb{R}^{2N})$. Since, the weak limit is independent of the choice of the subsequence, as a consequence of (3.4) and (3.5), we have

$$\lim_{n\to\infty}\langle A_\alpha(v_n),w\rangle=\langle A_\alpha v,w\rangle$$

for every $w\in X_\alpha$. Thus $A_\alpha$ is continuous. Similarly, we obtain $B$ is continuous.

(*ii*) **Boundedness:** First using Cauchy-Schwartz inequality, for every $v,w\in$

$X_\alpha$, we obtain

$$\begin{aligned}\langle A_\alpha(v), w\rangle &= \int_\Omega H(\nabla v)^{p-1}\nabla H(\nabla v)\nabla w\,dx\\&\quad+\alpha\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|v(x)-v(y)|^{p-2}(v(x)-v(y))(w(x)-w(y))}{|x-y|^{N+ps}}\,dxdy\\&\qquad\leq\int_\Omega H(\nabla v)^{p-1}H_0(\nabla H(\nabla v))H(\nabla w)\,dx\\&\qquad\quad+\alpha\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|v(x)-v(y)|^{p-1}|w(x)-w(y)|}{|x-y|^{N+ps}}\,dxdy.\end{aligned}$$

Now using Hölder's inequality with exponents $\frac{p}{p-1}$ and $p$ we have

$$\begin{aligned}&\langle A_\alpha(v), w\rangle \qquad (3.6)\\&\leq\int_\Omega H(\nabla v)^{p-1}H(\nabla w)\,dx+\alpha\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|v(x)-v(y)|^{p-1}|w(x)-w(y)|}{|x-y|^{N+ps}}\,dxdy\\&\qquad\leq\Big(\int_\Omega H(\nabla v)^p\,dx\Big)^{\frac{p-1}{p}}\Big(\int_\Omega H(\nabla w)^p\,dx\Big)^{\frac{1}{p}}\\&\quad+\alpha\Big(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|v(x)-v(y)|^p}{|x-y|^{N+ps}}\,dxdy\Big)^{\frac{p-1}{p}}\Big(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|w(x)-w(y)|^p}{|x-y|^{N+ps}}\,dxdy\Big)^{\frac{1}{p}}\\&\leq\left[\Big(\int_\Omega H(\nabla v)^p\,dx\Big)^{\frac{p-1}{p}}+\alpha\Big(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|v(x)-v(y)|^p}{|x-y|^{N+ps}}\,dxdy\Big)^{\frac{p-1}{p}}\right]\|w\|_{X_\alpha}\\&\quad\leq\Big(\int_\Omega H(\nabla v)^p\,dx+\alpha\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|v(x)-v(y)|^p}{|x-y|^{N+ps}}\,dxdy\Big)^{\frac{p-1}{p}}\|w\|_{X_\alpha}\\&\qquad\qquad=\|v\|_{X_\alpha}^{p-1}\|w\|_{X_\alpha},\end{aligned}$$

where in the second and third step above, we have used Lemma 2.4.

Therefore, we have

$$\|A_\alpha(v)\|_{X_\alpha^*}=\sup_{\|w\|_{X_\alpha}\leq 1}|\langle Av, w\rangle|\leq\|v\|_{X_\alpha}^{p-1}\|w\|_{X_\alpha}\leq\|v\|_{X_\alpha}^{p-1}.$$

Thus, $A_\alpha$ is bounded.

**Coercivity:** We observe that

$$\langle A_\alpha(v), v\rangle=\int_\Omega|\nabla v|^p\,dx+\alpha\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|v(x)-v(y)|^p}{|x-y|^{N+ps}}\,dxdy=\|v\|_{X_\alpha}^p.$$

Since $p>1$, we have $A_\alpha$ is coercive.

**Monotonicity:** For $u \in X_\alpha$, let us denote by

$$\mathcal{A}(u(x,y)) = |u(x)-u(y)|^{p-2}(u(x)-u(y)), \quad d\mu = \frac{dxdy}{|x-y|^{N+ps}}.$$

Using the inequality (1.4), for every $v, w \in X_\alpha$, we have

$$\begin{aligned}
&\langle A_\alpha(v) - A_\alpha(w), v-w\rangle \\
&= \int_\Omega \langle H(\nabla v)^{p-1}\nabla H(\nabla v) - H(\nabla w)^{p-1}\nabla H(\nabla w), \nabla(v-w)\rangle\, dx \\
&\quad + \alpha \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \Big(\mathcal{A}(v(x,y)) - \mathcal{A}(w(x,y))\Big)((v(x)-w(x)) - (v(y)-w(y)))\, d\mu \\
&\geq 0.
\end{aligned}$$

Thus, $A_\alpha$ is monotone. □

Lemma 3.2. *Let $\alpha = 0$ or $1$. Then the operators $A_\alpha$ defined by* (3.1) *and $B$ defined by* (3.2) *satisfy the following properties:*

$(H_1)$ $A_\alpha(tv) = |t|^{p-2}tA_\alpha(v) \quad \forall t \in \mathbb{R}$ *and* $\forall v \in X_\alpha$.

$(H_2)$ $B(tv) = |t|^{q-2}tB(v) \quad \forall t \in \mathbb{R}$ *and* $\forall v \in L^q(\Omega)$.

$(H_3)$ $\langle A_\alpha(v), w\rangle \leq \|v\|_{X_\alpha}^{p-1}\|w\|_{X_\alpha}$ *for all $v, w \in X_\alpha$, where the equality holds if and only if $v = 0$ or $w = 0$ or $v = tw$ for some $t > 0$.*

$(H_4)$ $\langle B(v), w\rangle \leq \|v\|_{L^q(\Omega)}^{q-1}\|w\|_{L^q(\Omega)}$ *for all $v, w \in L^q(\Omega)$, where the equality holds if and only if $v = 0$ or $w = 0$ or $v = tw$ for some $t \geq 0$.*

$(H_5)$ *For every $w \in L^q(\Omega)\setminus\{0\}$ there exists $u \in X_\alpha \setminus \{0\}$ such that*

$$\langle A_\alpha(u), v\rangle = \langle B(w), v\rangle \quad \forall \quad v \in X_\alpha.$$

Proof. $(H_1)$ Follows by the definition of $A_\alpha$.

$(H_2)$ Follows by the definition of $B$.

$(H_3)$ First, we note that from (3.6) the inequality $\langle A_\alpha(v), w\rangle \leq \|v\|_{X_\alpha}^{p-1}\|w\|_{X_\alpha}$ holds for all $v, w \in X_\alpha$. Let the equality

$$\langle A_\alpha(v), w\rangle = \|v\|_{X_\alpha}^{p-1}\|w\|_{X_\alpha} \tag{3.7}$$

holds for every $v, w \in X_\alpha$. We claim that either $v = 0$ or $w = 0$ or $v = tw$ for some constant $t > 0$. Indeed, if $v = 0$ or $w = 0$, this is trivial. Therefore, we assume $v \neq 0$ and $w \neq 0$ and prove that $v = tw$ for some constant $t > 0$. By the estimate (3.6) if the equality (3.7) holds, then we have

$$\begin{aligned}(3.8)\qquad \langle A_\alpha(v), w\rangle = &\int_\Omega H(\nabla v)^{p-1} H(\nabla w)\, dx \\ &+ \alpha \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|v(x)-v(y)|^{p-1}|w(x)-w(y)|}{|x-y|^{N+ps}}\, dxdy,\end{aligned}$$

which gives us

$$(3.9)\qquad \int_\Omega f(x)\, dx + \alpha \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{g(x,y)\, dxdy}{|x-y|^{N+ps}} = 0,$$

where

$$f(x) = H(\nabla v)^{p-1} H(\nabla w) - H(\nabla v)^{p-1}\langle \nabla H(\nabla v), \nabla w\rangle$$

and

$$g(x,y) = |v(x)-v(y)|^{p-1}|w(x)-w(y)| - |v(x)-v(y)|^{p-2}(v(x)-v(y))w(x)-w(y).$$

By Cauchy-Schwartz inequality and Lemma 2.4, we have $f \geq 0$ in $\Omega$ and $g \geq 0$ in $\mathbb{R}^N \times \mathbb{R}^N$. Hence using these facts in (3.9), we have $f = 0$ in $\Omega$, which reduces to

$$(3.10)\qquad H(\nabla v)^{p-1} H(\nabla w) = H(\nabla v)^{p-1}\langle \nabla H(\nabla v), \nabla w\rangle \text{ in } \Omega.$$

On the other hand, if the equality (3.7) holds, then by the estimate (3.6) we have

$$(3.11)\qquad f_1 - f_2 = \alpha(g_2 - g_1),$$

where

$$f_1 = \int_\Omega H(\nabla v)^{p-1} H(\nabla w)\, dx, \quad f_2 = \Big(\int_\Omega H(\nabla v)^p\, dx\Big)^{\frac{p-1}{p}} \Big(\int_\Omega H(\nabla w)^p\, dx\Big)^{\frac{1}{p}},$$

$$g_1 = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|v(x)-v(y)|^{p-2}(v(x)-v(y))(w(x)-w(y))}{|x-y|^{N+ps}}\, dxdy$$

and

$$g_2 = \Big(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|v(x)-v(y)|^p}{|x-y|^{N+ps}}\, dxdy\Big)^{\frac{p-1}{p}} \Big(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|w(x)-w(y)|^p}{|x-y|^{N+ps}}\, dxdy\Big)^{\frac{1}{p}}.$$

If $\alpha = 0$, then from (3.11) we get $f_1 = f_2$. If $\alpha = 1$, then by Hölder's inequality, we know that $f_1 - f_2 \leq 0$ and $g_2 - g_1 \geq 0$. Therefore, we obtain from (3.11) that

$$f_1 = f_2 \text{ and } g_1 = g_2.$$

Since $f_1 = f_2$, the equality in Hölder's inequality holds, which gives

$$H(\nabla v) = H(t\nabla w) \text{ in } \Omega, \tag{3.12}$$

for some constant $t > 0$. Using (3.12), we observe that

$$\begin{aligned} L_1 &= H(t\nabla w)^p + (p-1)H(\nabla v)^p \\ &= H(\nabla v)^p + (p-1)H(\nabla v)^p = pH(\nabla v)^p. \end{aligned} \tag{3.13}$$

By (3.10) and (3.12), we obtain

$$\begin{aligned} L_2 &= p\langle t\nabla w, H(\nabla v)^{p-1}\nabla_\eta H(\nabla v)\rangle \\ &= pH(\nabla v)^{p-1}H(t\nabla w) = pH(\nabla v)^p. \end{aligned} \tag{3.14}$$

Thus, from (3.13) and (3.14), we have

$$L = L_1 - L_2 = 0. \tag{3.15}$$

Noting (3.12) and (3.15), by Lemma 2.3 to obtain $\nabla v = t\nabla w$ in $\Omega$. Therefore, $v = tw$ in $\Omega$ for some constant $t > 0$. Hence, the property $(H3)$ is verified.

$(H_4)$ This property can be verified similarly as in $(H_3)$.

$(H_5)$ Note that by Lemma 2.2, it follows that $X_\alpha$ is a separable and reflexive Banach space. By Lemma 3.1, the operator $A_\alpha : X_\alpha \to X_\alpha^*$ is bounded, continuous, coercive and monotone.

By the Sobolev embedding theorem, we have $X_\alpha$ is continuously embedded in $L^q(\Omega)$. Therefore, $B(w) \in X_\alpha^*$ for every $w \in L^q(\Omega) \setminus \{0\}$.

Hence, by Theorem 2.5, for every $w \in L^q(\Omega) \setminus \{0\}$, there exists $u \in X_\alpha \setminus \{0\}$ such that

$$\langle A_\alpha(u), v\rangle = \langle B(w), v\rangle \quad \forall v \in X_\alpha.$$

Hence the property $(H_5)$ holds. This completes the proof. □

## 4 - Proof of the main results:

**Proof of Theorem 2.8:**

$(a)$ First we recall the definition of the operators $A_\alpha : X_\alpha \to X_\alpha^*$ from (3.1) and $B : L^q(\Omega) \to (L^q(\Omega))^*$ from (3.2) respectively. Then, taking into account the property $(H_5)$ from Lemma 3.2 and proceeding along the lines of the proof in [**18**, page 579 and pages 584 – 585], the result follows.

($b$) Note that by Lemma 2.2, $X_\alpha$ is uniformly convex Banach space and by the Sobolev embedding theorem, $X$ is compactly embedded in $L^q(\Omega)$. Next, using Lemma 3.1-($i$), the operators $A_\alpha : X_\alpha \to X_\alpha^*$ and $B : L^q(\Omega) \to (L^q(\Omega))^*$ are continuous and by Lemma 3.2, the properties $(H_1) - (H_5)$ holds. Taking into account these facts, the result follows from [**18**, page 579, Theorem 1]. □

**Proof of Theorem 2.9:** The proof follows due to the same reasoning as in the proof of Theorem 2.8-($b$) except that here we apply [**18**, page 583, Proposition 2] in place of [**18**, page 579, Theorem 1].

**Proof of Theorem 2.10:**

($a$) Due to the homogeneity of the equation (1.1), without loss of generality, we assume that $\|u\|_{L^q(\Omega)} = 1$. Let $k \geq 1$ and set $L(k) := \{x \in \Omega : u(x) > k\}$. Choosing $v = (u-k)^+$ as a test function in (2.6), we obtain using (1.2) that

(4.1)
$$c_1 \int_{L(k)} |\nabla u|^p \, dx + \alpha \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+ps}} \, dxdy$$
$$\leq \lambda \int_{L(k)} u^{q-1}(u-k) \, dx \leq \lambda \int_{L(k)} u^{q-1}(u-k) \, dx.$$

Now proceeding along the lines of the proof of [**24**, Theorem 2.8], the result follows.

($b$) By [**27**, Theorem 3.59] (for $\alpha = 0$) and by [**22**, Theorem 3.10] (for $\alpha = 1$), the result follows. □

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