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# Mertens type formulas based on density

**Abstract.** We introduce a new density among sets of prime numbers which is called the Mertens density. Building on the works of Olofsson, Pollack and Wirsing, it is shown, in complete contrast with the cases of relative natural density and Dirichlet density, that the existence of Mertens density of a set of prime numbers turns out to be equivalent to Mertens type formulae and the limiting behaviors of the associated zeta function at one together with the size of the corresponding semigroup, all formed according to the underlying set of primes. Various constants, such as the Meissel-Mertens constants, appearing in the equivalent statements are shown to be related with each other through elementary formulas. This allows us to study specific partitioning properties between sets of primes taking into account their density and the asymptotics of the generated semigroup. It is further demonstrated that the Mertens density neither implies nor is implied by the relative natural density. Assuming explicit forms of the error terms, sharper versions of some of our results are also obtained.

**Keywords.** Mertens type formula, Mertens density, relative natural density, Dirichlet density, associated zeta function, size of semigroup.

**Mathematics Subject Classification:** 11N05, 11M41, 11Y60.

## 1 - Introduction

After the pioneering works of Chebyshev [**7**], [**8**] and Riemann [**37**] on the distribution of prime numbers during the period 1850–1860, up until the proof of the prime number theorem (abbreviated as PNT throughout) by Hadamard [**16**] and de la Vallée Poussin [**41**] in 1896, the only major progress in the field was due to Mertens [**30**], [**31**] who proved the following three important asymptotic formulas using mainly elementary methods.

$$\sum_{p\leq x}\frac{\log p}{p}=\log x+O(1), \tag{1.1}$$

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$$\sum_{p\le x}\frac{1}{p}=\log\log x+B_1+O\left(\frac{1}{\log x}\right), \tag{1.2}$$

$$\prod_{p\le x}\left(1-\frac{1}{p}\right)^{-1}=e^{\gamma}\log x+O(1), \tag{1.3}$$

where $p$ always denotes a prime number and $x\ge 2$. In (1.3), $\gamma$ is the Euler-Mascheroni constant defined by

$$\gamma:=\lim_{x\to\infty}\left(\sum_{n\le x}\frac{1}{n}-\log x\right)=0.5772\ldots,$$

$B_1$ in (1.2) is the Meissel-Mertens constant defined by

$$B_1:=\lim_{x\to\infty}\left(\sum_{p\le x}\frac{1}{p}-\log\log x\right)=0.2614\ldots,$$

and furthermore, $B_1$ and $\gamma$ are related by the formula

$$B_1=\gamma+\sum_{p}\left(\log\left(1-\frac{1}{p}\right)+\frac{1}{p}\right).$$

For the analysis of another series connecting $B_1$ and $\gamma$, see [**27**]. In the rest of the paper, we will refer to (1.1), (1.2) and (1.3) as (M1), (M2) and (M3), respectively. Recently, (M2) was extended to the cases of products of two and three primes by Popa [**34**], [**35**]. The most subtle feature of Mertens formulas is the unexpected appearance of $e^{\gamma}$ in (M3). Despite the initial interest and curiosity that these formulas spiked, Mertens was not able to deduce the PNT from them. Today we know that stronger versions of (M1) and (M2) such as the existence of the limit

$$\lim_{x\to\infty}\left(\sum_{p\le x}\frac{\log p}{p}-\log x\right),$$

and

$$\sum_{p\le x}\frac{1}{p}=\log\log x+B_1+o\left(\frac{1}{\log x}\right)$$

as $x\to\infty$, both imply the PNT. For versions of (M3) with better error terms taking advantage of the Vinogradov-Korobov type zero-free region for the Riemann zeta function, see [**43**] and [**44**]. We should also mention that a very useful variant of (M2) that applies to number fields was first obtained by Rosen [**38**]. Although falling short of implying the PNT in their original forms, striking connections between (M3) and other aspects of the theory were recently discovered by Olofsson [**32**] and Pollack [**33**]. To be precise, let $P$ be any set of prime numbers and $N_P(x)$ be the number of elements not exceeding $x$ that are in the multiplicative semigroup generated by $P$. The associated zeta function can then be represented as

$$\zeta_P(s) = \sum_{n\in\langle P\rangle} \frac{1}{n^s} = \prod_{p\in P}\left(1-\frac{1}{p^s}\right)^{-1}$$

for $\Re(s) > 1$, where

$$\langle P\rangle := \{n = p_1^{k_1}...p_r^{k_r} \ : \ r\geq 1,\ p_i\in P,\ k_i\geq 0\} \tag{1.4}$$

denotes the semigroup of positive integers generated by $P$. Olofsson [**32**] then showed that if $N_P(x)\sim cx$ for some constant $c>0$ as $x\to\infty$, then

$$\prod_{\substack{p\leq x\\ p\in P}}\left(1-\frac{1}{p}\right)^{-1} \sim ce^{\gamma}\log x \tag{1.5}$$

holds as $x\to\infty$. Pollack [**33**] further enhanced this result by demonstrating that (1.5) is indeed equivalent to

$$\zeta_P(s)\sim\frac{c}{s-1} \tag{1.6}$$

when $s\downarrow 1$ (by $s\downarrow 1$, we mean that $s$ approaches to 1 from the right through real values). The interesting feature of these results is that extensions of (M3) to general sets of prime numbers are strong enough to dictate the simple pole behavior of the corresponding zeta function at $s=1$ as pointed out by (1.6). In addition, both authors obtained the above asymptotics in the setting of Beurling's generalized prime number systems [**5**]. For different aspects of Beurling systems see [**3**], [**4**]. Other types of far reaching analogs of (M2) and (M3) (also of (M1), but we will not need it in the sequel) hold for all arithmetic progressions of primes. In particular, if $1\leq a\leq q$ and $(a,q)=1$, then we have

$$\sum_{\substack{p\leq x\\ p\equiv a \pmod q}} \frac{1}{p} = \frac{1}{\phi(q)}\log\log x + B_1(q,a) + O_q\left(\frac{1}{\log x}\right) \tag{1.7}$$

and

$$\prod_{\substack{p\leq x\\ p\equiv a \pmod q}}\left(1-\frac{1}{p}\right)^{-1} = C_1(q,a)(\log x)^{1/\phi(q)}\left(1+O_q\left(\frac{1}{\log x}\right)\right), \tag{1.8}$$

where $\phi(q)$ is Euler's function, and the constants $B_1(q,a)$ and $C_1(q,a)$ are given by explicit but complicated expressions involving $\gamma$, values of Dirichlet characters modulo $q$ and values of Dirichlet $L$-functions at $s=1$. Here (1.7) can be referenced to define $B_1(q,a)$ as the generalized Meissel-Mertens constant. Both (1.7) and (1.8) reveal quantitative forms of Dirichlet's theorem on the infinitude of primes in a progression. For better forms of (1.7) and (1.8) with improved remainder terms, the reader may consult [**42**]. Further research on the finer structure of Mertens type formulas in progressions and fruitful numerical studies on the behavior of $B_1(q,a), C_1(q,a)$ were carried out by Languasco and Zaccagnini [**22**], [**23**], [**24**], [**25**], [**26**].

If $P$ is any set of prime numbers, then the relative natural density and the Dirichlet density of $P$ are defined by the limits, when they exist,

$$\lim_{x\to\infty}\frac{P(x)}{\pi(x)}$$

and

$$\lim_{s\downarrow 1}\frac{\sum_{p\in P}\frac{1}{p^s}}{\log\frac{1}{s-1}},$$

respectively, where $P(x)$ and $\pi(x)$ are the counting functions of $P$ and the set of all primes, respectively. Moreover, if $P$ has relative natural density $\tau\in[0,1]$, then using $P(x)=(\tau+o(1))\pi(x)$ as $x\to\infty$, one obtains by partial summation that

$$\sum_{\substack{p\le x\\ p\in P}}\frac{1}{p}=(\tau+o(1))\log\log x. \tag{1.9}$$

However, conversely, (1.9) is far from implying that $P$ has relative natural density $\tau$. A classical theorem of Landau (see pages 641–669 of [**21**]) concerning the relative natural density states that if $P$ is taken to be the union of $r$ distinct arithmetic progressions modulo $q$, so that the relative natural density of $P$ is $r/\phi(q)$, then

$$N_P(x)\sim\frac{cx}{(\log x)^{1-\frac{r}{\phi(q)}}}$$

holds for some constant $c>0$. This was greatly extended by Wirsing [**45**] who showed for $0<\tau<1$ that

$$N_P(x)\sim\frac{cx}{(\log x)^{1-\tau}} \tag{1.10}$$

holds whenever $P$ has relative natural density $\tau$. Once again, the converse of this statement is not true. Wirsing [**46**] further showed for a general class of multiplicative arithmetic functions $f(n)$ satisfying mild conditions that

$$\sum_{p\le x} f(p) \sim \frac{\tau x}{\log x}$$

implies

$$\sum_{n\le x} f(n) \sim \frac{e^{-\gamma\tau}}{\Gamma(\tau)} \frac{x}{\log x} \prod_{p\le x} \left(1 + \frac{f(p)}{p} + \frac{f(p^2)}{p^2} + \dots\right),$$

where $\Gamma$ denotes the gamma function. At this point, it should be said that the relative natural density is not the right type of density suitable for the characterization of asymptotic behavior such as the ones encountered in (1.9) and (1.10).

A major goal of this paper is to supply a general framework which extends and unifies the above mentioned theorems of Olofsson, Pollack and Wirsing by the consideration of specific Mertens type formulae based on a different formulation of the density of the underlying set of primes. Our critical innovation in this effort is to introduce a new weight on sets of prime numbers which forms an equivalent condition at once for all Mertens type formulae, for the behavior of the corresponding zeta function at $s = 1$, and for the asymptotic size of $N_P(x)$. To make our point more transparent, let us define the Mertens density of a set $P$ of prime numbers to be the number $\tau \in [0, 1]$ whenever

$$\sum_{p\in P} \frac{1}{p^s} = \tau \log \frac{1}{s-1} + K_P + o(1) \tag{1.11}$$

holds for some constant $K_P$ as $s \downarrow 1$. In this case, we write $\tau = \delta(P)$. It is obvious from (1.11) that the Mertens density is a stringent form of the Dirichlet density, implying that the Dirichlet density of $P$ is also $\tau$. In light of the above remarks, we can now give our main contribution which completely settles the quest about the characterization of Mertens type formulas in terms of the Mertens density of the relevant set of primes.

Theorem 1. *Let $P$ be a set of prime numbers, and consider the following statements.*
*$(i)$ The asymptotic formula*

$$\sum_{\substack{p\le x\\ p\in P}} \frac{1}{p} = \tau \log\log x + c_P + o(1) \tag{1.12}$$

*holds for some constants $c_P$ and $0 \le \tau \le 1$ as $x \to \infty$.*

*(ii) The asymptotic formula*

$$\prod_{\substack{p\le x\\ p\in P}}\left(1-\frac{1}{p}\right)^{-1}=(1+o(1))a_P(\log x)^{\tau} \tag{1.13}$$

*holds for some constants $a_P>0$ and $0\le\tau\le 1$ as $x\to\infty$, where $(\log x)^{\tau}$ is taken to be 1 when $\tau=0$.*

*(iii) The asymptotic formula*

$$\zeta_P(s)\sim\frac{A}{(s-1)^{\tau}} \tag{1.14}$$

*holds for some constants $A>0$ and $0\le\tau\le 1$ as $s\downarrow 1$, where $(s-1)^{\tau}$ is taken to be 1 when $\tau=0$.*

*(iv) The Mertens density of $P$ is $\tau\in[0,1]$ so that $\delta(P)=\tau$.*

*(v) The asymptotic formula*

$$N_P(x)\sim\frac{cx}{(\log x)^{1-\tau}} \tag{1.15}$$

*holds for some constants $c>0$ and $0<\tau\le 1$ as $x\to\infty$, where $(\log x)^{1-\tau}$ is taken to be 1 when $\tau=1$.*

*Then the statements $(i),(ii),(iii)$ and $(iv)$ are equivalent, and all of the statements are equivalent when $0<\tau\le 1$. Furthermore, if we define a constant $c_P^{\times}$ by*

$$c_P^{\times}=c_P-\sum_{p\in P}\left(\log\left(1-\frac{1}{p}\right)+\frac{1}{p}\right), \tag{1.16}$$

*then the constants appearing in the above asymptotic formulas are subject to the following relations.*

$$a_P=e^{c_P^{\times}},\ A=a_Pe^{-\gamma\tau} \tag{1.17}$$

*holds when $0\le\tau\le 1$, and*

$$c=\frac{A}{\Gamma(\tau)} \tag{1.18}$$

*holds when $0<\tau\le 1$, where*

$$\Gamma(t)=\int_0^{\infty}e^{-x}x^{t-1}\,dx$$

*is the gamma function for $t>0$.*

Some comments on Theorem 1 are now in order. First, when $\tau = 0$, we may take $P$ to be the empty set so that all of $(i), (ii), (iii)$ and $(iv)$ hold with $c_P = 0$, $a_P = 1 = A$ and $\delta(P) = 0$. But $N_P(x) = 1$ for all $x \geq 1$ so that $(v)$ does not hold. Moreover, when $\tau = 1$, the equivalence of $(i)$ and $(v)$ gives that $N_P(x) \sim cx$ is equivalent to the general Mertens type formula

$$\sum_{\substack{p \leq x \\ p \in P}} \frac{1}{p} = \log\log x + c_P + o(1).$$

But since, using (M2),

$$\sum_{\substack{p \leq x \\ p \in Q}} \frac{1}{p} = \sum_{p \leq x} \frac{1}{p} - \sum_{\substack{p \leq x \\ p \in P}} \frac{1}{p} = B_1 - c_P + o(1)$$

holds, where $Q$ is the complement of $P$ in the set of all prime numbers, we infer that $N_P(x) \sim cx$ is equivalent to the convergence of

$$\sum_{p \in Q} \frac{1}{p}. \tag{1.19}$$

Consequently, we also have

$$c = \prod_{p \in Q} \left(1 - \frac{1}{p}\right). \tag{1.20}$$

Note that (1.20) is a manifestation of a fundamental principle in sieve theory, noticed and elaborated by Erdös [**13**], as $N_P(x)$ is the number of remaining integers not exceeding $x$ when we sift out by the primes in $Q$. Concerning $N_P(x)$, it is important to realize that Wirsing's formula in (1.10) is not true in general for a set of relative natural density 1. To give a counterexample, one needs to show the existence of a set $P$ of primes with relative natural density 1 such that (1.19) diverges. We refer to part $(i)$ of Theorem 2 and its proof below for the actual construction of such a $P$. To make another remark, let $P$ and $Q$ be complementary sets of primes such that $\delta(P) = \tau \in (0, 1)$ (so that $\delta(Q) = 1 - \tau$). Then by the equivalence of $(iv)$ and $(v)$, we may let $c_1 > 0$ and $c_2 > 0$ be the corresponding constants in (1.15) for $N_P(x)$ and $N_Q(x)$, respectively. Since $\zeta_P(s)\zeta_Q(s) = \zeta(s)$ for $\Re(s) > 1$, we readily deduce from (1.18) that

$$\lim_{\tau \downarrow 0} c_1 c_2 = \lim_{\tau \downarrow 0} \frac{1}{\Gamma(\tau)\Gamma(1 - \tau)} = 0.$$

To illustrate the numerology in Theorem 1, consider the following sets of prime numbers: $P_1 = \{3\} \cup \{p : p \equiv 1 \pmod 3\}$, $P_1' = \{p : p \equiv 1 \pmod 3\}$ and $P_2 = \{p : p \equiv 2 \pmod 3\}$. Using (M3), (1.8) and the notation of (1.13) with $\tau = 1/2$, we deduce that

$$a_{P_1'} a_{P_2} = \frac{2e^{\gamma}}{3}.$$

The instrumental $L$-function in this situation is given by

$$L(s,\chi) = \sum_{n=0}^{\infty} \left( \frac{1}{(3n+1)^s} - \frac{1}{(3n+2)^s} \right)$$

for $\Re(s) > 0$, where $\chi$ is the odd character modulo 3. It is well-known that

$$L(1,\chi) = \frac{\pi}{3\sqrt{3}}.$$

Therefore, using the Euler product

$$L(s,\chi) = \prod_{p} \left(1 - \frac{\chi(p)}{p^s}\right)^{-1}$$

when $s \downarrow 1$, we see that

$$\prod_{\substack{p \le x \\ p \equiv 1 \pmod 3}} \left(1 - \frac{1}{p}\right)^{-1} \cdot \prod_{\substack{p \le x \\ p \equiv 2 \pmod 3}} \left(1 + \frac{1}{p}\right)^{-1} = \frac{\pi}{3\sqrt{3}} + o(1),$$

and consequently that

$$\frac{a_{P_1'}}{a_{P_2}} = \frac{\pi}{3\sqrt{3}} \prod_{p \equiv 2 \pmod 3} \left(1 - \frac{1}{p^2}\right).$$

Note that we derived two equations in the unknown constants $a_{P_1'}$ and $a_{P_2}$. In this way, we obtain

$$a_{P_2} = \frac{\sqrt{2}\sqrt[4]{3}}{\sqrt{\pi}} e^{\gamma/2} \left( \prod_{p \equiv 2 \pmod 3} \left(1 - \frac{1}{p^2}\right) \right)^{-\frac{1}{2}}.$$

Since $a_{P_1} a_{P_2} = e^{\gamma}$,

$$a_{P_1} = \frac{\sqrt{\pi}}{\sqrt{2}\sqrt[4]{3}} e^{\gamma/2} \left( \prod_{p \equiv 2 \pmod 3} \left(1 - \frac{1}{p^2}\right) \right)^{\frac{1}{2}}$$

follows. Let us mention that this way of obtaining the constants $a_{P_1}$ and $a_{P_2}$ using the relevant $L$-function was first noticed by Uchiyama [**40**] who applied it to the more classical case of progressions 1 and 3 modulo 4. Recently, Languasco and Zaccagnini [**24**], [**25**] refined and elaborated on his approach by giving many other interesting evaluations with respect to other moduli. For evaluations of $L$-functions with periodic coefficients, see [**2**]. It follows from (1.17) and (1.18) of Theorem 1 that

$$\zeta_{P_1}(s) \sim \frac{A_1}{(s-1)^{1/2}}, \ \zeta_{P_2}(s) \sim \frac{A_2}{(s-1)^{1/2}}$$

when $s \downarrow 1$ with

$$A_1 = \frac{\sqrt{\pi}}{\sqrt{2}\sqrt[4]{3}} \left( \prod_{p \equiv 2 \pmod 3} \left(1 - \frac{1}{p^2}\right)\right)^{\frac{1}{2}},$$

$$A_2 = \frac{\sqrt{2}\sqrt[4]{3}}{\sqrt{\pi}} \left( \prod_{p \equiv 2 \pmod 3} \left(1 - \frac{1}{p^2}\right)\right)^{-\frac{1}{2}},$$

and

$$N_{P_1}(x) \sim \frac{c_1 x}{\sqrt{\log x}}, \ N_{P_2}(x) \sim \frac{c_2 x}{\sqrt{\log x}}$$

as $x \to \infty$ with

$$c_1 = \frac{A_1}{\Gamma(1/2)} = \frac{1}{\sqrt{2}\sqrt[4]{3}} \left( \prod_{p \equiv 2 \pmod 3} \left(1 - \frac{1}{p^2}\right)\right)^{\frac{1}{2}},$$

$$c_2 = \frac{\sqrt{2}\sqrt[4]{3}}{\pi} \left( \prod_{p \equiv 2 \pmod 3} \left(1 - \frac{1}{p^2}\right)\right)^{-\frac{1}{2}}.$$

Finally, we have

$$c_{P_1}^{\times} = \frac{1}{2}\left(\gamma + \log(\pi/2) - \frac{\log 3}{2} + \sum_{p \equiv 2 \pmod 3} \log\left(1 - \frac{1}{p^2}\right)\right),$$

$$c_{P_2}^{\times} = \frac{1}{2}\left(\gamma + \log(2/\pi) + \frac{\log 3}{2} - \sum_{p \equiv 2 \pmod 3} \log\left(1 - \frac{1}{p^2}\right)\right).$$

Note that $c_1c_2 = 1/\pi$, $A_1A_2 = 1$, $a_{P_1}a_{P_2} = e^\gamma$ and $c_{P_1}^\times + c_{P_2}^\times = \gamma$, as they should be, since $P_1$ and $P_2$ form a partition of the set of all primes. Lastly, we have

$$\zeta_{P_1'}(s) \sim \frac{A_1'}{(s-1)^{1/2}}, \ N_{P_1'}(x) \sim \frac{c_1' x}{\sqrt{\log x}}$$

when $s \downarrow 1$ and $x \to \infty$, respectively, with $A_1' = \frac{2}{3}A_1$ and $c_1' = \frac{2}{3}c_1$. The evident inequality $c_1 < c_2$ can be seen as a heuristic evidence supporting the claim that there are more primes congruent to 2 modulo 3 than congruent to 1 modulo 3 in a certain sense. This is an analog of Chebyshev's bias [**9**] (see also the paper of Fujii [**14**] for generalizations of Chebyshev's bias) which claims a similar favor for primes congruent to 3 modulo 4 over primes congruent to 1 modulo 4. As a final remark, let us say that the above approach can be adapted to the progressions 1 and 5 modulo 6 with the help of the $L$-function

$$L(s,\chi_1) = \sum_{n=0}^{\infty}\left(\frac{1}{(6n+1)^s} - \frac{1}{(6n+5)^s}\right)$$

for $\Re(s) > 0$, where $\chi_1$ is the odd character modulo 6, together with the evaluation

$$L(1,\chi_1) = \frac{\pi}{2\sqrt{3}}.$$

As suggested by the above comments about the rich interplay between the various constants appearing in Theorem 1, we are well prepared on our way to look at partitioning properties between sets of prime numbers with a viewpoint towards Mertens type formulas based on density.

Theorem 2. (*i*) *There exists two complementary sets of primes $P$ and $Q$ such that $P$ has relative natural density* 1 *and*

$$\sum_{p\in Q}\frac{1}{p}$$

*diverges. Therefore,* (1.10) *is not true in general whenever the relative natural density of the underlying set of primes is* 1.

(*ii*) *For $1 \le k \le 9$, let $P^k$ be the set of all primes whose first digit in the decimal expansion is $k$ (when viewed from left to right). Let $Q^k$ be any subset of $P^k$ such that*

$$\sum_{p\in Q^k}\frac{1}{p} \tag{1.21}$$

*converges. Then* $P^k - Q^k$ *can never be partitioned into finitely many subsets* $P_1, \ldots, P_m$ *such that*

$$N_{P_i}(x) \sim \frac{c_i x}{(\log x)^{a_i}} \tag{1.22}$$

*holds for all* $i$ *as* $x \to \infty$*, where* $c_i$*'s are positive constants and each* $a_i$ *is an algebraic number belonging to* $[0,1)$*.*

*(iii) Let* $Q$ *be any set of primes containing* $P^k$ *with the property that*

$$N_Q(x) \sim \frac{cx}{(\log x)^a}$$

*holds with a positive constant* $c$ *as* $x \to \infty$*, where* $a \in [0,1)$ *is an algebraic number. Then* $Q - P^k$ *can not be partitioned into finitely many subsets* $Q_1, \ldots, Q_m$ *such that*

$$N_{Q_i}(x) \sim \frac{c_i x}{(\log x)^{a_i}}$$

*holds for all* $i$ *as* $x \to \infty$*, where* $c_i$*'s are positive constants and each* $a_i \in [0,1)$ *is an algebraic number.*

*(iv) Let* $P_1, \ldots, P_m$ *be pairwise disjoint sets of primes having Dirichlet densities* $\tau_1, \ldots, \tau_m$*, respectively. Assume that each* $\tau_i$ *is a transcendental number. Then the maximal number of unordered pairs* $(P_i, P_j)$ *with* $i \neq j$ *such that*

$$N_{P_i \cup P_j}(x) \sim \frac{c_{ij} x}{(\log x)^{a_{ij}}} \tag{1.23}$$

*holds with positive constants* $c_{ij}$ *as* $x \to \infty$*, where each* $a_{ij} \in [0,1)$ *is an algebraic number, can not exceed* $[m^2/4]$*.*

*(v) Assume that* $P_1, \ldots, P_m$ *form a partition of the set of all prime numbers in such a way that*

$$N_{P_i}(x) \sim \frac{c_i x}{(\log x)^{a_i}} \tag{1.24}$$

*holds for all* $i$ *as* $x \to \infty$*, where* $c_i$ *is a positive constant and* $a_i \in (0,1)$*. Then*

$$\prod c_i = \left( \int_{\mathcal{R}} \prod x_i^{-a_i} \, dX \right)^{-1}, \tag{1.25}$$

*where* $dX = \prod dx_i$ *and* $\mathcal{R} \subseteq \mathbb{R}^m$ *is the region defined by the conditions*

$$x_i \geq 0, \ \sum x_i \leq 1.$$

*In particular, when $m = 2$, we have*

$$c_1 c_2 = \frac{\sin \pi a_1}{\pi}. \tag{1.26}$$

*Consequently, if $a_1$ is rational, then $c_1 c_2$ is transcendental.*

We leave it as an interesting bonus problem to decide whether the product

$$\prod_{i=1}^{m} c_i$$

is always transcendental or not when $m \geq 3$ in case all the $a_i \in (0,1)$ are rational numbers adding up to 1. As we shall verify later (see (4.21) below), this problem is equivalent to deciding whether the product

$$\prod_{i=1}^{m} \Gamma(\tau_i)$$

is transcendental or not whenever $\tau_i \in (0,1)$ are all rational numbers whose sum happens to be 1. Our next objective is to clarify the status of Mertens density. In particular, we show that the relative natural density and the Mertens density are incomparable. However, they are both stronger than the Dirichlet density. The precise formulation of this goes as follows.

Theorem 3. *There exists a set of prime numbers with relative natural density but having no Mertens density. Conversely, there exists a set of prime numbers with Mertens density but having no relative natural density.*

In case the $o(1)$ terms in (1.12) and (1.13) are given in the form of explicit error terms, we can then push for more quantitative versions of some of the implications in Theorem 1. In this vein, we are led to the following, where as usual, $(\log x)^\tau$ and $(s-1)^\tau$ are taken to be 1 when $\tau = 0$, and $(\log x)^{1-\tau}$ is taken to be 1 when $\tau = 1$.

Theorem 4. *Let $P$ be a set of prime numbers. If either*

$$\sum_{\substack{p \leq x \\ p \in P}} \frac{1}{p} = \tau \log\log x + c_P + O\left(\frac{1}{(\log\log x)^\alpha}\right) \tag{1.27}$$

*holds for some constants $c_P$, $0 \leq \tau \leq 1$ and $\alpha > 0$ as $x \to \infty$ or*

$$\prod_{\substack{p \leq x \\ p \in P}} \left(1 - \frac{1}{p}\right)^{-1} = \left(1 + O\left(\frac{1}{(\log\log x)^\alpha}\right)\right) a_P (\log x)^\tau \tag{1.28}$$

*holds for some constants* $a_P > 0$, $0 \le \tau \le 1$ *and* $\alpha > 0$ *as* $x \to \infty$, *then we have*

$$\zeta_P(s) = \left(1 + O\left(\left(\log \frac{1}{s-1}\right)^{-\alpha}\right)\right) \frac{A}{(s-1)^\tau} \tag{1.29}$$

*for some constant* $A > 0$ *as* $s \downarrow 1$. *Conversely, if*

$$\zeta_P(s) \sim \frac{A}{(s-1)^\tau}$$

*holds for some constants* $A > 0$ *and* $0 \le \tau \le 1$ *as* $s \downarrow 1$, *and the estimate*

$$\left| \int_x^\infty \left( \frac{P(t)}{t^2} - \frac{\tau}{t \log t} \right) dt \right| = O\left( \frac{1}{(\log \log x)^\alpha} \right) \tag{1.30}$$

*holds as* $x \to \infty$, *where* $P(t)$ *is the number of primes in* $P$ *not exceeding* $t$, *then* (1.27) *and* (1.28) *both hold as* $x \to \infty$.

*Finally, assuming that either*

$$\sum_{\substack{p \le x \\ p \in P}} \frac{1}{p} = \tau \log \log x + c_P + O\left( \frac{1}{\log x (\log \log x)^2} \right) \tag{1.31}$$

*holds for some constants* $c_P$ *and* $0 < \tau \le 1$ *as* $x \to \infty$ *or*

$$\prod_{\substack{p \le x \\ p \in P}} \left(1 - \frac{1}{p}\right)^{-1} = \left(1 + O\left( \frac{1}{\log x (\log \log x)^2} \right)\right) a_P (\log x)^\tau \tag{1.32}$$

*holds for some constants* $a_P > 0$ *and* $0 < \tau \le 1$ *as* $x \to \infty$, *we have*

$$N_P(x) = \frac{cx}{(\log x)^{1-\tau}} + O\left( \frac{x}{(\log x)^{1-\tau} \log \log x} \right) \tag{1.33}$$

*for some constant* $c > 0$ *as* $x \to \infty$.

## 2 - Preliminaries

In this section we collect all of the necessary ingredients that will be used in the proofs of our claims. Our first result serves for the equivalence of the additive and multiplicative versions of Mertens formula under a fairly general span of error terms.

Lemma 1. *Let $m(x)$ be a function such that $m(x)=O(x)$ and $m(x)\to\infty$ as $x\to\infty$. If $P$ is a set of prime numbers, then the asymptotic formulas*

$$\sum_{\substack{p\le x\\ p\in P}}\frac{1}{p}=\tau\log\log x+c_P+O\left(\frac{1}{m(x)}\right), \tag{2.1}$$

$$\prod_{\substack{p\le x\\ p\in P}}\left(1-\frac{1}{p}\right)^{-1}=\left(1+O\left(\frac{1}{m(x)}\right)\right)a_P(\log x)^{\tau} \tag{2.2}$$

*are equivalent for any constants $0\le\tau\le 1$, $a_P>0$ and $c_P$, where $a_P=e^{c_P^{\times}}$ and $c_P^{\times}$ is defined as in (1.16).*

Proof. Taking logarithm, as $a_P>0$, we see that (2.2) is equivalent to

$$\sum_{\substack{p\le x\\ p\in P}}\frac{1}{p}+\sum_{\substack{p\le x\\ p\in P}}\sum_{k\ge 2}\frac{1}{kp^k}=\tau\log\log x+\log a_P+O\left(\frac{1}{m(x)}\right). \tag{2.3}$$

Since

$$\sum_{p\in P}\sum_{k\ge 2}\frac{1}{kp^k}=-\sum_{p\in P}\left(\log\left(1-\frac{1}{p}\right)+\frac{1}{p}\right) \tag{2.4}$$

is convergent and

$$\sum_{\substack{p>x\\ p\in P}}\sum_{k\ge 2}\frac{1}{kp^k}=O\left(\frac{1}{x}\right)=O\left(\frac{1}{m(x)}\right), \tag{2.5}$$

we may deduce from (2.3)-(2.5) that (2.2) is equivalent to

$$\sum_{\substack{p\le x\\ p\in P}}\frac{1}{p}=\tau\log\log x+\log a_P+\sum_{p\in P}\left(\log\left(1-\frac{1}{p}\right)+\frac{1}{p}\right)+O\left(\frac{1}{m(x)}\right). \tag{2.6}$$

From (2.6), equivalence of (2.1) and (2.2) is proven with

$$c_P=\log a_P+\sum_{p\in P}\left(\log\left(1-\frac{1}{p}\right)+\frac{1}{p}\right). \tag{2.7}$$

Finally, (2.7) is equivalent to $a_P=e^{c_P^{\times}}$, where $c_P^{\times}$ is defined as in (1.16). This completes the proof of Lemma 1. □

We need the following classical result from Tauberian theory (see page 30 of [**20**]).

Lemma 2. *Assume that $a(v)$ is a locally Riemann integrable function and the improper integral*

$$F(r) := \int_0^\infty a(v)e^{-rv}\, dv$$

*exists for all $r > 0$. If $F(r) \to L$ as $r \downarrow 0$, and*

$$a(v) \geq -\frac{C}{v}$$

*holds for all sufficiently large values of $v$, where $C$ is a positive constant, then*

$$\int_0^\infty a(v)\, dv = L.$$

Our last preliminary is an elementary but useful fact that we will require about the prime counting function.

Lemma 3. *Let $\{x_n\}$ and $\{y_n\}$ be sequences of positive real numbers tending to infinity such that $x_n = o(y_n)$. Then $\pi(x_n) = o(\pi(y_n))$.*

Proof. Given $\epsilon > 0$, we know that $x_n \leq \epsilon y_n$ for all $n \geq N(\epsilon)$. But then by the monotonicity of $\pi(x)$ and Chebyshev estimates, one obtains for some constants $U > 1$ and $0 < V < 1$ that

$$\pi(x_n) \leq \pi(\epsilon y_n) \leq \frac{U\epsilon y_n}{\log(\epsilon y_n)} \leq \frac{2U\epsilon y_n}{\log(y_n)} \leq \frac{2U\epsilon}{V}\pi(y_n) \tag{2.8}$$

for all large enough $n$, since $\log(\epsilon y_n) \sim \log y_n$ as as $n \to \infty$. Result follows from (2.8). □

## 3 - Proof of Theorem 1

$(i) \Leftrightarrow (ii)$ : The equivalence of $(i)$ and $(ii)$ for $0 \leq \tau \leq 1$ follows immediately from Lemma 1 since we may take $m(x)$ to satisfy $m(x) \to \infty$ when $x \to \infty$ in Lemma 1 so that $O(1/m(x)) = o(1)$ in (2.1) and (2.2).

$(i) \Rightarrow (iii)$ : Let us assume that (1.12) holds for some constants $c_P$ and $0 \leq \tau \leq 1$. Consider the decomposition,

$$\sum_{p\in P}\sum_{k\geq 2}\frac{1}{kp^k} = \sum_{\substack{p\in P\\ p\leq x}}\sum_{\substack{k\geq 2\\ p^k\leq x}}\frac{1}{kp^k} + \sum_{\substack{p\in P\\ p\leq x}}\sum_{\substack{k\geq 2\\ p^k> x}}\frac{1}{kp^k} + \sum_{\substack{p\in P\\ p> x}}\sum_{k\geq 2}\frac{1}{kp^k}. \tag{3.1}$$

Note that

$$\sum_{\substack{p\in P\\ p\le x}}\sum_{\substack{k\ge 2\\ p^k>x}}\frac{1}{kp^k}\le\sum_p\frac{\log p}{\log x}\sum_{k\ge 2}\frac{1}{p^k}=O\left(\frac{1}{\log x}\sum_p\frac{\log p}{p^2}\right)=O\left(\frac{1}{\log x}\right),\tag{3.2}$$

and clearly

$$\sum_{\substack{p\in P\\ p>x}}\sum_{k\ge 2}\frac{1}{kp^k}=O\left(\sum_{p>x}\frac{1}{p^2}\right)=O\left(\frac{1}{x}\right).\tag{3.3}$$

Combining (3.1)-(3.3) and taking into account (2.4), we have

$$\sum_{p\in P}\sum_{\substack{k\ge 2\\ p^k\le x}}\frac{1}{kp^k}=-\sum_{p\in P}\left(\log\left(1-\frac{1}{p}\right)+\frac{1}{p}\right)+O\left(\frac{1}{\log x}\right).\tag{3.4}$$

Based on (1.4), let us define the von Mangoldt function associated to $P$ as follows.

$$\Lambda_P(n)=\begin{cases}\Lambda(n) & \text{if } n\in\langle P\rangle\\ 0 & \text{if } n\notin\langle P\rangle\,,\end{cases}$$

where $\Lambda(n)$ is the classical von Mangoldt function. Adding (1.12) to (3.4) and using (1.16), we obtain

$$\sum_{1<n\le x}\frac{\Lambda_P(n)}{n\log n}=\tau\log\log x+c_P^{\times}+o(1).\tag{3.5}$$

We now assume that $0<\tau\le 1$. Since

$$\sum_{n\le(\log x)^\tau}\frac{1}{n}=\tau\log\log x+\gamma+o(1),$$

(3.5) can be rewritten as

$$S_P(x):=\sum_{1<n\le x}\frac{\Lambda_P(n)}{n\log n}=\sum_{n\le(\log x)^\tau}\frac{1}{n}+c_P^{\times}-\gamma+o(1).\tag{3.6}$$

Note that the Mellin transform of both sides of (3.6) should coincide. It will suffice to compare the Mellin transforms only at real arguments. First, one has

$$\delta\int_1^\infty\frac{S_P(x)}{x^{\delta+1}}\,dx=\sum_{n=2}^{\infty}\frac{\Lambda_P(n)}{n^{\delta+1}\log n}=\log\zeta_P(\delta+1)\tag{3.7}$$

for all $\delta > 0$. We first deal with the case when $\tau = 1$. Thus, coming to the sum on the right hand side of (3.6), one computes its transform as

(3.8)
$$\delta \int_1^\infty x^{-1-\delta} \sum_{n \le \log x} \frac{1}{n}\, dx = \delta \sum_{n=1}^\infty \frac{1}{n} \int_{e^n}^\infty x^{-1-\delta}\, dx = \sum_{n=1}^\infty \frac{e^{-\delta n}}{n} = -\log(1 - e^{-\delta})$$

for $\delta > 0$. Of course we have

$$\delta \int_1^\infty \frac{c_P^\times - \gamma}{x^{\delta+1}}\, dx = c_P^\times - \gamma. \tag{3.9}$$

Let $g(x) = o(1)$ be the function representing the $o(1)$ term on the right hand side of (3.6). The fact that this property of $g(x)$ is preserved under the Mellin transform deserves some justification. To this end, note that, given $\epsilon > 0$, $|g(x)| < \epsilon$ holds when $x \ge M$ for some constant $M$ depending on $\epsilon$. We may write

$$\delta \int_1^\infty \frac{g(x)}{x^{\delta+1}}\, dx = \delta \int_1^M \frac{g(x)}{x^{\delta+1}}\, dx + \delta \int_M^\infty \frac{g(x)}{x^{\delta+1}}\, dx. \tag{3.10}$$

Next we have

$$\left| \delta \int_M^\infty \frac{g(x)}{x^{\delta+1}}\, dx \right| < \epsilon. \tag{3.11}$$

Since $g(x) = O(1)$,

$$\delta \int_1^M \frac{g(x)}{x^{\delta+1}}\, dx = O\left( \delta \int_1^M \frac{1}{x}\, dx \right) = O(\delta \log M). \tag{3.12}$$

Assembling (3.10)-(3.12), we easily justify that

$$\delta \int_1^\infty \frac{g(x)}{x^{\delta+1}}\, dx = o(1) \tag{3.13}$$

as $\delta \downarrow 0$. Observing that

$$-\log(1 - e^{-\delta}) = \log\left(\frac{1}{\delta}\right) + O(\delta)$$

when $\delta \downarrow 0$, we may gather (3.7), (3.8), (3.9) and (3.13) to arrive at the formula

$$\log \zeta_P(\delta + 1) = \log\left(\frac{1}{\delta}\right) + c_P^\times - \gamma + o(1) \tag{3.14}$$

when $\delta \downarrow 0$. By exponentiating (3.14), we see that (1.14) and (1.17) hold with $A = e^{c_P^{\times} - \gamma} = a_P e^{-\gamma}$ when $\tau = 1$. The case $0 < \tau < 1$ is more involved. In this case, we are lacking the logarithmic function in (3.8) representing the Mellin transform. Instead, we consider

$$\delta \int_1^\infty x^{-1-\delta} \sum_{n \le (\log x)^\tau} \frac{1}{n}\, dx = \delta \sum_{n=1}^\infty \frac{1}{n} \int_{e^{n^{1/\tau}}}^\infty x^{-1-\delta}\, dx = \sum_{n=1}^\infty \frac{e^{-\delta n^{\frac{1}{\tau}}}}{n}. \tag{3.15}$$

Motivated by (3.15), we define the function

$$h(\delta) := \sum_{n=1}^\infty \frac{e^{-\delta n^{\frac{1}{\tau}}}}{n} \tag{3.16}$$

for $\delta > 0$. Because of uniform convergence of (3.16), we obtain

$$h'(\delta) = -\sum_{n=1}^\infty \frac{n^{\frac{1}{\tau}}}{n} e^{-\delta n^{\frac{1}{\tau}}}. \tag{3.17}$$

Our task is to recover $h(\delta)$ from its derivative. As an approximation to the series on the right hand side of (3.17), a good candidate is the improper integral

$$I_\delta := \int_0^\infty x^{\frac{1}{\tau}-1} e^{-\delta x^{\frac{1}{\tau}}}\, dx. \tag{3.18}$$

On the one hand, by direct evaluation of the integral in (3.18), we know that $I_\delta = \tau/\delta$. The function $v(x) = x^{\frac{1}{\tau}-1} e^{-\delta x^{\frac{1}{\tau}}}$ for $x \ge 0$ has a global maximum at the point

$$x = \frac{(1-\tau)^\tau}{\delta^\tau}.$$

Moreover, $v(x)$ is increasing before this point and decreasing after. It makes sense to define a cut off parameter for the approximation of $I_\delta$ as the integer

$$m = \left[\frac{(1-\tau)^\tau}{\delta^\tau}\right]. \tag{3.19}$$

As a result of (3.19) and the monotonicity properties of $v(x)$, we find that

$$\sum_{n=1}^{m-1} n^{\frac{1}{\tau}-1} e^{-\delta n^{\frac{1}{\tau}}} \le \int_0^m x^{\frac{1}{\tau}-1} e^{-\delta x^{\frac{1}{\tau}}}\, dx \le \sum_{n=1}^{m} n^{\frac{1}{\tau}-1} e^{-\delta n^{\frac{1}{\tau}}} \tag{3.20}$$

and

$$\sum_{n=m+2}^{\infty} n^{\frac{1}{\tau}-1} e^{-\delta n^{\frac{1}{\tau}}} \le \int_{m+1}^\infty x^{\frac{1}{\tau}-1} e^{-\delta x^{\frac{1}{\tau}}}\, dx \le \sum_{n=m+1}^{\infty} n^{\frac{1}{\tau}-1} e^{-\delta n^{\frac{1}{\tau}}}. \tag{3.21}$$

Adding (3.20), (3.21) and rearranging, one obtains that

$$\begin{aligned}(3.22)\qquad -(v(m)+v(m+1)) &= -m^{\frac{1}{\tau}-1}e^{-\delta m^{\frac{1}{\tau}}} - (m+1)^{\frac{1}{\tau}-1}e^{-\delta(m+1)^{\frac{1}{\tau}}}\\ &\le I_\delta - \sum_{n=1}^{\infty} n^{\frac{1}{\tau}-1}e^{-\delta n^{\frac{1}{\tau}}} \le \int_m^{m+1} v(x)\,dx.\end{aligned}$$

We also know that

$$(3.23)\qquad v(m)+v(m+1) \le 2\int_m^{m+1} v(x)\,dx.$$

It follows from (3.22) and (3.23) that

$$(3.24)\qquad h'(\delta) = -\sum_{n=1}^{\infty} n^{\frac{1}{\tau}-1}e^{-\delta n^{\frac{1}{\tau}}} = -\frac{\tau}{\delta} + O\left(\int_m^{m+1} v(x)\,dx\right).$$

Next we have, by the mean value theorem, that

$$(3.25)\qquad \int_m^{m+1} v(x)\,dx = -\frac{\tau}{\delta}\left(e^{-\delta(m+1)^{\frac{1}{\tau}}} - e^{-\delta m^{\frac{1}{\tau}}}\right) = c_0^{\frac{1}{\tau}-1}e^{-\delta c_0^{\frac{1}{\tau}}}$$

for some $m < c_0 < m+1$. Using (3.19), we have

$$(3.26)\qquad c_0 = \left(\frac{1-\tau}{\delta}\right)^{\tau} + O(1).$$

Therefore, from (3.26), we infer as $\delta \downarrow 0$ that

$$(3.27)\qquad c_0^{\frac{1}{\tau}-1} \sim \left(\frac{1-\tau}{\delta}\right)^{1-\tau}.$$

Since $e^{-\delta c_0^{\frac{1}{\tau}}} = O(1)$, gathering (3.24), (3.25) and (3.27), we finally reach the formula

$$(3.28)\qquad h'(\delta) = -\frac{\tau}{\delta} + O\left(\frac{1}{\delta^{1-\tau}}\right)$$

when $\delta \downarrow 0$. By integration of (3.28), one gets

$$(3.29)\qquad h(\delta) - C = \tau\log\left(\frac{1}{\delta}\right) + O(\delta^{\tau})$$

for some constant $C$. Thus (3.29) can be formulated as

$$(3.30)\qquad h(\delta) = \tau\log\left(\frac{1}{\delta}\right) + C + o(1).$$

Using now (3.30), (3.14) becomes

$$\log \zeta_P(\delta+1) = \tau \log\left(\frac{1}{\delta}\right) + c_P^{\times} - \gamma + C + o(1). \tag{3.31}$$

Thus (1.14) follows from (3.31) when $0 < \tau < 1$. This completes the proof that ($iii$) follows from ($i$) when $0 < \tau < 1$. The above argument further gives that, when $\tau = 0$, (1.14) holds with $A = e^{c_P^{\times}} = a_P$. This finishes the proof that ($i$) implies ($iii$) for all $0 \le \tau \le 1$. However, apart from the cases $\tau = 0$ and $\tau = 1$, the value of $A$, which is

$$A = e^{c_P^{\times} - \gamma + C},$$

stays unspecified due to the indeterminacy of $C$ in the above argument. Our next goal is to determine the exact value of $C$ by a refined analysis. Indeed we will show that $C = \gamma - \gamma\tau$ for all $0 < \tau < 1$. To this end, let $P$ be a set of prime numbers with counting function $P(x)$ satisfying

$$P(x) = \frac{\tau x}{\log x} + O\left(\frac{x}{\log x(\log\log x)^2}\right) \tag{3.32}$$

for some $0 < \tau < 1$. Applying partial summation and using (3.32), it is easy to verify that

$$\sum_{\substack{p \le x \\ p \in P}} \frac{1}{p} = \tau \log\log x + c_P + o(1) \tag{3.33}$$

holds for some constant $c_P$. Similarly as above, (3.33) leads to (3.31). Next we study the asymptotic behavior of the left hand side of (3.31) in an alternative way. For this purpose, let us recall a result of Wirsing [**45**] who showed, assuming (3.32), that

$$N_P(x) = \frac{cx}{(\log x)^{1-\tau}} + O\left(\frac{x}{(\log x)^{1-\tau}\log\log x}\right) \tag{3.34}$$

holds, where $c > 0$ in (3.34) satisfies

$$c = \frac{a_P e^{-\gamma\tau}}{\Gamma(\tau)}. \tag{3.35}$$

Note that for $\delta > 0$, we have

$$\zeta_P(\delta+1) = (\delta+1)\int_1^{\infty} \frac{N_P(x)}{x^{2+\delta}}\,dx. \tag{3.36}$$

Noting that $N_P(x) = 1$ when $1 \le x < 2$, and using (3.34) in (3.36), we obtain that

(3.37)
$$\zeta_P(\delta+1) = 1 - \frac{1}{2^{1+\delta}} + (\delta+1)\int_2^\infty \frac{c}{x^{1+\delta}(\log x)^{1-\tau}}\,dx + (\delta+1)\int_2^\infty \frac{E(x)}{x^{2+\delta}}\,dx,$$

where

$$E(x) := N_P(x) - \frac{cx}{(\log x)^{1-\tau}}$$

in (3.37) is subject to the estimate

$$E(x) = O\left(\frac{x}{(\log x)^{1-\tau}\log\log x}\right)$$

when $x \ge 3$. By a change of variable, one gets

$$\begin{aligned}(3.38)\qquad \int_2^\infty \frac{c}{x^{1+\delta}(\log x)^{1-\tau}}\,dx &= \frac{c}{\delta^\tau}\int_{\delta\log 2}^\infty e^{-t}t^{\tau-1}\,dt \\ &= \frac{c}{\delta^\tau}\left(\Gamma(\tau) - \int_0^{\delta\log 2} e^{-t}t^{\tau-1}\,dt\right).\end{aligned}$$

When $\delta \downarrow 0$, we may write

$$(3.39)\quad \int_0^{\delta\log 2} e^{-t}t^{\tau-1}\,dt = \int_0^{\delta\log 2}(1+O(t))t^{\tau-1}\,dt = (\log 2)^\tau\frac{\delta^\tau}{\tau} + O(\delta^{\tau+1}).$$

Consequently from (3.38) and (3.39), we have

$$(3.40)\quad (\delta+1)\int_2^\infty \frac{c}{x^{1+\delta}(\log x)^{1-\tau}}\,dx = \frac{c(\delta+1)\Gamma(\tau)}{\delta^\tau} - \frac{c(\delta+1)(\log 2)^\tau}{\tau} + O(\delta).$$

Again by a change of variable, we obtain from the estimate on $E(x)$ that

$$\begin{aligned}(3.41)\qquad (\delta+1)\int_2^\infty \frac{E(x)}{x^{2+\delta}}\,dx &= O\left(\int_3^\infty \frac{1}{x^{1+\delta}(\log x)^{1-\tau}\log\log x}\,dx\right) \\ &= O\left(\frac{1}{\delta^\tau}\int_{\delta\log 3}^\infty \frac{e^{-t}t^{\tau-1}}{\log(t/\delta)}\,dt\right).\end{aligned}$$

For any $\epsilon > \delta\log 3$, consider the decomposition

$$(3.42)\qquad \int_{\delta\log 3}^\infty \frac{e^{-t}t^{\tau-1}}{\log(t/\delta)}\,dt = \int_{\delta\log 3}^\epsilon \frac{e^{-t}t^{\tau-1}}{\log(t/\delta)}\,dt + \int_\epsilon^\infty \frac{e^{-t}t^{\tau-1}}{\log(t/\delta)}\,dt.$$

It is plain that

$$\int_{\epsilon}^{\infty} \frac{e^{-t}t^{\tau-1}}{\log(t/\delta)}\, dt \le \frac{\Gamma(\tau)}{\log(\epsilon/\delta)} = o(1) \tag{3.43}$$

as $\delta \downarrow 0$. Moreover, we also have

$$\int_{\delta \log 3}^{\epsilon} \frac{e^{-t}t^{\tau-1}}{\log(t/\delta)}\, dt \le \frac{1}{\log\log 3} \int_0^{\epsilon} e^{-t}t^{\tau-1}\, dt = o(1) \tag{3.44}$$

as $\epsilon > \delta \log 3$ can be arbitrarily small when $\delta \downarrow 0$. Combining (3.41)-(3.44), we justify that

$$(\delta+1)\int_2^{\infty} \frac{E(x)}{x^{2+\delta}}\, dx = o\left(\frac{1}{\delta^{\tau}}\right). \tag{3.45}$$

Now from (3.37), (3.40) and (3.45), we arrive at the formula

(3.46)
$$\zeta_P(\delta+1) = \frac{c}{\delta^{\tau}}\left(\frac{\delta^{\tau}}{c}\left(1 - \frac{1}{2^{1+\delta}}\right) + (\delta+1)\Gamma(\tau) - \frac{(\delta+1)(\delta\log 2)^{\tau}}{\tau} + o(1)\right)$$

when $\delta \downarrow 0$. Therefore, from (3.46), we verify that

$$\log \zeta_P(\delta+1) = \tau \log\left(\frac{1}{\delta}\right) + \log c + \log\Gamma(\tau) + o(1). \tag{3.47}$$

Comparing (3.31) and (3.47), we must have

$$\log c + \log\Gamma(\tau) = c_P^{\times} - \gamma + C. \tag{3.48}$$

But also referring to (3.35), we have

$$\log c = \log a_P - \gamma\tau - \log\Gamma(\tau) = c_P^{\times} - \gamma\tau - \log\Gamma(\tau). \tag{3.49}$$

From (3.48) and (3.49), we complete the demonstration of our claim that $C = \gamma - \gamma\tau$ holds for $0 < \tau < 1$. This completes the proof of (1.17) for all $0 \le \tau \le 1$.

*(iii)* $\Rightarrow$ *(ii)* : First assume $0 < \tau \le 1$ and $s > 1$. We follow the approach taken in [**32**] and [**33**] with some critical adjustments. Benefitting from Stieltjes

integration, we have

(3.50)
$$\log \zeta_P(s) = -\sum_{p\in P} \log\left(1-\frac{1}{p^s}\right) = -\int_{p_1}^{\infty} \log\left(1-\frac{1}{t^s}\right)\, dP(t)$$
$$= s\int_{p_1}^{\infty} \frac{P(t)}{t(t^s-1)}\, dt$$
$$= s\int_{p_1}^{\infty} \left(P(t)-\frac{\tau t}{\log t}\right)\frac{1}{t^{s+1}}\, dt + \tau s\int_{p_1}^{\infty} \frac{1}{t^s\log t}\, dt + s\int_{p_1}^{\infty} \frac{P(t)}{t^{s+1}(t^s-1)}\, dt,$$

where $p_1$ represents the least prime number in $P$. Since $P(t) = O(t/\log t)$,

$$s\int_{p_1}^{\infty} \frac{P(t)}{t^{s+1}(t^s-1)}\, dt = \int_{p_1}^{\infty} \frac{P(t)}{t^2(t-1)}\, dt + o(1) \tag{3.51}$$

holds when $s \downarrow 1$. Let
$$\mathrm{Ei}(x) = \int_{-\infty}^{x} \frac{e^u}{u}\, du$$
be the Eulerian integral. It is known that (see page 884 of [**15**])

$$\mathrm{Ei}(x) = \log(-x) + \gamma + o(1) \tag{3.52}$$

when $x < 0$ and $x \to 0$. Using the change of variable $e^{\frac{u}{1-s}} = t$ and applying (3.52), we obtain as $s \downarrow 1$ that

(3.53)
$$\tau s\int_{p_1}^{\infty} \frac{1}{t^s\log t}\, dt = -\tau s\mathrm{Ei}((1-s)\log p_1) = -\tau s\log((s-1)\log p_1) - \gamma\tau s + o(1)$$
$$= \log\frac{1}{(s-1)^{\tau}} - \tau\log\log p_1 - \gamma\tau + o(1).$$

Combining (3.50), (3.51) and (3.53), we infer that

$$s\int_{p_1}^{\infty} \left(\frac{P(t)}{t^2} - \frac{\tau}{t\log t}\right)\frac{1}{t^{s-1}}\, dt = \log((s-1)^{\tau}\zeta_P(s)) + \tau\log\log p_1 + \gamma\tau - \int_{p_1}^{\infty} \frac{P(t)}{t^2(t-1)}\, dt + o(1). \tag{3.54}$$

It is clear from (1.14) that

$$\log((s-1)^{\tau}\zeta_P(s)) = \log A + o(1). \tag{3.55}$$

Using the change of variable $t = e^v$, we may write

(3.56)
$$\int_{p_1}^{\infty} \left( \frac{P(t)}{t^2} - \frac{\tau}{t \log t} \right) \frac{1}{t^{s-1}} \, dt = \int_{\log p_1}^{\infty} \left( \frac{P(e^v)}{e^v} - \frac{\tau}{v} \right) e^{-(s-1)v} \, dv := F(r)$$

with $r = s - 1$. Note that $F(r)$ exists for all $r > 0$ and

$$F(r) \to \log A + \tau \log \log p_1 + \gamma\tau - \int_{p_1}^{\infty} \frac{P(t)}{t^2(t-1)} \, dt$$

as $r \downarrow 0$ by (3.54) and (3.55). Since

$$\frac{P(e^v)}{e^v} - \frac{\tau}{v} \geq -\frac{\tau}{v},$$

we may apply Lemma 2 to $F(r)$ in (3.56) and justify the formula

$$\int_{p_1}^{\infty} \left( \frac{P(t)}{t^2} - \frac{\tau}{t \log t} \right) dt = \log A + \tau \log \log p_1 + \gamma\tau - \int_{p_1}^{\infty} \frac{P(t)}{t^2(t-1)} \, dt. \tag{3.57}$$

Next we have, using (3.57) and Chebyshev's estimate, that

$$\log \prod_{\substack{p \leq x \\ p \in P}} \left(1 - \frac{1}{p}\right)^{-1} = \int_{p_1}^{x} \frac{P(t)}{t(t-1)} + O\left(\frac{P(x)}{x}\right) \tag{3.58}$$

$$\begin{aligned}
&= \int_{p_1}^{x} \left( \frac{P(t)}{t^2} - \frac{\tau}{t \log t} \right) dt + \int_{p_1}^{x} \frac{\tau}{t \log t} \, dt + \int_{p_1}^{x} \frac{P(t)}{t^2(t-1)} \, dt + O\left(\frac{1}{\log x}\right) \\
&= \tau \log \log x - \tau \log \log p_1 + \int_{p_1}^{\infty} \frac{P(t)}{t^2(t-1)} \, dt + \int_{p_1}^{\infty} \left( \frac{P(t)}{t^2} - \frac{\tau}{t \log t} \right) dt \\
&\quad + O\left( \left| \int_{x}^{\infty} \left( \frac{P(t)}{t^2} - \frac{\tau}{t \log t} \right) dt \right| \right) + O\left(\frac{1}{\log x}\right) + O\left(\frac{1}{x}\right) \\
&= \tau \log \log x + \log A + \gamma\tau + o(1)
\end{aligned}$$

as $x \to \infty$. Therefore, (1.13) is an immediate consequence of (3.58) with $a_P = Ae^{\gamma\tau}$. In the case when $\tau = 0$, we may repeat the above argument just by removing the role of the Eulerian integral. Assuming $\zeta_P(s) \sim A$ as $s \downarrow 1$, and using the decomposition

$$\log \zeta_P(s) = s \int_{p_1}^{\infty} \frac{P(t)}{t^2} \frac{1}{t^{s-1}} \, dt + s \int_{p_1}^{\infty} \frac{P(t)}{t^{s+1}(t^s - 1)} \, dt,$$

we deduce from Lemma 2 that

$$\lim_{s\downarrow 1}\left(s\int_{p_1}^{\infty}\frac{P(t)}{t^2}\frac{1}{t^{s-1}}\,dt\right)=\log A-\int_{p_1}^{\infty}\frac{P(t)}{t^2(t-1)}\,dt.$$

Consequently, we have

$$\int_{p_1}^{\infty}\frac{P(t)}{t^2}=\log A-\int_{p_1}^{\infty}\frac{P(t)}{t^2(t-1)}\,dt. \tag{3.59}$$

Finally, with the help of (3.59), we obtain as above that

(3.60)
$$\log\prod_{\substack{p\le x\\ p\in P}}\left(1-\frac{1}{p}\right)^{-1}=\int_{p_1}^{x}\frac{P(t)}{t^2}\,dt+\int_{p_1}^{x}\frac{P(t)}{t^2(t-1)}\,dt+O\left(\frac{1}{\log x}\right)=\log A+o(1)$$

when $x\to\infty$. Thus (3.60) implies (1.13) with $a_P=A$. Moreover, (1.17) holds for all $0\le\tau\le 1$.

$(iii)\Leftrightarrow(iv)$ : First assume that (1.14) holds. Then

$$\log\zeta_P(s)=\tau\log\frac{1}{s-1}+\log A+o(1) \tag{3.61}$$

follows when $s\downarrow 1$. We also know for $s>1$ that

$$\log\zeta_P(s)=\sum_{p\in P}\frac{1}{p^s}+\sum_{p\in P}\sum_{k\ge 2}\frac{1}{kp^{ks}}. \tag{3.62}$$

Comparing (3.61) and (3.62), it is plain that

$$\sum_{p\in P}\frac{1}{p^s}=\tau\log\frac{1}{s-1}+\log A-\sum_{p\in P}\sum_{k\ge 2}\frac{1}{kp^k}+o(1) \tag{3.63}$$

as $s\downarrow 1$. Clearly, (3.63) gives that (1.11) holds with

$$K_P=\log A-\sum_{p\in P}\sum_{k\ge 2}\frac{1}{kp^k}.$$

Thus the Mertens density of $P$ is $\tau$. An inspection of the above argument reveals that all of the steps are reversible. Thus, conversely, if the Mertens density of $P$ is $\tau$, then (1.14) must hold.

$(i)\Leftrightarrow(v)$ : First assume that $0<\tau<1$. Then Wirsing showed that (see [**45**]) (1.12) and (1.15) are equivalent, where $c$ is as in (3.35). Note that

(1.17) and (3.35) imply that (1.18) holds when $0 < \tau < 1$. To complete the proof, we have to treat the case $\tau = 1$. If we assume (1.15) so that $N_P(x) \sim cx$ holds, then using Olofsson's theorem (see [**32**]), (1.13) follows with $\tau = 1$ and $a_P = ce^{\gamma}$. Therefore, (1.12) is valid and moreover, since $A = a_P e^{-\gamma}$ by (1.17) and $\Gamma(1) = 1$, it follows that $c = A$ and (1.18) holds. Conversely, assume that (1.12) holds with $\tau = 1$. Then (1.13) holds with $\tau = 1$ and some positive constant $a_P$. If $Q$ is the complement of $P$ in the set of all primes, then using (M3), we deduce that

$$\prod_{p \in Q} \left(1 - \frac{1}{p}\right) = a_P e^{-\gamma}. \tag{3.64}$$

But as $N_P(x)$ corresponds to the number of unsifted integers not exceeding $x$ with respect to sieving by the primes in $Q$, quoting a fundamental principle in sieve theory (see the concept of $B$-free numbers in [**1**] and [**13**]), we know from (3.64) that

$$N_P(x) \sim \left(\prod_{p \in Q} \left(1 - \frac{1}{p}\right)\right) x. \tag{3.65}$$

But then (1.15) follows from (3.64) and (3.65) with $c = a_P e^{-\gamma} = A$. Once again (1.18) holds. Proof of Theorem 1 is now complete.

## 4 - Proof of Theorem 2

($i$) We will construct a desired set $Q$ of primes inductively as follows. To get started, let $x_1 > 3$ be a large enough real number, and we randomly pick

$$\left[\frac{x_1}{(\log x_1)\sqrt{\log\log x_1}}\right]$$

many prime numbers that are $\leq x_1$. Note that this is possible as there is a supply of $\pi(x_1) \gg \frac{x_1}{\log x_1}$ many primes. Call this first set of primes $Q_1$. Thus

$$Q(x_1) = |Q_1| = \left[\frac{x_1}{(\log x_1)\sqrt{\log\log x_1}}\right].$$

As our inductive hypothesis, let us assume that the numbers $3 < x_1 < x_2 < \ldots < x_k$ are chosen such that

$$Q(x_j) = \left[\frac{x_j}{(\log x_j)\sqrt{\log\log x_j}}\right] \tag{4.1}$$

holds for all $1 \leq j \leq k$, and furthermore we have a monotone sequence of sets of chosen primes $Q_1 \subseteq Q_2 \subseteq \ldots \subseteq Q_k$ with $|Q_j| = Q(x_j)$ for all $1 \leq j \leq k$. For the inductive step of the construction, let $x_{k+1} > x_k$ be a large enough number such that it is possible to choose exactly

$$\left[\frac{x_{k+1}}{(\log x_{k+1})\sqrt{\log\log x_{k+1}}}\right] - |Q_k| \tag{4.2}$$

many prime numbers from the interval $(x_k, x_{k+1}]$. Once this is guaranteed, then we may let $Q'_k$ be the newly chosen set of primes from $(x_k, x_{k+1}]$ with size as in (4.2), and we may put $Q_{k+1} = Q_k \cup Q'_k$. This would complete the inductive step of our construction since one obtains by the disjointness of $Q_k$ and $Q'_k$ together with (4.1), (4.2) that

$$Q(x_{k+1}) = |Q_{k+1}| = \left[\frac{x_{k+1}}{(\log x_{k+1})\sqrt{\log\log x_{k+1}}}\right], \tag{4.3}$$

where $Q_1 \subseteq Q_2 \subseteq \ldots \subseteq Q_k \subseteq Q_{k+1}$. This certainly allows us to take

$$Q = \bigcup_{k=1}^{\infty} Q_k.$$

To guarantee that it is possible to choose as many primes as promised in (4.2) and satisfy (4.3), we only require the condition

$$\left[\frac{x_{k+1}}{(\log x_{k+1})\sqrt{\log\log x_{k+1}}}\right] - |Q_k| \leq \pi(x_{k+1}) - \pi(x_k). \tag{4.4}$$

Indeed, we aim to satisfy the following condition which is stronger than (4.4).

$$\left[\frac{x_{k+1}}{(\log x_{k+1})\sqrt{\log\log x_{k+1}}}\right] \leq \pi(x_{k+1}) - \pi(x_k). \tag{4.5}$$

We may now impose the condition that our sequence $\{x_k\}$ is subject to the recursive formula

$$x_{k+1} = x_k + \frac{4x_k}{\sqrt{\log\log x_k}} \tag{4.6}$$

for all $k \geq 1$, where $x_1$ can be taken as large as we wish. Using the asymptotic behavior of the prime counting function over short intervals, it is known that

$$\pi(x+y) - \pi(x) \sim \frac{y}{\log x} \tag{4.7}$$

holds as $x \to \infty$, where $y \geq x^{\theta}$ for some $\theta < 1$ (We owe the existence of such a $\theta$ to a method first developed by Hoheisel [**18**]. The best value of $\theta$ is due to Huxley [**19**] who showed that (4.7) holds for any $7/12 < \theta < 1$. Heath-Brown's [**17**] improvement gave that it is possible to even take

$$y = x^{7/12-\epsilon(x)}$$

for some function $\epsilon(x)$ tending to zero as $x \to \infty$). Certainly, we are allowed to take

$$y = \frac{4x}{\sqrt{\log\log x}}.$$

Therefore, from (4.6) and (4.7), we deduce by picking $x_1$ large enough that

$$\pi(x_{k+1}) - \pi(x_k) \geq \frac{2x_k}{(\log x_k)\sqrt{\log\log x_k}}. \tag{4.8}$$

is true for all $k \geq 1$. Moreover, we may also assume that

$$1 + \frac{4}{\sqrt{\log\log x_k}} \leq 2 \tag{4.9}$$

for all $k$, by taking $x_1$ large enough. It is now clear from (4.9) that for all $k$

$$\begin{aligned} \left[\frac{x_{k+1}}{(\log x_{k+1})\sqrt{\log\log x_{k+1}}}\right] &= \left[\frac{x_k\left(1+\frac{4}{\sqrt{\log\log x_k}}\right)}{(\log x_{k+1})\sqrt{\log\log x_{k+1}}}\right] \\ &\leq \frac{2x_k}{(\log x_k)\sqrt{\log\log x_k}}. \end{aligned} \tag{4.10}$$

Finally, (4.5) is achieved by combining (4.8) and (4.10). The construction of $Q$ is complete. Let $P$ be the complement of $Q$ in the set of all primes. Next we show that $P$ and $Q$ have the desired properties. By partial summation, we obtain for any $m \geq 1$ that

$$\sum_{\substack{p \leq x_m \\ p \in Q}} \frac{1}{p} = \frac{Q(x_m)}{x_m} + \int_2^{x_m} \frac{Q(t)}{t^2}\, dt. \tag{4.11}$$

Note that $\frac{Q(x_m)}{x_m} = o(1)$ as $m \to \infty$. We may write, by taking $x_0 = 2$,

$$\int_2^{x_m} \frac{Q(t)}{t^2}\, dt = \sum_{k=0}^{m-1} \int_{x_k}^{x_{k+1}} \frac{Q(t)}{t^2}\, dt \geq \sum_{k=1}^{m-1} Q(x_k)\left(\frac{1}{x_k} - \frac{1}{x_{k+1}}\right) + O(1). \tag{4.12}$$

Since $x_k \sim x_{k+1}$ as $k \to \infty$, one obtains that

$$Q(x_k)\left(\frac{1}{x_k} - \frac{1}{x_{k+1}}\right) \gg \frac{1}{(\log x_k)\log\log x_k}. \tag{4.13}$$

However, by (4.9), we know that $x_{k+1} \le 2x_k$, and consequently $x_k \le 2^k x_1$ follows for all $k$. Using this observation together with (4.11)-(4.13), it is plain that

$$\sum_{p\in Q}\frac{1}{p} \gg \sum_{k=1}^{\infty}\frac{1}{(\log x_k)\log\log x_k} \gg \sum_{k=2}^{\infty}\frac{1}{k\log k}$$

all diverge. Since $Q(x_k) = o(x_k)$, given any $\delta > 0$, we have

$$\frac{P(x_k)}{x_k} \ge 1 - \delta$$

for all $k \ge K(\delta)$. Thus if $x_k \le x \le x_{k+1}$, then (again using $x_k \sim x_{k+1}$)

$$\frac{P(x)}{x} \ge \frac{P(x_k)}{x_{k+1}} = \frac{P(x_k)}{x_k}\frac{x_k}{x_{k+1}} \ge 1 - 2\delta$$

when $k$ is large enough in terms of $\delta$ so consequently when $x$ is large enough in terms of $\delta$. This shows that $P$ must have relative natural density 1. The proof of $(i)$ is complete.

$(ii)$ From a distributional point of view, $P^k$ is a peculiar set of primes having no relative natural density. In spite of this, the Dirichlet density of $P^k$, denoted by $D(P^k)$, exists, and is given by

$$D(P^k) = \log_{10}\left(\frac{k+1}{k}\right)$$

for any $1 \le k \le 9$. It turns out that the distribution of $P^k$ is strikingly related to what is known as Benford's law in probability and the theory surrounding it. For justifications and further enlightening discussions on these issues, the reader is referred to the papers of Bumby and Ellentuck [**6**], Raimi [**36**] and Cohen and Katz [**10**], [**11**] (see also the remarks in [**39**] concerning Dirichlet's theorem on primes in progressions). Let us verify first that $D(P^k)$ is transcendental. Recall that, according to the celebrated Gelfond-Schneider theorem, if $\alpha \ne 0, 1$ and $\beta$ are algebraic real numbers, and $\beta$ is irrational, then $\alpha^\beta$ must be transcendental. First

$$10^{D(P^k)} = \frac{k+1}{k}$$

is not transcendental. Moreover, $D(P^k)$ is irrational, since otherwise

$$\log_{10}\left(\frac{k+1}{k}\right) = \frac{a}{b}$$

with positive integers $a, b$ leads to $10^a k^b = (k+1)^b$ which forces $k = 1$. Thus we get $10^a = 2^b$ which is not possible. Therefore, $D(P^k)$ can not be algebraic. For a contradiction, assume that a partition of the form

$$P^k - Q^k = P_1 \cup ... \cup P_m \tag{4.14}$$

exists, where each $P_i$ is subject to (1.22). Because of (1.22), we may use Theorem 1 to deduce that each $P_i$ has Mertens density, and

$$\delta(P_i) = \tau_i = 1 - a_i \in (0, 1].$$

Note that, since each $a_i \in [0, 1)$ is algebraic, each $\tau_i$ is algebraic as well. Moreover, $D(P_i) = \tau_i$ for all $i$, and as a result of (1.21), again using Theorem 1, $\delta(Q^k) = 0$. Thus $D(Q^k) = 0$ as well. Using the fact that the Dirichlet density is finitely additive, (4.14) leads to

$$D(P^k) = D(Q^k) + \sum_{i=1}^{m} D(P_i) = \sum_{i=1}^{m} \tau_i. \tag{4.15}$$

However, a contradiction arises, since the left hand side of (4.15) is transcendental but the right hand side is not. This completes the proof of (*ii*).

The proof of (*iii*) is very similar to the proof of (*ii*) so will be omitted.

(*iv*) First using (1.23), pairwise disjointness of $P_i$ and $P_j$ together with Theorem 1, we obtain that

$$\delta(P_i \cup P_j) = \delta(P_i) + \delta(P_j) = 1 - a_{ij} \in (0, 1]$$

for all $i \neq j$. It follows from this that

$$\tau_i + \tau_j = D(P_i) + D(P_j) = 1 - a_{ij}. \tag{4.16}$$

Since each $a_{ij}$ is algebraic, we require from (4.16) that each $\tau_i + \tau_j$ is algebraic. At this point, we may formulate this situation in the language of graph theory. Consider a graph $G$ with $m$ vertices, where the each vertex is labeled as a $P_i$. We draw an edge between $P_i$ and $P_j$ with $i \neq j$ when and only when $\tau_i + \tau_j$ is an algebraic number. This defines a graph $G$, where each unordered pair $(P_i, P_j)$ with $i \neq j$ can be identified with the edge joining the vertices $P_i$ and $P_j$. To complete the proof, we need to show that the maximum number of edges in $G$ is at most $[m^2/4]$. The critical observation here is the fact that our graph $G$ is triangle-free. Since otherwise there would exist three edges connecting the distinct vertices $P_i$, $P_j$ and $P_k$ in $G$. But then each of $\tau_i + \tau_j$, $\tau_i + \tau_k$ and $\tau_j + \tau_k$ would be algebraic, and this would easily imply that $\tau_i$, $\tau_j$, $\tau_k$ are algebraic. This is not possible as each $\tau_i$ is transcendental. It is known from a result originally due to Mantel [**29**] that the maximum number of edges of an $m$-vertex triangle-free graph is

$$\leq \left[\frac{m^2}{4}\right], \tag{4.17}$$

and the bound in (4.17) is sharp. This completes the proof of $(iv)$.

$(v)$ Using (1.24) and Theorem 1, we know that

$$\delta(P_i) = 1 - a_i = \tau_i \in (0, 1)$$

holds for all $1 \leq i \leq m$. Since $P_i$'s form a partition of the set of all primes, it follows by finite additivity that

$$\sum_{i=1}^{m} \tau_i = 1, \tag{4.18}$$

and from Euler products that

$$\prod_{i=1}^{m} \zeta_{P_i}(s) = \zeta(s) \tag{4.19}$$

for $\Re(s) > 1$. Again by Theorem 1, we have

$$\zeta_{P_i}(s) \sim \frac{A_i}{(s-1)^{\tau_i}} \tag{4.20}$$

as $s \downarrow 1$ for some constant $A_i > 0$ when $1 \leq i \leq m$. Assembling (4.18)-(4.20) and using (1.18), it is clear that

$$\left(\prod_{i=1}^{m} c_i\right)\left(\prod_{i=1}^{m} \Gamma(\tau_i)\right) = \prod_{i=1}^{m} A_i = 1. \tag{4.21}$$

To finish the argument, let us invoke a classical relation from the theory of the gamma function due to Dirichlet [**12**] and Liouville [**28**] given in the form

$$\int_{\mathcal{R}} f(x_1 + \cdots + x_m) \prod_{i=1}^{m} x_i^{\tau_i - 1}\, dX = \frac{\prod_{i=1}^{m} \Gamma(\tau_i)}{\Gamma\left(\sum_{i=1}^{m} \tau_i\right)} \int_0^1 f(x) x^{(\sum_{i=1}^{m} \tau_i) - 1}\, dx, \tag{4.22}$$

where $f(x)$ is any continuous real valued function defined on $\mathbb{R}$, $dX = \prod_{i=1}^{m} dx_i$ and $\mathcal{R}$ is the region in $m$-space defined by the inequalities $x_i \geq 0$, $\sum_{i=1}^{m} x_i \leq 1$. Next taking $f(x)$ to be the constant function 1 in (4.22) and referring to (4.18), we may write

$$\prod_{i=1}^{m} \Gamma(\tau_i) = \int_{\mathcal{R}} \prod_{i=1}^{m} x_i^{-a_i} \; dX. \tag{4.23}$$

Therefore, (1.25) is a consequence of (4.21) and (4.23). Specifically when $m = 2$, the above calculations reduce to

$$c_1 c_2 = \frac{1}{\Gamma(\tau_1)\Gamma(1-\tau_1)} = \frac{1}{\Gamma(a_1)\Gamma(1-a_1)}. \tag{4.24}$$

Thus (1.26) follows from (4.24) and the reflection formula

$$\Gamma(a_1)\Gamma(1-a_1) = \frac{\pi}{\sin \pi a_1}$$

for the gamma function. Note that if $a_1$ is rational, then it is clear that $\sin \pi a_1$ is an algebraic number. Since $\pi$ is well-known to be transcendental, it follows that $c_1 c_2$ is always transcendental in this case.

## 5 - Proof of Theorem 3

Let $P$ and $Q$ be complementary sets of prime numbers as constructed in part ($i$) of Theorem 2. Thus $P$ has relative natural density 1 and

$$\sum_{p \in Q} \frac{1}{p} \tag{5.1}$$

diverges. For a contradiction, assume that the Mertens density of $P$ exists. Put $\delta(P) = \tau \in [0, 1]$. Since $P$ has relative natural density 1, the Dirichlet density of $P$ should also be 1, and this forces $\delta(P) = 1$. Using the equivalence of ($i$) and ($iv$) in Theorem 1, we know that

$$\sum_{\substack{p \leq x \\ p \in P}} \frac{1}{p} = \log\log x + c_P + o(1) \tag{5.2}$$

holds for some constant $c_P$. But from (M2) and (5.2), one obtains that

$$\sum_{\substack{p \leq x \\ p \in Q}} \frac{1}{p} = B_1 - c_P + o(1). \tag{5.3}$$

Clearly, (5.1) and (5.3) are in contradiction. In conclusion, $P$ is the desired set of primes having relative natural density but no Mertens density.

Next we construct a set of primes having Mertens density but no relative natural density. Consider the terms of the sequence $x_n = 2^{n^3}$ for all large enough $n$. Let $P$ be the set of all prime numbers except the ones that belong to the intervals

$$I_n := (2^{\frac{n^3}{1+\frac{1}{n^2}}}, 2^{n^3}] = (x_n^{\frac{1}{1+\frac{1}{n^2}}}, x_n]$$

for all large enough $n$. Using (M2), it is easy to verify that

$$\sum_{p \in I_n} \frac{1}{p} = \log\left(1 + \frac{1}{n^2}\right) + O\left(\frac{1}{n^3}\right) = O\left(\frac{1}{n^2}\right) \tag{5.4}$$

when $n$ is large. Consequently from (5.4), we easily infer that the contribution of all the discarded primes belonging to the $I_n$'s is at most

$$\sum_{n=M}^{\infty} \sum_{p \in I_n} \frac{1}{p} = O\left(\sum_{n=1}^{\infty} \frac{1}{n^2}\right) = O(1), \tag{5.5}$$

where the positive integer $M$ can be taken as large as we please. It follows from (M2) and (5.5) that

$$\sum_{\substack{p \le x \\ p \in P}} \frac{1}{p} = \log\log x + c_P + o(1) \tag{5.6}$$

holds for some constant $c_P$. Again using the equivalence of $(i)$ and $(iv)$ from Theorem 1, (5.6) guarantees that $P$ has Mertens density and $\delta(P) = 1$. Since

$$\lim_{n\to\infty} \left(\frac{n^5}{n^2+1} - (n-1)^3\right) = \infty,$$

$I_n$'s do not overlap when $n$ is large enough. Moreover, we have

$$x_{n-1} = o(x_n^{\frac{1}{1+\frac{1}{n^2}}}), \tag{5.7}$$

and

$$\frac{x_n^{\frac{1}{1+\frac{1}{n^2}}}}{x_n} = x_n^{-\frac{1}{n^2+1}} = o(1). \tag{5.8}$$

Since all the primes in the interval $(x_{n-1}, x_n^{\frac{1}{1+\frac{1}{n^2}}})$ belong to $P$, (5.7) and Lemma 3 imply that

$$P(x_n^{\frac{1}{1+\frac{1}{n^2}}}) \sim \pi(x_n^{\frac{1}{1+\frac{1}{n^2}}}). \tag{5.9}$$

It is plain from (5.9) that

$$\limsup \frac{P(x)}{\pi(x)} = 1. \tag{5.10}$$

On the other hand, all primes in $(x_n^{\frac{1}{1+\frac{1}{n^2}}}, x_n]$ are not in $P$ so that by (5.8) and Lemma 3,

$$P(x_n) \leq \pi(x_n^{\frac{1}{1+\frac{1}{n^2}}}) = o(\pi(x_n))$$

follows, and we have

$$\liminf \frac{P(x)}{\pi(x)} = 0. \tag{5.11}$$

Finally, (5.10) and (5.11) show that $P$ has no relative natural density. This completes the proof of Theorem 3.

## 6 - Proof of Theorem 4

Note that by Lemma 1, (1.27) and (1.28) are equivalent so it is enough to assume only (1.27). It suffices to make the $o(1)$ terms in (3.14) and (3.31) explicit. We define a function $g(x)$ by the formula

$$S_P(x) := \sum_{1<n\leq x} \frac{\Lambda_P(n)}{n \log n} = \sum_{n\leq(\log x)^\tau} \frac{1}{n} + c_P^\times - \gamma + g(x) \tag{6.1}$$

for $x \geq 1$, where we may assume in (6.1) that

$$g(x) = O\left(\frac{1}{(\log\log x)^\alpha}\right) \tag{6.2}$$

holds when $x \geq 3$. To show (1.29), we have to analyze the Mellin transform of $g(x)$. To this end, we have from (6.2) that

(6.3)
$$\delta \int_1^\infty \frac{g(x)}{x^{1+\delta}}\, dx = \delta \int_3^\infty \frac{1}{x^{1+\delta}(\log\log x)^\alpha}\, dx + O(\delta) = \delta \int_{\log 3}^\infty \frac{e^{-\delta u}}{(\log u)^\alpha}\, du + O(\delta).$$

Consider the decomposition

$$\delta\int_{\log 3}^{\infty}\frac{e^{-\delta u}}{(\log u)^{\alpha}}\,du=\delta\int_{\log 3}^{\frac{1}{\delta}}\frac{e^{-\delta u}}{(\log u)^{\alpha}}\,du+\delta\int_{\frac{1}{\delta}}^{\infty}\frac{e^{-\delta u}}{(\log u)^{\alpha}}\,du. \tag{6.4}$$

Then we have

$$\delta\int_{\frac{1}{\delta}}^{\infty}\frac{e^{-\delta u}}{(\log u)^{\alpha}}\,du\ll\frac{\delta}{(\log(1/\delta))^{\alpha}}\int_{\frac{1}{\delta}}^{\infty}e^{-\delta u}\,du\ll\frac{1}{(\log(1/\delta))^{\alpha}}. \tag{6.5}$$

Moreover, integrating by parts, we get

$$\begin{aligned}\delta\int_{\log 3}^{\frac{1}{\delta}}\frac{e^{-\delta u}}{(\log u)^{\alpha}}\,du&\ll\delta\int_{\log 3}^{\frac{1}{\delta}}\frac{1}{(\log u)^{\alpha}}\,du\\&=\frac{1}{(\log(1/\delta))^{\alpha}}+O(\delta)+\delta\alpha\int_{\log 3}^{\frac{1}{\delta}}\frac{1}{(\log u)^{1+\alpha}}\,du.\end{aligned} \tag{6.6}$$

We may further decompose

$$\delta\int_{\log 3}^{\frac{1}{\delta}}\frac{1}{(\log u)^{1+\alpha}}\,du=\delta\int_{\sqrt{\frac{1}{\delta}}}^{\frac{1}{\delta}}\frac{1}{(\log u)^{1+\alpha}}\,du+\delta\int_{\log 3}^{\sqrt{\frac{1}{\delta}}}\frac{1}{(\log u)^{1+\alpha}}\,du. \tag{6.7}$$

Clearly,

$$\delta\int_{\log 3}^{\sqrt{\frac{1}{\delta}}}\frac{1}{(\log u)^{1+\alpha}}\,du\ll\sqrt{\delta} \tag{6.8}$$

and

$$\delta\int_{\sqrt{\frac{1}{\delta}}}^{\frac{1}{\delta}}\frac{1}{(\log u)^{1+\alpha}}\,du\ll\frac{1}{(\log(1/\delta))^{1+\alpha}}. \tag{6.9}$$

Assembling (6.3)-(6.9), we can justify that

$$\delta\int_{1}^{\infty}\frac{g(x)}{x^{1+\delta}}\,dx=O\left(\frac{1}{(\log(1/\delta))^{\alpha}}\right). \tag{6.10}$$

Using (6.10), (3.14) and (3.31) become

$$\log\zeta_P(\delta+1)=\log\frac{1}{\delta}+c_P^{\times}-\gamma+O\left(\frac{1}{(\log(1/\delta))^{\alpha}}\right). \tag{6.11}$$

for $\tau = 1$ and

$$\log \zeta_P(\delta+1) = \tau \log \frac{1}{\delta} + c_P^{\times} - \gamma + C + O\left(\frac{1}{(\log(1/\delta))^{\alpha}}\right) \tag{6.12}$$

for $0 < \tau < 1$, respectively. When $\tau = 0$, we have

$$\log \zeta_P(\delta+1) = c_P^{\times} + O\left(\frac{1}{(\log(1/\delta))^{\alpha}}\right). \tag{6.13}$$

Now (1.29) easily follows from (6.11)-(6.13) for all $0 \leq \tau \leq 1$ by exponentiation.

Let us now assume that $\zeta_P(s)$ satisfies the given asymptotic behavior as $s \downarrow 1$ and (1.30) holds. Then (3.58) and (3.60) become

$$\log \prod_{\substack{p \leq x \\ p \in P}} \left(1 - \frac{1}{p}\right)^{-1} = \tau \log\log x + \log A + \gamma\tau + O\left(\frac{1}{(\log\log x)^{\alpha}}\right) \tag{6.14}$$

when $0 < \tau \leq 1$ and

$$\log \prod_{\substack{p \leq x \\ p \in P}} \left(1 - \frac{1}{p}\right)^{-1} = \log A + O\left(\frac{1}{(\log\log x)^{\alpha}}\right) \tag{6.15}$$

when $\tau = 0$, respectively. Exponentiating (6.14) and (6.15), we obtain (1.28) (and also (1.27)) for all $0 \leq \tau \leq 1$.

Note that (1.31) and (1.32) are equivalent by Lemma 1 so it is enough to assume only (1.31). Then we may write

$$P(x) = x \sum_{\substack{p \leq x \\ p \in P}} \frac{1}{p} - \int_3^x \sum_{\substack{p \leq t \\ p \in P}} \frac{1}{p}\, dt + O(1). \tag{6.16}$$

From (1.31), one has

$$x \sum_{\substack{p \leq x \\ p \in P}} \frac{1}{p} = \tau x \log\log x + c_P x + O\left(\frac{x}{\log x (\log\log x)^2}\right). \tag{6.17}$$

Again using (1.31), one obtains that

(6.18)
$$\int_3^x \sum_{\substack{p \leq t \\ p \in P}} \frac{1}{p}\, dt = \tau \int_3^x \log\log t\, dt + c_P(x-3) + O\left(\int_3^x \frac{1}{\log t (\log\log t)^2}\, dt\right).$$

Moreover, we easily get

$$\int_3^x \log\log t \, dt = \int_2^x \log\log t \, dt + O(1) = x \log\log x - \mathrm{li}\ x + O(1), \tag{6.19}$$

where

$$\mathrm{li}\ x = \int_2^x \frac{1}{\log t}\, dt$$

is the logarithmic integral. It is also plain that

$$\int_3^x \frac{1}{\log t (\log\log t)^2}\, dt = O\left(\frac{x}{\log x (\log\log x)^2}\right). \tag{6.20}$$

Gathering (6.16)-(6.20), it is now clear that

$$P(x) = \tau\ \mathrm{li}\ x + O\left(\frac{x}{\log x (\log\log x)^2}\right) \tag{6.21}$$

holds for $0 < \tau \leq 1$. However, since

$$\mathrm{li}\ x = \frac{x}{\log x} + O\left(\frac{x}{(\log x)^2}\right),$$

(6.21) leads us to the formula

$$P(x) = \frac{\tau x}{\log x}\left(1 + O\left(\frac{1}{(\log\log x)^2}\right)\right) \tag{6.22}$$

when $0 < \tau \leq 1$. Assuming (6.22), Wirsing [**45**] showed that

$$N_P(x) = \frac{cx}{(\log x)^{1-\tau}}\left(1 + O\left(\frac{1}{\log\log x}\right)\right)$$

for some positive constant $c$, where $(\log x)^{1-\tau}$ is taken to be 1 when $\tau = 1$. This ends the proof of (1.33) and the proof of Theorem 4.

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## References

[**1**] E. Alkan and A. Zaharescu, *B-free numbers in short arithmetic progressions*, J. Number Theory **113** (2005), 226–243.

[**2**] E. Alkan, *On Dirichlet L-functions with periodic coefficients and Eisenstein series*, Monatsh. Math. **163** (2011), 249–280.

[**3**] E. Alkan, *Distribution of averages of Ramanujan sums*, Ramanujan J. **29** (2012), 385–408.

[**4**] E. Alkan, *Ramanujan sums and the Burgess zeta function*, Int. J. Number Theory **8** (2012), 2069–2092.

[**5**] A. Beurling, *Analyse de la loi asymptotique de la distribution des nombres premiers generalises*, Acta Math. **68** (1937), 255–291.

[**6**] R. Bumby and E. Ellentuck, *Finitely additive measures and the first digit problem*, Fund. Math. **65** (1969), 33–42.

[**7**] P. L. Chebyshev, *Sur la fonction qui détermine la totalité des nombres premiers inférieurs à une limite donnée*, Mem. Acad. Sci. St. Petersburg **6** (1848), 1–19.

[**8**] P. L. Chebyshev, *Mémorie sur nombres premiers*, Mem. Acad. Sci. St. Petersburg **7** (1850), 17–33.

[**9**] P. L. Chebyshev, *Lettre de M. le professeur Tchébychev à M. Fuss, sur un nouveau théorème relatif aux nombres premiers contenus dans la formes* $4n+1$ *et* $4n+3$, Bull. Classe Phys. Acad. Imp. Sci. St. Petersburg **11** (1853), 208.

[**10**] D. I. A. Cohen, *An explanation of the first digit phenomenon*, J. Combinatorial Theory Ser. A **20** (1976), 367–370.

[**11**] D. I. A. Cohen and T. M. Katz, *Prime numbers and the first digit phenomenon*, J. Number Theory **18** (1984), 261–268.

[**12**] P. G. L. Dirichlet, *Sur une nouvelle méthode pour la détermination des intégrales multiples*, J. Math. Pures Appl. **4** (1839), 164–168.

[**13**] P. Erdős, *On the difference of consecutive terms of sequences defined by divisibility properties*, Acta Arith. **12** (1966/1967), 175–182.

[**14**] A. Fujii, *Some generalizations of Chebyshev's conjecture*, Proc. Japan Acad. Ser. A Math. Sci. **64** (1988), 260–263.

[**15**] I. S. Gradshteyn and I. M. Ryzhik, *Table of integrals, series, and products*, Seventh ed., Elsevier/Academic Press, Amsterdam, 2007.

[**16**] J. Hadamard, *Sur la distribution des zéros de la fonction* $\zeta(s)$ *et ses conséquences arithmétiques*, Bull. Soc. Math. France **24** (1896), 199–220.

[**17**] D. R. Heath-Brown, *The number of primes in a short interval*, J. Reine Angew. Math. **389** (1988), 22–63.

[**18**] G. Hoheisel, *Primzahlprobleme in der Analysis*, Sitzungberichte Akad. Berlin (1930), 580–588.

[**19**] M. N. Huxley, *On the difference between consecutive primes*, Invent. Math. **15** (1972), 164–170.

[**20**] J. Korevaar, *Tauberian theory: A century of developments*, Grundlehren der Math. Wiss., **329**, Springer-Verlag, Berlin, 2004.

[**21**] E. Landau, *Handbuch der Lehre von der Verteilung der Primzahlen*, **2**, Leipzig, Berlin: B. G. Teubner, 1909.

[**22**] A. Languasco and A. Zaccagnini, *A note on Mertens' formula for arithmetic progressions*, J. Number Theory **127** (2007), 37–46.

[**23**] A. Languasco and A. Zaccagnini, *Some estimates for the average of the error term of the Mertens product for arithmetic progressions*, Funct. Approx. Comment. Math. **38** (2008), 41–47.

[**24**] A. Languasco and A. Zaccagnini, *On the constant in the Mertens product for arithmetic progressions. II. Numerical values*, Math. Comp. **78** (2009), 315–326.

[**25**] A. Languasco and A. Zaccagnini, *On the constant in the Mertens product for arithmetic progressions. I. Identities*, Funct. Approx. Comment. Math. **42** (2010), 17–27.

[**26**] A. Languasco and A. Zaccagnini, *Computing the Mertens and Meissel-Mertens constants for sums over arithmetic progressions*, Experiment. Math. **19** (2010), 279–284.

[**27**] P. Lindqvist and J. Peetre, *On the remainder in a series of Mertens*, Exposition. Math **15** (1997), 467–478.

[**28**] J. Liouville, *Note sur quelques intégrales définies*, J. Math. Pures Appl. **4** (1839), 225–235.

[**29**] W. Mantel, *Problem* 28, Wiskundige Opgaven **10(60-61)** (1907), 320.

[**30**] F. Mertens, *Ueber einige asymptotische Gesetze der Zahlentheorie*, J. Reine Angew. Math. **77** (1874), 289–338.

[**31**] F. Mertens, *Ein Beitrag zur analytischen Zahlentheorie*, J. Reine Angew. Math. **78** (1874), 46–62.

[**32**] R. Olofsson, *Properties of the Beurling generalized primes*, J. Number Theory **131** (2011), 45–58.

[**33**] P. Pollack, *On Mertens' theorem for Beurling primes*, Canad. Math. Bull. **56** (2013), 829–843.

[**34**] D. Popa, *A double Mertens type evaluation*, J. Math. Anal. Appl. **409** (2014), 1159–1163.

[**35**] D. Popa, *A triple Mertens evaluation*, J. Math. Anal. Appl. **444** (2016), 464–474.

[**36**] R. A. Raimi, *The first digit problem*, Amer. Math. Monthly **83** (1976), 521–538.

[**37**] B. Riemann, *Ueber die Anzahl der Primzahlen unter einer gegebenen Grösse*, Monatsberichte Königlichen Preussischen Akademie der Wissenschaften Berlin (1859), 671–680.

[**38**] M. Rosen, *A generalization of Mertens' theorem*, J. Ramanujan Math. Soc. **14** (1999), 1–19.

[**39**] J.-P. Serre, *A course in Arithmetic*, Grad. Texts in Math., **7**, Springer-Verlag, New York-Heidelberg, 1973.

[**40**] S. Uchiyama, *On some products involving primes*, Proc. Amer. Math. Soc. **28** (1971), 629–630.

[**41**] C.-J. de la Vallée Poussin, *Recherches analytiques sur la théorie des nombres premiers, I, II, III*, Ann. Soc. Sci. Bruxelles **20** (1896), 183–256, 281–362, 363–397.

[**42**] E. A. Vasil'kovskaja, *Mertens' formula for an arithmetic progression*, Taškent. Gos. Univ. Naučn. Trudy **548** Voprosy Mat (1977), 14–17, 139–140.

[**43**] A. I. Vinogradov, *On Mertens' theorem*, Dokl. Akad. Nauk. SSSR **143** (1962), 1020–1021.

[**44**] A. I. Vinogradov, *On the remainder in Mertens' formula*, Dokl. Akad. Nauk. SSSR **148** (1963), 262–263.

[**45**] E. Wirsing, *Über die Zahlen, deren Primteiler einer gegebenen Menge angehören*, Arch. Math. **7** (1956), 263–272.

[**46**] E. Wirsing, *Das asymptotische Verhalten von Summen über multiplikative Funktionen*, Math. Ann. **143** (1961), 75–102.

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