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Powers of Fibonacci numbers which are products of repdigits

Abstract. In this article we solve the equation $F_n^k = \left(d_1 \cdot \frac{10^m-1}{9}\right) \cdot \left(d_2 \cdot \frac{10^q-1}{9}\right)$, with $n, k, d_1, d_2, m, q \in \mathbb{N}, d_1, d_2 = 1, \ldots, 9, m, q \geq 2$, $k \geq 2$, showing that the only perfect power of a Fibonacci number which is a product of two repdigits is $F_{10}^2 = 55 \cdot 55$.
In order to do this we use only elementary methods, like divisibility properties of Fibonacci numbers, periodicity, results on prime factorizations and an application of Nagell-Ljunggren equations.

Keywords. Exponential Diophantine equations, Fibonacci numbers.

Mathematics Subject Classification: 11D61, 11B39.

1 - Introduction

Let F_n be the Fibonacci sequence defined by $F_1 = 1$, $F_2 = 1$, and $F_n = F_{n-1} + F_{n-2}$, for $n \geq 3$.

A repdigit is a number written with only one distinct digit and at least two digits.

Problems involving Fibonacci numbers and their products which are repdigits and products of such numbers have been studied in several articles, for example in [**2**, **3**].

The common strategy in those papers is to use linear forms in logarithms to bound the largest index in a solution, and then applying Baker-Davenport's reduction method to lower that bound and search for possible solutions through computer calculations. See [**1**, **8**].

For instance, in [**2**] it is shown that $F_1 = 1$, $F_2 = 1$, $F_3 = 2$, $F_4 = 3$, $F_5 = 5$, $F_6 = 8$ and $F_{10} = 55$ are the only Fibonacci numbers which are products of two numbers written with only one distinct digit.

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In this article we extend that result for powers of Fibonacci numbers, using only divisibility properties, periodicity, results on prime factorizations, and a result on the Nagell-Ljunggren equation, showing that the only solution of the equation

$$F_n^k = \left(d_1 \cdot \frac{10^m - 1}{9}\right) \cdot \left(d_2 \cdot \frac{10^q - 1}{9}\right)$$

with $n, k, d_1, d_2, m, q \in \mathbb{N}, d_1, d_2 = 1, \ldots, 9, m, q \geq 2,\ k \geq 2$ is given by $(n, k, d_1, d_2, m, q) = (10, 2, 5, 5, 2, 2)$.

2 - On the equation $F_n^k = \left(d_1 \cdot \frac{10^m-1}{9}\right) \cdot \left(d_2 \cdot \frac{10^q-1}{9}\right)$

First we state some useful lemmas and propositions that will be used in the solution and prove some of them.

The well known divisibility properties of Fibonacci numbers are used throughout the article, like if $2 \mid F_n$ then $3 \mid F_n$, if $3 \mid F_n$, then $4 \mid n$, if $5 \mid n$ then $5 \mid F_n$, if $6 \mid n$, then $8 \mid F_n$.

Lemma 2.1. *Let n be an odd number. If p is an odd prime with $p \mid F_n$, then $p \equiv 1 \pmod 4$. See* [**4**].

Definition 2.2. *Let $\left(\frac{a}{p}\right)$ denote the Legendre Symbol of a mod p.*

Lemma 2.3. *Let n be an odd number. Then $\left(\frac{L_n}{5}\right) = 1$, where L_n is the n-th Lucas Number. See* [**7**].

Lemma 2.4. *The only solutions to the Nagell-Ljunggren equation $y^2 = \frac{x^n-1}{x-1}$, for $x, y, n \in \mathbb{N}$, $n \geq 3$ are given by $x = 3, n = 5, y = 11$ and $x = 7, n = 4, y = 20$. See* [**5**, **9**].

Proposition 2.5. *The product $\left(\frac{10^n-1}{9}\right) \cdot \left(\frac{10^m-1}{9}\right)$ is a perfect square if and only if $n = m$.*

Proof. If $n = m$ clearly the product is a perfect square. Suppose without loss of generality that $n > m$.

We have $(10^n - 1, 10^m - 1) = 10^{(n,m)} - 1$, where (n, m) denotes the greatest common divisor of n, m.

Hence, $10^n - 1 = x \cdot \left(10^{(n,m)} - 1\right)$ and $10^m - 1 = y \cdot \left(10^{(n,m)} - 1\right)$ for naturals x, y, with $(x, y) = 1$.

Therefore,

$$9^2 \cdot \left(\frac{10^n - 1}{9}\right) \cdot \left(\frac{10^m - 1}{9}\right) = x \cdot y \cdot \left(10^{(n,m)} - 1\right)^2.$$

In order for this number to be a perfect square, x and y must be perfect squares.

Let $d = (n, m)$. We can write $n = q \cdot d$ and $m = p \cdot d$ for some natural numbers q, p. If $p = 1$, then $q \geq 2$ since $n \neq m$. Suppose $q = 2$. It follows that

$$x = \frac{10^{2d} - 1}{10^d - 1} = 10^d + 1.$$

But $10^d + 1 \equiv 2 \pmod 3$, and 2 is not a quadratic residue modulo 3, so that x is not a perfect square.

Suppose $q \geq 3$. Then,

$$x = \frac{(10^d)^q - 1}{10^d - 1}.$$

If $x = k^2$, then

$$k^2 = \frac{(10^d)^q - 1}{10^d - 1}.$$

But this is a Nagell-Ljunggren equation and by Lemma 2.4 this equation has no solution.

Consider the case $p = 2$. Then

$$y = \frac{10^{2d} - 1}{10^d - 1} = 10^d + 1$$

and analogously y can not be a square.

If $p \geq 3$, similarly

$$y = \frac{(10^d)^p - 1}{10^d - 1}$$

and y can not be a square.

Therefore, for $n \neq m$ the product is not a perfect square. $\square$

Proposition 2.6. *If $F_n^k = \left(d_1 \cdot \frac{10^m - 1}{9}\right) \cdot \left(d_2 \cdot \frac{10^q - 1}{9}\right)$, $m, q \geq 2$, then necessarily n is even.*

Proof. If $m, q \geq 2$, then $11\ldots1 \equiv 3 \pmod 4$, so that there exists a prime $p \equiv 3 \pmod 4$ with $p \mid 11\ldots1$. Therefore, $p \mid F_n^k$ and necessarily $p \mid F_n$. By Lemma 2.1, it follows that n is even. $\square$

Proposition 2.7. *The only solution to the equation $F_n^k = \left(d_1 \cdot \frac{10^m - 1}{9}\right) \cdot \left(d_2 \cdot \frac{10^q - 1}{9}\right)$, with $k \geq 2$, $m, q \geq 2$ and at least one of m, q less than or equal to 4, is given by $(n, k, d_1, d_2, m, q) = (10, 2, 5, 5, 2, 2)$.*

Proof. It can be checked by direct calculations that for $2 \leq m, q \leq 4$ the only solution is $(n, k, d_1, d_2, m, q) = (10, 2, 5, 5, 2, 2)$.

Hence, we have to consider the cases in which one of m, q is larger than or equal to 5 and the other is 2, 3 or 4. By Proposition 2.6, it follows that n is even, so this occurs in all the cases.

By symmetry, we can suppose without loss of generality that $m = 2, 3, 4$ and $q \geq 5$. Suppose $m = 2$. Then the equation is

$$F_n^k = d_1 \cdot 11 \cdot d_2 \cdot \frac{10^q - 1}{9}.$$

Since $11 \mid F_n^k$, $11 \mid F_n$ and then $10 \mid n$ and $5 \mid F_n$, so that either d_1 or d_2 is equal to 5.

If only one of d_1, d_2 is equal to 5, then clearly $5 \mid\mid F_n^k$, and necessarily $k = 1$. If both are equal to 5, by Proposition 2.5 the right-hand side is not a square, and since $5^2 \mid\mid F_n^k$, necessarily $k = 1$.

Therefore, there is no solution in this case.

Suppose $m = 3$. Then the equation is

$$F_n^k = d_1 \cdot 111 \cdot d_2 \cdot \frac{10^q - 1}{9}.$$

Since $37 \mid 111$, $37 \mid F_n^k$ and $19 \mid n$. Thus $113 \mid F_n$.

The order of 10 modulo 113 is 112, so if $113 \mid 10^q - 1$, then $112 \mid q$, and q must be even. But q even implies $11 \mid 10^q - 1$, so that $11 \mid F_n$ and $10 \mid n$. Hence, the right-hand side is multiple of 5 and at least one of d_1, d_2 is equal to 5. Analogously to the previous case, $k = 1$.

Therefore, there is no solution in this case.

Suppose $m = 4$. Then the equation is

$$F_n^k = d_1 \cdot 1111 \cdot d_2 \cdot \frac{10^q - 1}{9}.$$

Since $11 \mid 1111$, $11 \mid F_n$ and then $10 \mid n$, so that $5 \mid F_n$. By the same argument of the previous cases, $k = 1$.

Therefore, there is no solution in this case. $\square$

Lemma 2.8. *The only Fibonacci numbers that are written with only one distinct digit are* $F_1 = 1$, $F_2 = 1$, $F_3 = 2$, $F_4 = 3$, $F_5 = 5$, $F_6 = 8$, $F_{10} = 55$. *See* [**7**].

Lemma 2.9. *The only solutions to the equation* $F_n^k = d \cdot \frac{10^m - 1}{9}$ *with* $n, k, d, m \in \mathbb{N}, k \geq 2$, $d = 1, \ldots, 9$ *are given by* $(n, d, m) = (1, 1, 1)$ *and any* $k \geq 2$, $(n, d, m) = (2, 1, 1)$ *and any* $k \geq 2$, $(n, k, d, m) = (3, 2, 4, 1)$, $(n, k, d, m) = (3, 3, 8, 1)$, $(n, k, d, m) = (4, 2, 9, 1)$. *See* [**6**].

Now we prove the claim.

Theorem 2.10. *The only solution to the equation*

$$F_n^k = \left(d_1 \cdot \frac{10^m - 1}{9}\right) \cdot \left(d_2 \cdot \frac{10^q - 1}{9}\right)$$

with $n, k, d_1, d_2, m, q \in \mathbb{N}$, $k \geq 2$, $d_1, d_2 = 1, \dots, 9$, $m, q \geq 2$, *is given by* $(n, k, d_1, d_2, m, q) = (10, 2, 5, 5, 2, 2)$.

Proof. In view of the previous lemmas and propositions, it is necessary to consider only the cases in which n is even, and both $m, q \geq 5$.

Case $d_1 = 1$:

If d_2 is even, then $2 \mid F_n$ and $3 \mid n$, so that $6 \mid n$. Thus, $8 \mid F_n$ and necessarily $k = 1$, since the highest power of 2 that divides the right-hand side is $2^3 = 8$, corresponding to $d_2 = 8$.

If $d_2 = 1$, the equation becomes

$$F_n^k = \left(\frac{10^m - 1}{9}\right) \cdot \left(\frac{10^q - 1}{9}\right).$$

First we consider $m \neq q$.

By Proposition 2.5, the right-hand side is not a square, so that k is odd.

Rewrite this equation as

$$F_n \cdot F_n^{k-1} = \left(\frac{10^m - 1}{9}\right) \cdot \left(\frac{10^q - 1}{9}\right).$$

Since k is odd, F_n^{k-1} is a perfect square. Also $\frac{10^m-1}{9} \equiv 7 \pmod{32}$, $\frac{10^q-1}{9} \equiv 7 \pmod{32}$, $\frac{10^m-1}{9} \equiv 7 \pmod{16}$ and $\frac{10^q-1}{9} \equiv 7 \pmod{16}$ for $m, q \geq 5$.

It follows that m and q must be odd, otherwise 11 divides the right-hand side, so that $11 \mid F_n$, which implies $10 \mid n$ and then $5 \mid F_n$, contradiction since the right-hand side is not multiple of 5.

The only odd squares modulo 16 are 1 and 9. If $F_n^{k-1} \equiv 9 \pmod{16}$, by the equation it follows that $F_n \equiv 9 \pmod{16}$. By periodicity and since n is even, the only possibility would be $n \equiv 14 \pmod{24}$.

Put $n = 24x + 14$. Next, consider the equation modulo 5 and use the well known relation $F_{2n} = F_n \cdot L_n$. Since the Legendre Symbol is a completely multiplicative function, we have

$$\left(\frac{F_n}{5}\right) = 1.$$

Thus, $\left(\frac{F_{12x+7} \cdot L_{12x+7}}{5}\right) = 1$, and by Lemma 2.3 it follows that $\left(\frac{F_{12x+7}}{5}\right) = 1$.

Let p be any prime factor of F_{12x+7}. Then, $p \mid F_n$ so that $p \mid \frac{10^m-1}{9}$ or $p \mid \frac{10^q-1}{9}$. From

$$\frac{10^m - 1}{9} \equiv 0 \pmod{p},$$

or

$$\frac{10^q - 1}{9} \equiv 0 \pmod{p},$$

it follows that

$$10^{m+1} \equiv 10 \pmod{p},$$

or

$$10^{q+1} \equiv 10 \pmod{p},$$

so that $\left(\frac{10}{p}\right) = 1$. This splits into the possibilities $\left(\frac{2}{p}\right) = 1$ and $\left(\frac{5}{p}\right) = 1$, or $\left(\frac{2}{p}\right) = -1$ and $\left(\frac{5}{p}\right) = -1$.

By the Law of Quadratic Reciprocity $\left(\frac{p}{5}\right) = \left(\frac{5}{p}\right)$, and by Lemma 2.1, $p \equiv 1, 5 \pmod{8}$. Hence, in order to have $\left(\frac{F_{12x+7}}{5}\right) = 1$ it is necessary that the amount of primes congruent to 5 modulo 8 counted with multiplicity be even.

This shows that $F_{12x+7} \equiv 1 \pmod{8}$, which is a contradiction, since the period of Fibonacci numbers modulo 8 is 12 and for $r \equiv 7 \pmod{12}$, $F_r \equiv 5 \pmod{8}$.

Hence, the only possibility is $F_n^{k-1} \equiv 1 \pmod{16}$, by periodicity and since n is even, it follows from the equation that $n \equiv 2 \pmod{24}$.

The only odd squares modulo 32 are $1, 9, 17, 25$. Since $F_n^{k-1} \equiv 1 \pmod{16}$, necessarily $F_n^{k-1} \equiv 1, 17 \pmod{32}$.

For $F_n^{k-1} \equiv 1 \pmod{32}$, it follows from the equation that $F_n \equiv 17 \pmod{32}$. By periodicity and since n is even, $n \equiv 26 \pmod{48}$.

Thus, $F_n \equiv 6 \pmod{7}$. Hence, the left-hand side is congruent to 6 modulo 7. Clearly, m and q are not divisible by 3, otherwise $3 \mid F_n$ and $4 \mid n$, which does not occur. Since m and q are odd, they can be only of the form $6x + 1$ or $6x + 5$. Hence, modulo 7 the right-hand side can be congruent only to 1, 2 or 4. Therefore, the equality can not hold.

For $F_n^{k-1} \equiv 17 \pmod{32}$, it follows that $F_n \equiv 1 \pmod{32}$. But any power of 1 is congruent to 1 modulo 32, so this is not a possibility.

Now we consider $m = q$.

We have k even, put $k = 2z$. The equation becomes

$$F_n^z = \left(\frac{10^m - 1}{9}\right).$$

If $z \geq 2$, by Lemma 2.9 this does not give a solution. If $z = 1$, by Lemma 2.8 this does not give a solution.

If $d_2 = 3$, then if $3 \nmid m$ and $3 \nmid q$, it follows that $3 \mid\mid F_n^k$, which implies $k = 1$.

If $3 \mid m$ or $3 \mid q$ then $111 \mid F_n$, so that $19 \mid n$, which implies $113 \mid F_n$. Similarly to the proof of the Proposition 2.7, this implies m or q even, and then $11 \mid F_n$, $10 \mid n$ and $5 \mid F_n$, but 5 does not divide the right-hand side.

If $d_2 = 5$, then $5 \mid\mid F_n^k$, which implies $k = 1$.

If $d_2 = 7$, then if $6 \nmid m$ and $6 \nmid q$, $7 \mid\mid F_n^k$ which implies $k = 1$. If $6 \mid m$ or $6 \mid q$, then m or q is even, and analogously to the previous proofs $5 \mid F_n$ resulting in the same contradiction.

If $d_2 = 9$, then the right-hand side is congruent to 1 mod 8. If $m \neq q$, by Proposition 2.5 it follows that k is odd. Rewrite the equation as

$$F_n \cdot F_n^{k-1} = 9 \cdot \left(\frac{10^m - 1}{9}\right) \cdot \left(\frac{10^q - 1}{9}\right).$$

The only odd square modulo 8 is 1, so that $F_n^{k-1} \equiv 1 \pmod 8$. Hence, $F_n \equiv 1 \pmod 8$. By periodicity and since n is even, $n \equiv 2 \pmod{12}$. But this is a contradiction, since $3 \mid F_n$ so $4 \mid n$.

If $m = q$, $3 \nmid m$ and $3 \nmid q$, then $3^2 \mid\mid F_n^k$, so that $k = 1$ or $k = 2$. Suppose $k = 2$. The equation becomes

$$F_n^2 = 3 \cdot \left(\frac{10^m - 1}{9}\right).$$

By Lemma 2.9, this does not give a solution.

If $3 \mid m$ or $3 \mid q$, then $111 \mid F_n$, so that $37 \mid F_n$, which implies $19 \mid n$. Hence, $113 \mid F_n$ and m or q is even. Thus, $11 \mid F_n$, $10 \mid n$ and then $5 \mid F_n$, but this is a contradiction since the right-hand side is not multiple of 5.

Therefore, there is no solution in this case.

Case $d_1 = 2$:

By symmetry, the possibility $d_2 = 1$ has already been dealt in the previous case.

If d_2 is even, analogously to the previous proofs, $6 \mid n$ so $8 \mid F_n$. Since the highest power of 2 that divides the right-hand side is $2^4 = 16$, corresponding to $d_2 = 8$, this implies that $k = 1$.

If d_2 is odd, then clearly $2 \mid\mid F_n^k$, contradiction.

Therefore, there is no solution in this case.

Case $d_1 = 3$:

By symmetry the possibilities $d_2 = 1$ and $d_2 = 2$ have already been dealt in the other cases.

If d_2 is even, analogously to the previous proofs $6 \mid n$ so $8 \mid F_n$. Since the highest power of 2 that divides the right-hand side is $2^3 = 8$, corresponding to $d_2 = 8$, necessarily $k = 1$.

If $d_2 = 3$, $3 \nmid m$ and $3 \nmid q$, then $3^2 \mid\mid F_n^k$, and if $m \neq q$, by Proposition 2.5 the right-hand side is not a square, so that k is odd and $k = 1$.

If $m = q$, $3 \nmid m$ or $3 \nmid q$, then $3^2 \mid\mid F_n^k$, so that $k = 1$ or $k = 2$. Suppose $k = 2$. Then the equation becomes

$$F_n^2 = 3 \cdot \left(\frac{10^m - 1}{9}\right).$$

By Lemma 2.9, this does not give a solution.

If $3 \mid m$ or $3 \mid q$, analogously to the previous proofs $37 \mid F_n$ and $19 \mid n$, so that $113 \mid F_n$ and m or q is even, which implies $5 \mid F_n$ and the same contradiction.

If $d_2 = 5$, then clearly $5 \mid\mid F_n^k$ and necessarily $k = 1$.

If $d_2 = 7$, then if $6 \nmid m$ and $6 \nmid q$, $7 \mid\mid F_n^k$, and necessarily $k = 1$.

If $6 \mid m$ or $6 \mid q$, then m or q is even, and analogously to the previous proofs, $5 \mid F_n$, contradiction.

If $d_2 = 9$, $3 \nmid m$ and $3 \nmid q$, then $3^3 \mid\mid F_n^k$ so that $k = 1$ or $k = 3$. Suppose $k = 3$. Then the equation becomes

$$F_n^3 = 3 \cdot 9 \cdot \left(\frac{10^m - 1}{9}\right) \cdot \left(\frac{10^q - 1}{9}\right).$$

The right-hand side is congruent to 11 modulo 32. The only solution of the congruence $x^3 \equiv 11 \pmod{32}$ is $x \equiv 19 \pmod{32}$, so that $F_n \equiv 19 \pmod{32}$. By periodicity, $n \equiv 28 \pmod{48}$. This implies $n \equiv 12 \pmod{16}$, so that $F_n \equiv 4 \pmod{7}$.

We have m and q odd, otherwise similarly to the previous cases $11 \mid F_n$ and $5 \mid F_n$, contradiction.

Since m and q can be only of the forms $6x+1$ or $6x+5$, the right-hand side can be congruent only to 6, 5 or 3 modulo 7. But $4^3 \equiv 1 \pmod{7}$, contradiction.

Therefore, there is no solution in this case.

Case $d_1 = 4$:

Since $2 \mid F_n$, $3 \mid n$ and then $6 \mid n$, so that $8 \mid F_n$.

If d_2 is even then the highest power of 2 that divides the right-hand side is 2^5, corresponding to $d_2 = 8$. Hence, $k = 1$.

If d_2 is odd, then $4 \mid\mid F_n^k$, contradiction.

Therefore, there is no solution in this case.

Case $d_1 = 5$:

If $d_2 \neq 5$, then $5 \mid\mid F_n^k$ and necessarily $k = 1$.

If $d_2 = 5$, then $5^2 \mid\mid F_n^k$, which implies $k = 1$ or $k = 2$. Suppose $k = 2$. Then, by Proposition 2.5, $m = q$ and the equation becomes

$$F_n = 5 \cdot \left(\frac{10^m - 1}{9} \right).$$

By Lemma 2.8, the only solution is given by $n = 10$ and $m = 2$, which gives the solution $(n, k, d_1, d_2, m, q) = (10, 2, 5, 5, 2, 2)$, and since we are considering $m, q \geq 5$, there is no new solution in this case.

<u>Case $d_1 = 6$:</u>

Since $2 \mid F_n$, $3 \mid n$ and since $3 \mid F_n$, $4 \mid n$, then $12 \mid n$, so that $16 \mid F_n$.

If d_2 is odd, clearly $2 \mid\mid F_n^k$, contradiction.

If d_2 is even, the highest power of 2 that divides the right-hand side is 2^4 corresponding to $d_2 = 8$. Hence, $k = 1$.

Therefore, there is no solution in this case.

<u>Case $d_1 = 7$:</u>

By symmetry, the possibilities $d_2 = 1, 2, 3, 4, 5, 6$ have already been dealt in the previous cases.

If $d_2 = 7$, then m and q must be odd, since similarly to the previous proofs, if any of them is even, $5 \mid F_n$, but the right-hand side is not a multiple of 5.

Hence, $7^2 \mid\mid F_n^k$, and $k = 1$ or $k = 2$. Suppose $k = 2$. If $m \neq q$, by Proposition 2.5 the right-hand side is not a square, contradiction.

If $m = q$, the equation becomes

$$F_n = 7 \cdot \left(\frac{10^m - 1}{9} \right).$$

By Lemma 2.8, this does not give a solution.

If $d_2 \neq 7$, since also $d_2 \neq 5$, m and q must be odd and clearly $7 \mid\mid F_n^k$, so that $k = 1$.

Therefore, there is no solution in this case.

Case $d_1 = 8$:

By symmetry, the cases $d_2 = 1, 2, 3, 4, 5, 6, 7$ have already been dealt.

Since $2 \mid F_n$, $3 \mid n$, so that $6 \mid n$ and then $8 \mid F_n$.

If $d_2 = 8$, the highest power of 2 dividing the right-hand side is 2^6, so that $k = 1$ or $k = 2$. Suppose $k = 2$. By Proposition 2.5, $m = q$ and the equation becomes

$$F_n = 8 \cdot \left(\frac{10^m - 1}{9} \right).$$

By Lemma 2.8, this does not give a solution.

If $d_2 = 9$, then clearly $2^3 \mid\mid F_n^k$, so that $k = 1$.

Therefore, there is no solution in this case.

Case $d_1 = 9$:

By symmetry the cases $d_2 = 1, 2, 3, 4, , 6, 7, 8$ have already been dealt.

If $d_2 = 9$, then if $m \neq q$ by Proposition 2.5, k is odd.

Rewrite the equation as

$$F_n \cdot F_n^{k-1} = 9 \cdot 9 \cdot \left(\frac{10^m - 1}{9} \right) \cdot \left(\frac{10^q - 1}{9} \right).$$

The only odd square modulo 8 is 1, thus $F_n^{k-1} \equiv 1 \pmod 8$. The right-hand side is congruent to 1 modulo 8. Hence, by the equation $F_n \equiv 1 \pmod 8$. By periodicity and since n is even, $n \equiv 2 \pmod{12}$. But this is a contradiction since $3 \mid F_n$ so $4 \mid n$.

If $m = q$, then k is even. Put $k = 2z$. Then, the equation becomes

$$F_n^z = 9 \cdot \left(\frac{10^m - 1}{9} \right).$$

For $z \geq 2$, by Lemma 2.9 this does not give a solution.

For $z = 1$, by Lemma 2.8 this does not give a solution.

Therefore, there is no solution in this case. □

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