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The porous medium equation with capillary pressure effects

Abstract. We consider a third order equation, which includes pressure as a dissipative term, and describes the dynamics of two-phase flows in a porous media. It is a generalization of Benjamin-Bona-Mahony equation, which models long waves in a nonlinear dispersive system. We prove the well-posedness of the Cauchy problem, associated with this equation.

Keywords. Existence, uniqueness, stability, porous medium equation, Cauchy problem.

Mathematics Subject Classification: 35G25, 35K55.

1 - Introduction

In this paper, we investigate the existence of the classical solution of the following Cauchy problem:

$$\begin{cases} \partial_t u + \partial_x f(u) = \partial_x(g(u)\partial_x u) + \beta^2 \partial_t \partial_x^2 u, & t > 0, x \in \mathbb{R}, \\ u(0,x) = u_0(x), & x \in \mathbb{R}, \end{cases} \tag{1.1}$$

where $\beta \neq 0$ and $f : \mathbb{R} \to \mathbb{R}$ is smooth function, such that

$$f \in C^1(\mathbb{R}), \tag{1.2}$$

while, on the function g, we assume one within the following two

$$g \in C^1(\mathbb{R}), \quad |g(u)| \leq L, \quad \text{for every } u \in \mathbb{R}; \tag{1.3}$$

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$$g \in C^1(\mathbb{R}), \quad g(u) \geq 0. \tag{1.4}$$

On the initial datum, we assume

$$u_0 \in H^1(\mathbb{R}). \tag{1.5}$$

The nonnegativity assumption on the diffusion coefficient g in (1.4) is often present on papers on the porous medium equation. Here we consider the case of a purely bounded coefficients without any restriction on the sign (see (1.3)).

When $g = 0$ (1.1) becomes

$$\partial_t u + \partial_x f(u) = \beta^2 \partial_t \partial_x^2 u, \tag{1.6}$$

which is known as the Benjamin-Bona-Mahony equation [**7**].

The function $u(t, x)$, in (1.1), represents the saturation (volume fraction) of one of the phases. The flux $f(u)$, known as the fractional flow rate, depends on the ratio of relative permeabilities of the two phases. The function $g(u)$ represents the equilibrium capillary pressure. From a mathematical point of view, $g(u)$ is, usually, a positive and decreasing function of saturation. It approaches zero at $u = 0$, 1, where one phase is absent. Finally, β^2 is a relaxation time for the dynamic capillary pressure with a linear rate dependence.

There has been much interest recently in refining the Hassanizadeh-Gray dynamic capillary pressure model (see [**23**] and the references therein) and in exploring properties of wave-like solutions of (1.1) and related equations [**14**, **18**, **19**, **30**].

Much of the recent effort has focused on characterizing traveling wave solutions under various simplifications and constitutive assumptions. A striking novel feature of the analysis is the presence of traveling waves that are undercompressive in the sense of shock waves [**17**, **20**].

In [**28**,**29**], the authors analyze traveling wave solutions for (1.1), in the natural case in which relative permeabilities are quadratic functions of saturation. In particular, in [**28**] the authors prove that the structure of traveling waves suggests the form of a nonclassical Riemann solver (in the limit of negligible capillary pressure), in which shock waves are deemed admissible only if they are singular limits of traveling waves.

In [**1**,**21**,**31**,**33**,**34**], the authors develop a numerical scheme for (1.1), while, in [**30**], the existence of TW solution is studied. Instead, in [**32**], the existence of non-monotone travelling waves solutions is proven. The stability of travelling wave solutions and the asymptotic behavior for (1.1) are studied in [**2**, **13**, **24**] assuming

$$f(u) = u^2, \quad g(u) = 1. \tag{1.7}$$

Finally, in [**27**], the existence of the travelling waves solutions for (1.1) in the case

$$f(u) = u - u^3, \quad g(u) = \alpha. \tag{1.8}$$

Equation (1.1) is a generalization of the following one

$$\partial_t u = \partial_x(g(u)\partial_x u) + \beta^2 \partial_t \partial_x^2 u, \tag{1.9}$$

which is deduced in [**6**] to describe the seepage of homogeneous liquids in fissured rocks, and in [**4, 5**] to describe the fluid flow.

From a mathematical point of view, in [**15**], the initial and boundary value problem for (1.9) is studied, while, in [**26**] some existence result are proven.

If $g(u) = 0$, (1.1) is equivalent to (1.6), which models long waves in a nonlinear dispersive system and is also called the regularized long wave equation [**7**].

From a mathematical point of view, the Cauchy problem for (1.6) is studied in [**3, 16**], while, in [**8, 9, 10, 25**], the convergence of the solution of (1.6) to the unique entropy one of the following scalar conservation law

$$\partial_t u + \partial_x f(u) = 0, \quad f(u) = u^2,\, u^3,$$

is proven. We use the following definition of solution.

Definition 1.1. *We say that a function* $u : [0,\infty) \times \mathbb{R} \to \mathbb{R}$ *is a solution of* (1.1)*, if*

$$u \in H^1((0,T)\times\mathbb{R}) \cap L^\infty(0,T;H^2(\mathbb{R})), \quad \partial_t\partial_x u \in L^\infty(0,T;L^2(\mathbb{R})), \qquad T>0,$$
$$u(0,\cdot) = u_0 \text{ a.e. in } (0,\infty)\times\mathbb{R}$$

and for every test function $\varphi \in C^\infty(\mathbb{R}^2)$ *with compact support*

$$\int_0^\infty \int_{\mathbb{R}} \left(\partial_t u\varphi + \partial_x f(u)\varphi + g(u)\partial_x u \partial_x\varphi + \beta^2 \partial_t\partial_x u \partial_x \varphi\right) dtdx = 0.$$

The main result of this paper is the following theorem.

Theorem 1.1. *Fix* $T>0$*. Assume* (1.2), (1.5) *and one between* (1.3) *and* (1.4)*. There exists a solution* u *of* (1.1)*, such that*

$$\begin{aligned} &u \in H^1((0,T)\times\mathbb{R}) \cap L^\infty(0,T;H^1(\mathbb{R})) \cap W^{1,\infty}((0,T)\times\mathbb{R}),\\ &\partial_t\partial_x u \in L^\infty(0,T;L^2(\mathbb{R}). \end{aligned} \tag{1.10}$$

In particular, if $f \in C^2(\mathbb{R})$, u is unique. Moreover, if u_1 and u_2 are two solutions of (1.1), *we have that*

$$\|u_1(t,\cdot) - u_2(t,\cdot)\|^2_{H^1(\mathbb{R})} \leq \frac{\tau_2^2 e^{C(T)t}}{\tau_1^2} \|u_{1,0} - u_{2,0}\|^2_{H^1(\mathbb{R})}, \tag{1.11}$$

where

$$\tau_1^2 = \min\{1,\, \beta^2\}, \quad \tau_2^2 = \max\{1,\, \beta^2\}, \tag{1.12}$$

for some suitable $C(T) > 0$, and every $0 \leq t \leq T$.

Since (1.3) and (1.4) are satisfied by (1.6), Theorem 1.1 holds also for (1.6).

The paper is organized as follows. In Section 2, we prove several a priori estimates on a vanishing viscosity approximation of (1.1). Those play a key role in the proof of our main result, that is given in Section 3.

2 - Vanishing viscosity approximation

Our existence argument is based on passing to the limit in a vanishing viscosity approximation of (1.1).

Fix a small number $\varepsilon > 0$, and let $u_\varepsilon = u_\varepsilon(t,x)$ be the unique regular solution of the following problem (see [**22**]):

$$\begin{cases} \partial_t u_\varepsilon + \partial_x f(u_\varepsilon) = \partial_x(g(u_\varepsilon)\partial_x u_\varepsilon) + \beta^2 \partial_t \partial_x^2 u_\varepsilon - \varepsilon \partial_x^4 u_\varepsilon, & t > 0, x \in \mathbb{R}, \\ u_\varepsilon(0,x) = u_{0,\varepsilon}(x), & x \in \mathbb{R}, \end{cases} \tag{2.1}$$

where $u_{\varepsilon,0}$ is $C^\infty(\mathbb{R})$ approximations of u_0 such that

$$\|u_{\varepsilon,0}\|_{H^1(\mathbb{R})} \leq \|u_0\|_{H^1(\mathbb{R})}, \quad \sqrt{\varepsilon}\left\|\partial_x^2 u_{\varepsilon,0}\right\|_{L^2(\mathbb{R})} + \varepsilon \left\|\partial_x^3 u_{\varepsilon,0}\right\|_{L^2(\mathbb{R})} \leq C_0, \tag{2.2}$$

and C_0 is a positive constant, independent on ε.

Let us prove some a priori estimates on u_ε, denoting with C the constants which depend only on the initial data, and $C(T)$ the constants which depend also on T.

Lemma 2.1. *Fix $T > 0$ and assume* (1.3). *There exists a constant $C(T) > 0$, independent on ε, such that*

$$\begin{aligned} \|u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} &+ \beta^2 \|\partial_x u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} \\ &+ 2\varepsilon e^{\frac{2Lt}{\beta^2}} \int_0^t e^{-\frac{2Ls}{\beta^2}} \left\|\partial_x^2 u_\varepsilon(s,\cdot)\right\|^2_{L^2(\mathbb{R})} ds \leq C(T), \end{aligned} \tag{2.3}$$

for every $0 \leq t \leq T$. In particular, we have that

$$\|u_\varepsilon\|_{L^\infty((0,\infty)\times\mathbb{R})} \leq C(T). \tag{2.4}$$

Proof. Multiplying (2.1) by $2u_\varepsilon$, an integration on $\mathbb{R}$ gives

$$\frac{d}{dt}\left(\|u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})}+\beta^2\|\partial_x u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})}\right)=2\int_{\mathbb{R}}u_\varepsilon\partial_t u_\varepsilon dx-2\beta^2\int_{\mathbb{R}}u_\varepsilon\partial_t\partial_x^2 u_\varepsilon dx$$

$$=\underbrace{-2\int_{\mathbb{R}}u_\varepsilon f'(u_\varepsilon)\partial_x u_\varepsilon dx}_{=0}+2\int_{\mathbb{R}}u_\varepsilon\partial_x(g(u_\varepsilon)\partial_x u_\varepsilon)dx-2\varepsilon\int_{\mathbb{R}}u_\varepsilon\partial_x^4 u_\varepsilon dx$$

$$=-2\int_{\mathbb{R}}g(u_\varepsilon)(\partial_x u_\varepsilon)^2dx-2\varepsilon\left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}.$$

Hence, we have that

$$\frac{d}{dt}\left(\|u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})}+\beta^2\|\partial_x u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})}\right)+2\varepsilon\left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}=-2\int_{\mathbb{R}}g(u_\varepsilon)(\partial_x u_\varepsilon)^2dx. \tag{2.5}$$

Thanks to (1.3),

$$2\int_{\mathbb{R}}|g(u_\varepsilon)|(\partial_x u_\varepsilon)^2\le 2L\|\partial_x u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})}\le\frac{2L}{\beta^2}\beta^2\|\partial_x u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})}.$$

Consequently, by (2.5),

$$\frac{d}{dt}\left(\|u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})}+\beta^2\|\partial_x u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})}\right)+2\varepsilon\left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}$$
$$\le\frac{2L}{\beta^2}\beta^2\|\partial_x u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})}\le\frac{2L}{\beta^2}\left(\|u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})}+\beta^2\|\partial_x u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})}\right).$$

It follows from the Gronwall Lemma and (2.2) that

$$\|u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})}+\beta^2\|\partial_x u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})}+2\varepsilon e^{\frac{2Lt}{\beta^2}}\int_0^t e^{-\frac{2Ls}{\beta^2}}\left\|\partial_x^2 u_\varepsilon(s,\cdot)\right\|^2_{L^2(\mathbb{R})}ds\le Ce^{\frac{2Lt}{\beta^2}}\le C(T),$$

which gives (2.3).

Finally, we prove (2.4). Due to the Hölder inequality,

$$u_\varepsilon^2(t,x)=2\int_{-\infty}^x u_\varepsilon\partial_x u_\varepsilon dy\le 2\int_{\mathbb{R}}|u_\varepsilon||\partial_x u_\varepsilon|dx\le\|u_\varepsilon(t,\cdot)\|_{L^2(\mathbb{R})}\|\partial_x u_\varepsilon(t,\cdot)\|_{L^2(\mathbb{R})}. \tag{2.6}$$

Therefore, by (2.3),

$$\|u_\varepsilon\|^2_{L^\infty((0,\infty)\times\mathbb{R})}\le C(T),$$

which gives (2.4). □

Lemma 2.2. *Assume* (1.4). *For each* $t \geq 0$, *we have that*

$$\|u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} + \beta^2 \|\partial_x u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} + 2\int_0^t\int_{\mathbb{R}} g(u_\varepsilon)(\partial_x u_\varepsilon)^2 dsdx + 2\varepsilon \int_0^t \left\|\partial_x^2 u_\varepsilon(s,\cdot)\right\|^2_{L^2(\mathbb{R})} ds \leq C. \tag{2.7}$$

In particular, we get

$$\|u_\varepsilon(t,\cdot)\|_{L^\infty(\mathbb{R})} \leq C. \tag{2.8}$$

Proof. Arguing as in Lemma 2.1, thanks to (1.4), we have that

$$\frac{d}{dt}\left(\|u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} + \beta^2 \|\partial_x u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})}\right) + 2\int_{\mathbb{R}} g(u_\varepsilon)(\partial_x u_\varepsilon)^2 dx + 2\varepsilon \left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} = 0.$$

Integrating on $(0,t)$, by (2.2), we have (2.7).

Finally, we prove (2.8). Thanks to (2.6) and (2.7),

$$\|u_\varepsilon(t,\cdot)\|^2_{L^\infty(\mathbb{R})} \leq C,$$

which gives (2.8). □

Lemma 2.3. *Fix* $T > 0$ *and assume* (1.3), *or* (1.4). *There exist a constant* $C(T) > 0$, *independent on* ε, *such that*

$$\varepsilon \left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + \int_0^t \|\partial_t u_\varepsilon(s,\cdot)\|^2_{L^2(\mathbb{R})} ds + \beta^2 \int_0^t \|\partial_t\partial_x u_\varepsilon(s,\cdot)\|^2_{L^2(\mathbb{R})} ds \leq C(T), \tag{2.9}$$

for every $0 \leq t \leq T$.

Proof. Let $0 \leq t \leq T$. Multiplying (2.1) by $2\partial_t u_\varepsilon$, an integration on $\mathbb{R}$ gives

$$2\|\partial_t u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} = -2\int_{\mathbb{R}} f'(u_\varepsilon)\partial_x u_\varepsilon \partial_t u_\varepsilon dx + 2\int_{\mathbb{R}} \partial_x(g(u_\varepsilon)\partial_x u_\varepsilon)\partial_t u_\varepsilon + 2\beta^2 \int_{\mathbb{R}} \partial_t u_\varepsilon \partial_t \partial_x^2 u_\varepsilon dx - 2\varepsilon \int_{\mathbb{R}} \partial_t u_\varepsilon \partial_x^4 u_\varepsilon dx$$

$$
\begin{aligned}
&= -2\int_{\mathbb{R}} f'(u_\varepsilon)\partial_x u_\varepsilon \partial_t u_\varepsilon dx - 2\int_{\mathbb{R}} g(u_\varepsilon)\partial_x u_\varepsilon \partial_t \partial_x u_\varepsilon dx \\
&\quad - 2\beta^2 \left\|\partial_t\partial_x u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + 2\varepsilon \int_{\mathbb{R}} \partial_t\partial_x u_\varepsilon \partial_x^3 u_\varepsilon dx \\
&= -2\int_{\mathbb{R}} f'(u_\varepsilon)\partial_x u_\varepsilon \partial_t u_\varepsilon dx - 2\int_{\mathbb{R}} g(u_\varepsilon)\partial_x u_\varepsilon \partial_t \partial_x u_\varepsilon dx \\
&\quad - 2\beta^2 \left\|\partial_t\partial_x u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} - \varepsilon \frac{d}{dt}\left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}.
\end{aligned}
$$

Therefore, we have that

(2.10)

$$
\begin{aligned}
\varepsilon \frac{d}{dt}&\left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + 2\left\|\partial_t u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + 2\beta^2\left\|\partial_t\partial_x u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} \\
&= -2\int_{\mathbb{R}} f'(u_\varepsilon)\partial_x u_\varepsilon \partial_t u_\varepsilon dx - 2\int_{\mathbb{R}} g(u_\varepsilon)\partial_x u_\varepsilon \partial_t \partial_x u_\varepsilon dx.
\end{aligned}
$$

Due to Lemma 2.1, or 2.2 and the Young inequality,

$$
\begin{aligned}
2\int_{\mathbb{R}} &|f'(u_\varepsilon)||\partial_x u_\varepsilon||\partial_t u_\varepsilon| dx \le 2\left\|f'\right\|_{L^\infty(-C(T),C(T))}\int_{\mathbb{R}} |\partial_x u_\varepsilon||\partial_t u_\varepsilon| dx \\
&\le 2C(T)\int_{\mathbb{R}} |\partial_x u_\varepsilon||\partial_t u_\varepsilon| dx \le C(T)\left\|\partial_x u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + \left\|\partial_t u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} \\
&\le C(T) + \left\|\partial_t u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}, \\
2\int_{\mathbb{R}} &|g(u_\varepsilon)||\partial_x u_\varepsilon||\partial_t\partial_x u_\varepsilon| dx \le 2\left\|g\right\|_{L^\infty(-C(T),C(T))}\int_{\mathbb{R}} |\partial_x u_\varepsilon||\partial_t\partial_x u_\varepsilon| dx \\
&\le 2C(T)\int_{\mathbb{R}} |\partial_x u_\varepsilon||\partial_t\partial_x u_\varepsilon| dx = 2\int_{\mathbb{R}} \left|\frac{C(T)\partial_x u_\varepsilon}{\beta}\right| \left|\beta\partial_t\partial_x u_\varepsilon\right| dx \\
&\le C(T)\left\|\partial_x u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + \beta^2\left\|\partial_t\partial_x u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} \\
&\le C(T) + \beta^2\left\|\partial_t\partial_x u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}.
\end{aligned}
$$

It follows from (2.10) that

$$
\varepsilon \frac{d}{dt}\left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + \left\|\partial_t u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + \beta^2\left\|\partial_t\partial_x u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} \le C(T).
$$

Integrating on $(0,t)$, by (2.2), we get

$$
\varepsilon\left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + \int_0^t \left\|\partial_t u_\varepsilon(s,\cdot)\right\|^2_{L^2(\mathbb{R})} ds
$$

$$+\beta^2\int_0^t \|\partial_t\partial_x u_\varepsilon(s,\cdot)\|^2_{L^2(\mathbb{R})}\,ds \le C + C(T)t \le C(T),$$

which gives (2.9). □

Lemma 2.4. *Fix $T>0$ and assume* (1.3), *or* (1.4). *There exist a constant $C(T)>0$, independent on ε, such that*

$$\begin{aligned}\varepsilon^2\left\|\partial_x^3 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} &+ \varepsilon\int_0^t \left\|\partial_{tx}^2 u_\varepsilon(s,\cdot)\right\|^2_{L^2(\mathbb{R})}ds\\ &+\frac{\beta^2\varepsilon}{2}\int_0^t \left\|\partial_t\partial_x^2 u_\varepsilon(s,\cdot)\right\|^2_{L^2(\mathbb{R})}ds \le C(T).\end{aligned} \tag{2.11}$$

for every $0\le t\le T$.

Proof. Let $0\le t\le T$. We begin by observing that

$$\partial_x(g(u_\varepsilon)\partial_x u_\varepsilon) = g'(u_\varepsilon)(\partial_x u_\varepsilon)^2 + \partial_x^2 u_\varepsilon. \tag{2.12}$$

Multiplying (2.1) by $-2\varepsilon\partial_t\partial_x^2 u_\varepsilon$, thanks to (2.12), an integration on $\mathbb{R}$ gives

$$\begin{aligned}&2\beta^2\varepsilon\left\|\partial_t\partial_x^2 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\ =&2\varepsilon\int_{\mathbb{R}}\partial_t u_\varepsilon\partial_t\partial_x^2 u_\varepsilon dx + 2\varepsilon\int_{\mathbb{R}} f'(u_\varepsilon)\partial_x u_\varepsilon\partial_t\partial_x^2 u_\varepsilon dx\\ &-2\varepsilon\int_{\mathbb{R}} g'(u_\varepsilon)(\partial_x u_\varepsilon)^2\partial_t\partial_x^2 u_\varepsilon dx - 2\varepsilon\int_{\mathbb{R}} g(u_\varepsilon)\partial_x^2 u_\varepsilon\partial_t\partial_x^2 u_\varepsilon dx\\ &+2\varepsilon^2\int_{\mathbb{R}}\partial_x^4 u_\varepsilon\partial_t\partial_x^2 u_\varepsilon dx\\ =&-2\varepsilon\|\partial_t\partial_x u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} + 2\varepsilon\int_{\mathbb{R}} f'(u_\varepsilon)\partial_x u_\varepsilon\partial_t\partial_x^2 u_\varepsilon dx\\ &-2\varepsilon\int_{\mathbb{R}} g'(u_\varepsilon)(\partial_x u_\varepsilon)^2\partial_t\partial_x^2 u_\varepsilon dx - 2\varepsilon\int_{\mathbb{R}} g(u_\varepsilon)\partial_x^2 u_\varepsilon\partial_t\partial_x^2 u_\varepsilon dx\\ &-\varepsilon^2\frac{d}{dt}\left\|\partial_x^3 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}.\end{aligned}$$

Therefore, we have that

$$\begin{aligned}\varepsilon\frac{d}{dt}\left\|\partial_x^3 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} &+ 2\varepsilon\|\partial_t\partial_x u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} + 2\beta^2\varepsilon\left\|\partial_t\partial_x^2 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\ &=2\varepsilon\int_{\mathbb{R}} f'(u_\varepsilon)\partial_x u_\varepsilon\partial_t\partial_x^2 u_\varepsilon dx - 2\varepsilon\int_{\mathbb{R}} g'(u_\varepsilon)(\partial_x u_\varepsilon)^2\partial_t\partial_x^2 u_\varepsilon dx\end{aligned} \tag{2.13}$$

$$- 2\varepsilon \int_{\mathbb{R}} g(u_\varepsilon)\partial_x^2 u_\varepsilon \partial_t \partial_x^2 u_\varepsilon dx.$$

Since $0 < \varepsilon < 1$, thanks to (2.9), Lemmas 2.1 or (2.2) and the Young inequality,

$$\begin{aligned}
2\varepsilon \int_{\mathbb{R}} |f'(u_\varepsilon)||\partial_x u_\varepsilon||\partial_t\partial_x^2 u_\varepsilon| dx &\le 2\varepsilon \left\|f'\right\|_{L^\infty(-C(T),C(T))} \int_{\mathbb{R}} |\partial_x u_\varepsilon||\partial_t\partial_x^2 u_\varepsilon| dx\\
&\le \varepsilon C(T) \int_{\mathbb{R}} |\partial_x u_\varepsilon||\partial_t\partial_x^2 u_\varepsilon| dx = \varepsilon \int_{\mathbb{R}} \left|\frac{C(T)\partial_x u_\varepsilon}{\beta}\right| \left|\beta\partial_t\partial_x^2 u_\varepsilon\right| dx\\
&\le \varepsilon C(T) \left\|\partial_x u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + \frac{\beta^2\varepsilon}{2} \left\|\partial_t\partial_x^2 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
&\le C(T) + \frac{\beta^2\varepsilon}{2} \left\|\partial_t\partial_x^2 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})},\\
2\varepsilon \int_{\mathbb{R}} |g'(u_\varepsilon)|(\partial_x u_\varepsilon)^2|\partial_t\partial_x^2 u_\varepsilon| dx &\le 2\varepsilon \left\|g'\right\|_{L^\infty(-C(T),C(T))} \int_{\mathbb{R}} (\partial_x u_\varepsilon)^2|\partial_t\partial_x^2 u_\varepsilon| dx\\
&\le \varepsilon C(T) \int_{\mathbb{R}} (\partial_x u_\varepsilon)^2|\partial_t\partial_x^2 u_\varepsilon| dx = \varepsilon \int_{\mathbb{R}} \left|\frac{C(T)(\partial_x u_\varepsilon)^2}{\beta}\right| \left|\beta\partial_t\partial_x^2 u_\varepsilon\right| dx\\
&\le \varepsilon C(T) \left\|\partial_x u_\varepsilon(t,\cdot)\right\|^4_{L^4(\mathbb{R})} + \frac{\beta^2\varepsilon}{2} \left\|\partial_t\partial_x^2 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})},\\
2\varepsilon \int_{\mathbb{R}} |g(u_\varepsilon)||\partial_x^2 u_\varepsilon||\partial_t\partial_x^2 u_\varepsilon| dx &\le 2\varepsilon \left\|g\right\|_{L^\infty(-C(T),C(T))} \int_{\mathbb{R}} |\partial_x^2 u_\varepsilon||\partial_t\partial_x^2 u_\varepsilon| dx\\
&\le \varepsilon C(T) \int_{\mathbb{R}} |\partial_x^2 u_\varepsilon||\partial_t\partial_x^2 u_\varepsilon| dx = \varepsilon \int_{\mathbb{R}} \left|\frac{C(T)\partial_x^2 u_\varepsilon}{\beta}\right| \left|\beta\partial_t\partial_x^2 u_\varepsilon\right| dx\\
&\le \varepsilon C(T) \left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + \frac{\beta^2\varepsilon}{2} \left\|\partial_t\partial_x^2 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
&\le C(T) + \frac{\beta^2\varepsilon}{2} \left\|\partial_t\partial_x^2 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}.
\end{aligned}$$

It follows from (2.13) that

$$\begin{aligned}
&\varepsilon^2 \frac{d}{dt} \left\|\partial_x^3 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + 2\varepsilon \left\|\partial_t\partial_x u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
&\qquad + \frac{\beta^2\varepsilon}{2} \left\|\partial_t\partial_x^2 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} \le C(T) + \varepsilon C(T) \left\|\partial_x u_\varepsilon(t,\cdot)\right\|^4_{L^4(\mathbb{R})}
\end{aligned} \tag{2.14}$$

and using [**11**, Lemma 2.3]

$$\left\|\partial_x u_\varepsilon(t,\cdot)\right\|^4_{L^4(\mathbb{R})} \le 6 \left(\left\|u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + \left\|\partial_x u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\right) \left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}. \tag{2.15}$$

Thanks to (2.9), (2.15) and Lemmas 2.1, or 2.2,

$$\varepsilon C(T)\left\|\partial_x u_\varepsilon(t,\cdot)\right\|^4_{L^4(\mathbb{R})} \le \varepsilon C(T)\left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} \le C(T). \tag{2.16}$$

Consequently, by (2.11) and (2.16), we have that

$$\varepsilon\frac{d}{dt}\left\|\partial_x^3 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + 2\varepsilon\left\|\partial_t\partial_x u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + \frac{\beta^2\varepsilon}{2}\left\|\partial_t\partial_x^2 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} \le C(T).$$

Integrating on $(0,t)$, by (2.2), we get

$$\begin{aligned}\varepsilon^2\left\|\partial_x^3 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} &+ 2\varepsilon\int_0^t \left\|\partial_t\partial_x u_\varepsilon(s,\cdot)\right\|^2_{L^2(\mathbb{R})} ds\\ &+ \frac{\beta^2\varepsilon}{2}\int_0^t \left\|\partial_t\partial_x^2 u_\varepsilon(s,\cdot)\right\|^2_{L^2(\mathbb{R})} ds \le C + C(T)t \le C(T),\end{aligned}$$

which gives (2.11). □

Lemma 2.5. *Fix $T>0$ and assume* (1.3), *or* (1.4). *There exist a constant $C(T)>0$, independent on ε, such that*

$$\left\|\partial_t u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + \beta^2\left\|\partial_t\partial_x u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} \le C(T), \tag{2.17}$$

for every $0\le t\le T$. Moreover,

$$\left\|\partial_t u_\varepsilon\right\|_{L^\infty((0,T)\times\mathbb{R})} \le C(T). \tag{2.18}$$

Proof. Let $0\le t\le T$. Multiplying (2.1) by $2\partial_t u_\varepsilon$, an integration on $\mathbb{R}$ gives

$$\begin{aligned}2\left\|\partial_t u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} =& -2\int_{\mathbb{R}} f'(u_\varepsilon)\partial_x u_\varepsilon\partial_t u_\varepsilon dx + 2\int_{\mathbb{R}}\partial_x(g(u_\varepsilon)\partial_x u_\varepsilon)\partial_t u_\varepsilon dx\\ & 2\beta^2\int_{\mathbb{R}}\partial_t\partial_x^2 u_\varepsilon\partial_t u_\varepsilon dx - 2\varepsilon\int_{\mathbb{R}}\partial_x^4 u_\varepsilon\partial_t u_\varepsilon dx\\ =& -2\int_{\mathbb{R}} f'(u_\varepsilon)\partial_x u_\varepsilon\partial_t u_\varepsilon dx - 2\int_{\mathbb{R}} g(u_\varepsilon)\partial_x u_\varepsilon\partial_t\partial_x u_\varepsilon dx\\ & - 2\beta^2\left\|\partial_t\partial_x u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + 2\varepsilon\int_{\mathbb{R}}\partial_x^3 u_\varepsilon\partial_t\partial_x u_\varepsilon dx.\end{aligned}$$

Therefore, we have that

$$\begin{aligned}&2\left\|\partial_t u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + 2\beta^2\left\|\partial_t\partial_x u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\ &\quad= -2\int_{\mathbb{R}} f'(u_\varepsilon)\partial_x u_\varepsilon\partial_t u_\varepsilon dx - 2\int_{\mathbb{R}} g(u_\varepsilon)\partial_x u_\varepsilon\partial_t\partial_x u_\varepsilon dx - 2\varepsilon\int_{\mathbb{R}}\partial_x^3 u_\varepsilon\partial_t\partial_x u_\varepsilon dx.\end{aligned} \tag{2.19}$$

Due to (2.11), Lemmas 2.1, or 2.2 and the Young inequality,

$$\begin{aligned}
2\int_{\mathbb{R}} |f'(u_\varepsilon)||\partial_x u_\varepsilon||\partial_t u_\varepsilon|dx &\leq 2\left\|f'\right\|_{L^\infty(-C(T),C(T))}\int_{\mathbb{R}}|\partial_x u_\varepsilon||\partial_t u_\varepsilon|dx\\
&\leq 2C(T)\int_{\mathbb{R}}|\partial_x u_\varepsilon||\partial_t u_\varepsilon|dx = C(T)\left\|\partial_x u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + \left\|\partial_t u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
&\leq C(T) + \left\|\partial_t u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})},\\
2\int_{\mathbb{R}} |g(u_\varepsilon)||\partial_x u_\varepsilon||\partial_t\partial_x u_\varepsilon|dx &\leq 2\left\|g\right\|_{L^\infty(-C(T),C(T))}\int_{\mathbb{R}}|\partial_x u_\varepsilon||\partial_t\partial_x u_\varepsilon|dx\\
&\leq C(T)\int_{\mathbb{R}}|\partial_x u_\varepsilon||\partial_t\partial_x u_\varepsilon|dx = \int_{\mathbb{R}}\left|\frac{C(T)\partial_x u_\varepsilon}{\beta}\right|\left|\beta\partial_t\partial_x u_\varepsilon\right|dx\\
&\leq C(T)\left\|\partial_x u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + \frac{\beta^2}{2}\left\|\partial_t\partial_x u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
&\leq C(T) + \frac{\beta^2}{2}\left\|\partial_t\partial_x u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})},\\
2\varepsilon\int_{\mathbb{R}} |\partial_x^3 u_\varepsilon||\partial_t\partial_x u_\varepsilon|dx &= \int_{\mathbb{R}}\left|\frac{2\varepsilon\partial_x^3 u_\varepsilon}{\beta}\right|\left|\beta\partial_t\partial_x u_\varepsilon\right|dx\\
&\leq \frac{2\varepsilon^2}{\beta^2}\left\|\partial_x^3 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + \frac{\beta^2}{2}\left\|\partial_t\partial_x u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
&\leq C(T) + \frac{\beta^2}{2}\left\|\partial_t\partial_x u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}.
\end{aligned}$$

Consequently, by (2.19), we have that

$$\left\|\partial_t u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + \beta^2\left\|\partial_t\partial_x u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} \leq C(T),$$

which gives (2.17).

Finally, we prove (2.18). Thanks to (2.17) and the Young inequality,

$$\begin{aligned}
(\partial_t u_\varepsilon(t,x))^2 =& 2\int_{-\infty}^x \partial_t u_\varepsilon \partial_t\partial_x u_\varepsilon dy \leq 2\int_{\mathbb{R}}|\partial_t u_\varepsilon||\partial_t\partial_x u_\varepsilon|dx\\
\leq& 2\left\|\partial_t u_\varepsilon(t,\cdot)\right\|_{L^2(\mathbb{R})}\left\|\partial_t\partial_x u_\varepsilon(t,\cdot)\right\|_{L^2(\mathbb{R})} \leq C(T).
\end{aligned}$$

Hence,

$$\left\|\partial_t u_\varepsilon\right\|^2_{L^\infty((0,T)\times\mathbb{R})} \leq C(T),$$

which gives (2.18). □

3 - Proof of Theorem 1.1

Using the Sobolev Immersion Theorem, we prove the following result.

Lemma 3.1. *Fix $T > 0$. There exist a subsequence $\{u_{\varepsilon_k}\}_{k\in\mathbb{N}}$ of $\{u_\varepsilon\}_{\varepsilon>0}$ and an a limit function u which satisfies* (1.10) *such that*

$$u_{\varepsilon_k} \to u \text{ a.e. and in } L^p_{loc}((0,T)\times\mathbb{R}),\ 1 \le p < \infty. \tag{3.1}$$

Moreover, u is solution of (1.1).

Proof. Thanks to Lemmas 2.1, or 2.2, 2.3 and 2.5,

$$\{u_\varepsilon\}_{\varepsilon>0} \text{ is uniformly bounded in } H^1((0,T)\times\mathbb{R}), \tag{3.2}$$

which gives (3.1).

Observe that, thanks to Lemmas 2.1, or 2.7,

$$u \in L^\infty(0,T;H^1(\mathbb{R})),$$

while, by Lemma 2.5,

$$u \in W^{1,\infty}((0,T)\times\mathbb{R}).$$

Moreover, by Lemma 2.3, we have that

$$\partial_t\partial_x u \in L^2((0,T)\times\mathbb{R}).$$

Therefore, (1.10) holds and u is solution of (1.1). □

Following [**12**, Theorem 1.1], we prove the following result.

Lemma 3.2. *If $f \in C^2(\mathbb{R})$, then* (1.11) *holds.*

Proof. Let $T > 0$. Since $C^2(\mathbb{R}) \subset C^1(\mathbb{R})$, Lemma 3.1 gives the existence of a solution u of (1.1) such that (1.10) holds.

We prove (1.11). Let u_1, u_2 be two solutions of (1.1), which satisfy (1.10), that is

$$\begin{cases} \partial_t u_1 + f'(u_1)\partial_x u_1 - \beta^2\partial_t\partial_x^2 u_1 = \partial_x(g(u_1)\partial_x u_1), & t > 0,\ x \in \mathbb{R},\\ u_1(0,x) = u_{1,0}(x), & x \in \mathbb{R}, \end{cases}$$

$$\begin{cases} \partial_t u_2 + f'(u_2)\partial_x u_2 - \beta^2\partial_t\partial_x^2 u_2 = \partial_x(g(u_2)\partial_x u_2), & t > 0,\ x \in \mathbb{R},\\ u_2(0,x) = u_{2,0}(x), & x \in \mathbb{R}. \end{cases}$$

Then, the function

$$\omega = u_1 - u_2 \tag{3.3}$$

is the solution of the following Cauchy problem:

$$\begin{cases} \partial_t \omega - \beta^2 \partial_t \partial_x^2 \omega + f'(u_1)\partial_x u_1 - f'(u_2)\partial_x u_2 \\ \qquad = \partial_x (g(u_1)\partial_x u_1 - g(u_2)\partial_x u_2), & t > 0,\ x \in \mathbb{R}, \\ \omega_0(x) = u_{1,0}(x) - u_{2,0}(x), & x \in \mathbb{R}. \end{cases} \tag{3.4}$$

Since

$$2\int_{\mathbb{R}} \omega \partial_t \omega dx - 2\beta^2 \int_{\mathbb{R}} \omega \partial_t \partial_x^2 \omega dx = \frac{d}{dt}\left(\|\omega(t,\cdot)\|^2_{L^2(\mathbb{R})} + \beta^2 \|\partial_x \omega(t,\cdot)\|^2_{L^2(\mathbb{R})} \right),$$

$$2\int_{\mathbb{R}} \omega \partial_x (g(u_1)\partial_x u_1 - g(u_2)\partial_x u_2) dx = -2\int_{\mathbb{R}} (g(u_1)\partial_x u_1 - g(u_2)\partial_x u_2)\partial_x \omega dx,$$

multiplying (3.4) by 2ω, an integration on $\mathbb{R}$ gives

$$\begin{aligned} &\frac{d}{dt}\left(\|\omega(t,\cdot)\|^2_{L^2(\mathbb{R})} + \beta^2 \|\partial_x \omega(t,\cdot)\|^2_{L^2(\mathbb{R})} \right) \\ &= -2\int_{\mathbb{R}} \left(f'(u_1)\partial_x u_1 - f'(u_2)\partial_x u_2 \right) \omega dx - 2\int_{\mathbb{R}} (g(u_1)\partial_x u_1 - g(u_2)\partial_x u_2)\partial_x \omega dx. \end{aligned} \tag{3.5}$$

Observe that

$$\begin{aligned} f'(u_1)\partial_x u_1 - f'(u_2)\partial_x u_2 &= f'(u_1)\partial_x \omega + (f'(u_1) - f'(u_2))\partial_x u_2 \\ g(u_1)\partial_x u_1 - g(u_2)\partial_x u_2 &= g(u_1)\partial_x \omega + (g(u_1) - g(u_2))\partial_x u_2. \end{aligned}$$

Therefore, by (3.5),

$$\begin{aligned} &\frac{d}{dt}\left(\|\omega(t,\cdot)\|^2_{L^2(\mathbb{R})} + \beta^2 \|\partial_x \omega(t,\cdot)\|^2_{L^2(\mathbb{R})} \right) \\ &\quad = -2\int_{\mathbb{R}} f'(u_1)\omega \partial_x \omega dx - 2\int_{\mathbb{R}} \omega (f'(u_1) - f'(u_2))\partial_x u_2 \\ &\qquad - 2\int_{\mathbb{R}} g(u_1)(\partial_x \omega)^2 dx - 2\int (g(u_1) - g(u_2))\partial_x u_2 dx \\ &\quad = 2\int_{\mathbb{R}} f''(u_1)\partial_x u_1 \omega^2 dx - 2\int_{\mathbb{R}} \omega (f'(u_1) - f'(u_2))\partial_x u_2 dx \\ &\qquad - 2\int_{\mathbb{R}} g(u_1)(\partial_x \omega)^2 dx - 2\int (g(u_1) - g(u_2))\partial_x u_2 \partial_x \omega dx. \end{aligned} \tag{3.6}$$

Since, u_1, $u_2 \in L^\infty(0,T;H^1(\mathbb{R}))$, thanks to (3.3), we have

$$\begin{aligned} |f'(u_1)-f'(u_2)| \le & C(T)|u_1-u_2| = C(T)|\omega|,\\ |g(u_1)-g(u_2)| \le & C(T)|u_1-u_2| = C(T)|\omega|, \end{aligned} \tag{3.7}$$

where

$$C(T) = \sup_{(0,T)\times\mathbb{R}} \Big\{ f''(u_1)+f''(u_2)\Big\} + \sup_{(0,T)\times\mathbb{R}} \Big\{ g(u_1)+g(u_2)\Big\}. \tag{3.8}$$

Moreover, there exists a constant $C(T)>0$ such that

$$\|\partial_x u_1(t,\cdot)\|_{L^2(\mathbb{R})}, \|\partial_x u_2(t,\cdot)\|_{L^2(\mathbb{R})} \le C(T), \tag{3.9}$$

for every $0\le t\le T$. Due to (3.7), (3.8), (3.9) and the Young inequality that

$$\begin{aligned} & 2\int_{\mathbb{R}} |f''(u_1)||\partial_x u_1|\omega^2 dx \le 2\left\|f''\right\|_{L^\infty(-C(T),C(T))}\int_{\mathbb{R}}|\partial_x u_1|\omega^2 dx\\ &\quad \le 2C(T)\int_{\mathbb{R}}|\partial_x u_1|\omega^2 dx \le C(T)\int_{\mathbb{R}}\omega^2(\partial_x u_1)^2 dx + C(T)\left\|\omega(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\ &\quad \le C(T)\left\|\omega(t,\cdot)\right\|^2_{L^\infty(\mathbb{R})}\left\|\partial_x u_1(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + C(T)\left\|\omega(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\ &\quad \le C(T)\left\|\omega(t,\cdot)\right\|^2_{L^\infty(\mathbb{R})} + C(T)\left\|\omega(t,\cdot)\right\|^2_{L^2(\mathbb{R})},\\ & 2\int_{\mathbb{R}}|\omega||f'(u_1)-f'(u_2)||\partial_x u_2 dx| \le 2C(T)\int_{\mathbb{R}}\omega^2\partial_x u_1 dx\\ &\quad \le C(T)\int_{\mathbb{R}}\omega^2(\partial_x u_1)^2 dx + C(T)\left\|\omega(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\ &\quad \le C(T)\left\|\omega(t,\cdot)\right\|^2_{L^\infty(\mathbb{R})}\left\|\partial_x u_1(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + C(T)\left\|\omega(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\ &\quad \le C(T)\left\|\omega(t,\cdot)\right\|^2_{L^\infty(\mathbb{R})} + C(T)\left\|\omega(t,\cdot)\right\|^2_{L^2(\mathbb{R})},\\ & 2\int_{\mathbb{R}}|g(u_1)|(\partial_x\omega)^2 dx \le 2\left\|g\right\|_{L^\infty(-C(T),C(T))}\left\|\partial_x\omega(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\ &\quad \le C(T)\left\|\partial_x\omega(t,\cdot)\right\|^2_{L^2(\mathbb{R})},\\ & 2\int|g(u_1)-g(u_2)||\partial_x u_2||\partial_x\omega| dx \le 2C(T)\int_{\mathbb{R}}|\omega||\partial_x u_2||\omega| dx\\ &\quad \le C(T)\int_{\mathbb{R}}\omega^2(\partial_x u_2)^2 dx + C(T)\left\|\partial_x\omega(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\ &\quad \le C(T)\left\|\omega(t,\cdot)\right\|^2_{L^\infty(\mathbb{R})}\left\|\partial_x u_2(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + C(T)\left\|\partial_x\omega(t,\cdot)\right\|^2_{L^2(\mathbb{R})} \end{aligned}$$

$$\leq C(T)\|\omega(t,\cdot)\|^2_{L^\infty(\mathbb{R})} + C(T)\|\partial_x\omega(t,\cdot)\|^2_{L^2(\mathbb{R})}.$$

If follows from (3.6) that

(3.10)
$$\frac{d}{dt}\left(\|\omega(t,\cdot)\|^2_{L^2(\mathbb{R})} + \beta^2\|\partial_x\omega(t,\cdot)\|^2_{L^2(\mathbb{R})}\right)$$
$$\leq C(T)\|\omega(t,\cdot)\|^2_{L^\infty(\mathbb{R})} + C(T)\|\omega(t,\cdot)\|^2_{L^2(\mathbb{R})} + C(T)\|\partial_x\omega(t,\cdot)\|^2_{L^2(\mathbb{R})}.$$

Observe that

$$\omega^2(t,x) = 2\int_{-\infty}^{x}\omega\partial_x\omega dy \leq 2\int_{\mathbb{R}}|\omega||\partial_x\omega|dx.$$

Therefore, by the Young inequality,

$$\|\omega(t,\cdot)\|^2_{L^\infty(\mathbb{R})} \leq \|\omega(t,\cdot)\|^2_{L^2(\mathbb{R})} + \|\partial_x\omega(t,\cdot)\|^2_{L^2(\mathbb{R})}.$$

Consequently, by (3.10), we have that

$$\begin{aligned}\frac{d}{dt}&\left(\|\omega(t,\cdot)\|^2_{L^2(\mathbb{R})} + \beta^2\|\partial_x\omega(t,\cdot)\|^2_{L^2(\mathbb{R})}\right)\\ &\leq C(T)\|\omega(t,\cdot)\|^2_{L^2(\mathbb{R})} + C(T)\|\partial_x\omega(t,\cdot)\|^2_{L^2(\mathbb{R})}\\ &\leq C(T)\left(\|\omega(t,\cdot)\|^2_{L^2(\mathbb{R})} + \beta^2\|\partial_x\omega(t,\cdot)\|^2_{L^2(\mathbb{R})}\right).\end{aligned}$$

The Gronwall Lemma and (3.4) give

$$\|\omega(t,\cdot)\|^2_{L^2(\mathbb{R})} + \beta^2\|\partial_x\omega(t,\cdot)\|^2_{L^2(\mathbb{R})} \leq e^{C(T)t}\left(\|\omega_0\|^2_{L^2(\mathbb{R})} + \beta^2\|\partial_x\omega_0\|^2_{L^2(\mathbb{R})}\right).$$

By (1.12), we have that

$$\tau_1^2\|\omega\|^2_{H^1(\mathbb{R})} \leq \tau_2^2 e^{C(T)t}\|\omega_0\|^2_{H^1(\mathbb{R})}. \tag{3.11}$$

Therefore, (1.11) follows from (3.4) and (3.11). □

P r o o f o f T h e o r e m 1.1. Theorem 1.1 follows from Lemmas 3.1 and 3.2. □

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