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Some remarks about intrinsic Lipschitz and tame maps in Carnot groups of step 2

Abstract. This note concerns low-dimensional intrinsic Lipschitz graphs, in the sense of Franchi, Serapioni, and Serra Cassano, in Carnot groups of step 2. More precisely, we prove the equivalence between intrinsic Lipschitz and tame maps.

Keywords. Carnot groups of step 2, SubRiemannian geometry, intrinsic Lipschitz maps.

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1 - Introduction

In this paper, we focus our attention on low-dimensional intrinsic Lipschitz graphs in Carnot groups of step 2. They are important in order to study the notion of rectifiable set in Carnot groups.

More precisely, rectifiability, introduced by Besicovich in the plane, is a key notion in Geometric Measure Theory and Calculus of Variations. The classical definition was given by Federer: in Euclidean spaces, rectifiable sets are defined as being essentially contained in the countable union of Lipschitz graphs. In Carnot groups, the corresponding notion of Lipschitz graphs is *intrinsic Lipschitz graphs (iLG).*

iLG were introduced by Franchi, Serapioni, and Serra Cassano in [**13**] in the context of Heisenberg groups $\mathbb{H}^n$ which are the most important examples of Carnot groups of step 2. The definition of iLG makes perfect sense also for *low dimensions*, but there are fewer works that study specifically low-dimensional iLG. Recently, they have appeared in [**3, 4, 10, 12**]. In [**10**], in $\mathbb{H}^n$, we used this notion in order to obtain an easy proof about the extension of intrinsic Lipschitz maps. A first step of the proof is to show their equivalence with the

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so-called tame maps (see Definition 2.13). Hence, it is natural to ask if it is possible to have the same equivalence in a more general case, i.e., for instance, in Carnot groups of step 2. In this paper, we give a positive answer about this question. More precisely, our main result proves that

$$\phi \text{ is an intrinsic Lipschitz map} \quad \Longleftrightarrow \quad \psi \text{ is a tame map}$$

where there is an explicit link between ϕ and ψ : they have the same horizontal components and their vertical components are equal up to the sign. We defer the definitions to Section 2 and more precise statements to Section 3, see especially Theorems 3.4 and 3.7.

Regarding the extension of intrinsic Lipschitz maps in Carnot groups of step 2, the proof in [**10**, Theorem 1.1], in $\mathbb{H}^n$, where the vertical component is just one, is not applicable. The reason is purely algebraic and is due to the fact that in Carnot groups of step 2 the vertical components are more than one. However, in [**18**, Theorem 1.1] and [**18**, Theorem 4.5] the author provides a characterization of (locally) Lipschitz functions from (geodesically convex) subsets of Riemannian manifolds into graded groups through a system of first order PDEs known as *weak contact equations*. Here the target space belongs to a special class of Carnot groups of step 2, called Allcock groups which includes Heisenberg groups. Finally, we want to recall a negative answer about the extension in step 3 proved in [**5**].

2 - Preliminaries

2.1 - *Carnot groups of step 2*

We here introduce Carnot groups of step 2 in exponential coordinates. We adopt as a general reference [**6**, Chapter 3], but the interested reader could also read [**1**, **2**]. For a general introduction of Carnot groups we recall [**17**]. In this subsection $\mathbb{G}$ will always be an arbitrary Carnot group of step 2.

We denote with m the rank of $\mathbb{G}$ and we identify $\mathbb{G}$ with $(\mathbb{R}^{m+h}, \cdot)$ by means of exponential coordinates associated with an adapted basis $(X_1, \dots, X_m, Y_1, \dots, Y_h)$ of the Lie algebra $\mathfrak{g}$. In this coordinates, we will identify any point $q \in \mathbb{G}$ with $q \equiv (x_1, \dots, x_m, y_1, \dots, y_h)$. The group operation $\cdot$ between two elements $q = (x, y)$ and $q' = (x', y')$ is given by

$$q \cdot q' = qq' = \left(x + x', y + y' - \frac{1}{2} \langle \mathcal{B}x, x' \rangle \right), \tag{2.1}$$

where $\langle \mathcal{B}x, x' \rangle := (\langle \mathcal{B}^{(1)}x, x' \rangle, \dots, \langle \mathcal{B}^{(h)}x, x' \rangle)$ and $\mathcal{B}^{(\ell)}$ are linearly independent and skew-symmetric matrices in $\mathbb{R}^{m \times m}$, for $\ell = 1, \dots, h$. For any $\ell = 1, \dots, h$

and any $j, i = 1, \dots, m$, we set $(\mathcal{B}^{(\ell)})_{j\ell} =: (b^{(\ell)}_{ji})$, and we stress that $\langle \mathcal{B}^{(\ell)} x, x' \rangle := \sum_{j,i=1}^m b^{(\ell)}_{ji} x'_j x_i$.

Moreover the dilation $\delta_\lambda : \mathbb{R}^{m+h} \to \mathbb{R}^{m+h}$ defined as

$$\delta_\lambda(x, y) := (\lambda x, \lambda^2 y), \qquad \text{for all } (x, y) \in \mathbb{R}^{m+h},$$

is an automorphism of $(\mathbb{R}^{m+h}, \cdot)$, for all $\lambda > 0$.

The identity of $\mathbb{G}$ is the origin of $\mathbb{R}^{m+h}$ and $(x, y)^{-1} = (-x, -y)$. For any $p \in \mathbb{G}$ the intrinsic left translation $\tau_p : \mathbb{G} \to \mathbb{G}$ is defined as

$$q \mapsto \tau_p q := pq.$$

It is standard to observe that in these coordinates we can write

$$\begin{aligned} X_j(p) &= \partial_{x_j} - \frac{1}{2} \sum_{\ell=1}^h \sum_{i=1}^m b^{(\ell)}_{ji} x_i \, \partial_{y_\ell}, && \text{for } j = 1, \dots, m, \\ Y_\ell(p) &= \partial_{y_\ell}, && \text{for } \ell = 1, \dots, h. \end{aligned}$$

Moreover,

$$\begin{aligned} &[X_j, X_i] = \sum_{\ell=1}^h b^{(\ell)}_{ji} Y_\ell, \quad \text{and} \\ &[X_j, Y_\ell] = 0, \qquad \forall j, i = 1, \dots, m, \ \text{ and } \ \forall \ell = 1, \dots, h. \end{aligned} \tag{2.2}$$

R e m a r k 2.3. Note that the above arguments show that there exist step 2 Carnot groups of any dimension $m \in \mathbb{N}$ of the first layer and any dimension

$$h \leq \frac{m(m-1)}{2},$$

of the second layer: it suffices to choose h linearly independent matrices $\mathcal{B}^{(1)}, \dots, \mathcal{B}^{(h)}$ in the vector space of the skew-symmetric $m \times m$ matrices (which has dimension $m(m-1)/2$) and then define the composition law as in (2.1).

A *homogeneous norm* on $\mathbb{G}$ is a nonnegative function $p \mapsto \|p\|$ such that for all $p, q \in \mathbb{G}$ and for all $\lambda \geq 0$

$$\|p\| = 0 \quad \text{if and only if } p = 0$$
$$\|\delta_\lambda p\| = \lambda \|p\|, \qquad \|pq\| \leq \|p\| + \|q\|.$$

We make the following choice of the homogeneous norm in $\mathbb{G}$:

$$\|(x, y)\| := \max\{|x|_{\mathbb{R}^m}, \varepsilon |y|^{1/2}_{\mathbb{R}^h}\}, \tag{2.4}$$

for a suitable $\varepsilon \in (0,1]$ (for the existence of such an $\varepsilon > 0$ see Theorem 5.1 in [**14**]). From now on, with a bit abuse of notation, we will write the norm of $\mathbb{R}^s$ for every $s \in \mathbb{N}$ with the same symbol $|\cdot|$.

However, given any homogeneous norm $\|\cdot\|$, it is possible to introduce a distance in $\mathbb{G}$ given by

$$d(p,q) = d(p^{-1}q, 0) = \|p^{-1}q\| \qquad \text{for all } p, q \in G.$$

The metric d is well behaved with respect to left translations and dilations, i.e. for all $p, q, q' \in \mathbb{G}$ and $\lambda > 0$,

$$d(pq, pq') = d(q, q'), \qquad d(\delta_\lambda q, \delta_\lambda q') = \lambda d(q, q')$$

Moreover, for any bounded subset $\Omega \subset \mathbb{G}$ there exist positive constants $c_1 = c_1(\Omega), c_2 = c_2(\Omega)$ such that for all $p, q \in \Omega$

$$c_1|p-q| \leq d(p,q) \leq c_2|p-q|^{1/2},$$

and, in particular, the topology induced on $\mathbb{G}$ by d is the Euclidean topology. Moreover, the metric dimension is different w.r.t. the Euclidean one; more precisely, it is equal to the integer $\sum_{i=1,2} i \dim \mathbb{G}^i = m+2h$, called *homogeneous dimension* of $\mathbb{G}$.

2.2 - *Complementary subgroups*

A homogeneous subgroup $\mathbb{W}$ of $\mathbb{G}$ is a Lie subgroup such that $\delta_\lambda x \in \mathbb{W}$ for every $x \in \mathbb{W}$ and for all $\lambda > 0$. Homogeneous subgroups are linear subspaces of $\mathbb{R}^{m+h}$, when $\mathbb{G}$ is identified with $\mathbb{R}^{m+h}$.

Definition 2.5. We say that $\mathbb{V}$ and $\mathbb{W}$ are *complementary subgroups in* $\mathbb{G}$ if $\mathbb{V}$ and $\mathbb{W}$ are homogeneous subgroups of $\mathbb{G}$ such that $\mathbb{W} \cap \mathbb{V} = \{0\}$ and

$$\mathbb{G} = \mathbb{V}\mathbb{W}.$$

By this we mean that for every $p \in \mathbb{G}$ there are $p_\mathbb{W} \in \mathbb{W}$ and $p_\mathbb{V} \in \mathbb{V}$ such that $p = p_\mathbb{V} p_\mathbb{W}$.

If $\mathbb{V}$ and $\mathbb{W}$ are complementary subgroups of $\mathbb{G}$ and one of them is a normal subgroup then $\mathbb{G}$ is said to be the semi-direct product of $\mathbb{V}$ and $\mathbb{W}$. If both $\mathbb{V}$ and $\mathbb{W}$ are normal subgroups then $\mathbb{G}$ is said to be the direct product of $\mathbb{V}$ and $\mathbb{W}$.

The elements $p_{\mathbb{W}} \in \mathbb{W}$ and $p_{\mathbb{V}} \in \mathbb{V}$ such that $p = p_{\mathbb{V}} p_{\mathbb{W}}$ are unique because of $\mathbb{V} \cap \mathbb{W} = \{0\}$ and are denoted components of p along $\mathbb{V}$ and $\mathbb{W}$ or projections of p on $\mathbb{V}$ and $\mathbb{W}$. The projection maps $\mathbf{P}_{\mathbb{W}} : \mathbb{G} \to \mathbb{W}$ and $\mathbf{P}_{\mathbb{V}} : \mathbb{G} \to \mathbb{V}$ defined

$$\mathbf{P}_{\mathbb{W}}(p) = p_{\mathbb{W}}, \qquad \mathbf{P}_{\mathbb{V}}(p) = p_{\mathbb{V}}, \qquad \text{for all } p \in \mathbb{G},$$

are polynomial functions (see Proposition 2.2.14 in [**16**]) if we identify $\mathbb{G}$ with $\mathbb{R}^{m+h}$, hence are C^∞. Nevertheless in general they are not Lipschitz maps, when $\mathbb{V}$ and $\mathbb{W}$ are endowed with the restriction of the left invariant distance d of $\mathbb{G}$ (see Example 2.2.15 in [**16**]).

The stratification of $\mathbb{G}$ induces a stratifications on the complementary subgroups $\mathbb{V}$ and $\mathbb{W}$. If $\mathbb{G} = \mathbb{G}^1 \oplus \mathbb{G}^2$ then also $\mathbb{V} = \mathbb{V}^1 \oplus \mathbb{V}^2$, $\mathbb{W} = \mathbb{W}^1 \oplus \mathbb{W}^2$ and $\mathbb{G}^i = \mathbb{V}^i \oplus \mathbb{W}^i$. A subgroup is *horizontal* if it is contained in the first layer $\mathbb{G}^1$. If $\mathbb{V}$ is horizontal then the complementary subgroup $\mathbb{W}$ is normal. Moreover,

$$\mathbb{G} = \mathbb{V}\mathbb{W} = \mathbb{W}\mathbb{V},$$

but, obviously, the projections of any point are different.

In this paper, we choose the splitting $\mathbb{G} = \mathbb{V}\mathbb{W}$.

Proposition 2.6 ([**16**]). *If $\mathbb{V}$ and $\mathbb{W}$ are complementary subgroups in $\mathbb{G}$ there is $c_0 = c_0(\mathbb{V}, \mathbb{W}) \in (0,1)$ such that for every $p \in \mathbb{G}$, it holds*

$$c_0(\|p_{\mathbb{W}}\| + \|p_{\mathbb{V}}\|) \le \|p_{\mathbb{V}} p_{\mathbb{W}}\| \le \|p_{\mathbb{W}}\| + \|p_{\mathbb{V}}\|. \tag{2.7}$$

In the sequel, we denote by $\mathbb{V}$ and $\mathbb{W}$ two arbitrary complementary subgroups of $\mathbb{G}$ with $\mathbb{V}$ horizontal and k-dimensional. Up to choosing a proper adapted basis of the Lie algebra $\mathfrak{g}$, we may suppose that $\mathbb{V} = \exp(\text{span}\{X_1, \dots, X_k\})$ and $\mathbb{W} = \exp(\text{span}\{X_{k+1}, \dots, X_m, Y_1, \dots, Y_h\})$. Thus, by means of exponential coordinates we can identify $\mathbb{W}$ and $\mathbb{V}$ with $\mathbb{R}^{m+h-k}$ and $\mathbb{R}^k$, respectively, as follows

$$\begin{aligned} &\mathbb{V} \equiv \{(x_1, , \dots, x_k, 0 \dots, 0) \,:\, x_i \in \mathbb{R} \text{ for } i = 1, \dots, k\}, \\ &\mathbb{W} \equiv \{(0, \dots, 0, x_{k+1}, \dots, x_m, y_1, \dots, y_h) \,: \\ &\qquad\qquad x_i, y_\ell \in \mathbb{R} \text{ for } i = k+1, \dots, m;\ \ell = 1, \dots h\}. \end{aligned} \tag{2.8}$$

Once complementary subgroups as in (2.8) have been fixed, it is convenient to identify $\phi : E \subset \mathbb{V} \to \mathbb{W}$ with a function $\phi : E \subset \mathbb{R}^k \to \mathbb{R}^{m+h-k}$ in the obvious way. This identification applied to intrinsic Lipschitz functions leads to the notion of *tame maps* which we discuss in the next section, see especially Theorems 3.4 and 3.7.

2.3 - *Intrinsic graphs*

This concept was introduced by Franchi, Serapioni and Serra Cassano. The reader can see [**8**, **11**, **13**, **15**].

Definition 2.9. We say that $S \subset \mathbb{G}$ is a *left intrinsic graph* or more simply an *intrinsic graph* if there are complementary subgroups $\mathbb{W}$ and $\mathbb{V}$ in $\mathbb{G}$ and $\phi : E \subset \mathbb{V} \to \mathbb{W}$ such that

$$S = \mathrm{graph}(\phi) := \{a\phi(a) : \, a \in E\}.$$

Observe that, by uniqueness of the components along $\mathbb{V}$ and $\mathbb{W}$, if $S = \mathrm{graph}(\phi)$ then ϕ is uniquely determined among all functions from $\mathbb{V}$ to $\mathbb{W}$.

We call graph map of ϕ, the function $\Phi : E \to \mathbb{G}$ defined as

$$\Phi(a) := a\phi(a) \quad \text{for all } a \in E. \tag{2.10}$$

Hence $S = \Phi(E)$ is equivalent to $S = \mathrm{graph}(\phi)$.

The concept of intrinsic graph is preserved by translation and dilation, i.e.

Proposition 2.11 ([**16**]). *If S is an intrinsic graph then, for all $\lambda > 0$ and for all $q \in \mathbb{G}$, $q \cdot S$ and $\delta_\lambda S$ are intrinsic graphs. In particular, if $S = graph(\phi)$ with $\phi : E \subset \mathbb{V} \to \mathbb{W}$, then*

(1) *For all $\lambda > 0$,*
$$\delta_\lambda\left(graph(\phi)\right) = graph(\phi_\lambda)$$
where $\phi_\lambda : \delta_\lambda E \subset \mathbb{V} \to \mathbb{W}$ and $\phi_\lambda(a) := \delta_\lambda \phi(\delta_{1/\lambda} a)$, for $a \in \delta_\lambda E$.

(2) *For any $q \in \mathbb{G}$,*
$$q\, graph(\phi) = graph(\phi_q)$$
where $\phi_q : E_q \subset \mathbb{V} \to \mathbb{W}$ is defined as $\phi_q(a) := (\mathbf{P}_{\mathbb{W}}(q^{-1}a))^{-1}\phi(\mathbf{P}_{\mathbb{V}}(q^{-1}a))$, for all $a \in E_q := \{a \, : \, \mathbf{P}_{\mathbb{V}}(q^{-1}a) \in E\}$.

We conclude this section with the definition of intrinsic Lipschitz maps. Regarding their properties, the reader can see [**7**, **9**, **19**, **20**].

Definition 2.12. Assume that $\mathbb{V}$ and $\mathbb{W}$ are homogeneous subgroups of $\mathbb{G}$ as above. A map $\phi : E \subset \mathbb{V} \to \mathbb{W}$ is said to be *intrinsic L-Lipschitz* for a constant $L \geq 0$ if

$$\|\mathbf{P}_{\mathbb{W}}(\Phi(v')^{-1}\Phi(v))\| \leq L \|\mathbf{P}_{\mathbb{V}}(\Phi(v')^{-1}\Phi(v))\|, \quad v, v' \in E,$$

where $\Phi : E \subset \mathbb{V} \to \mathbb{G}$ is the graph map defined as (2.10).

2.4 - *Tame maps*

The notion of tame maps is studied in [**10**, **12**], when $\mathbb{G}$ is the Heisenberg group $\mathbb{H}^n$. Here we extend this definition to Carnot groups of step 2.

Definition 2.13. Let $m, h, k \in \mathbb{N}$ with $k < m$ and $h \leq \frac{m(m-1)}{2}$, $\mathcal{B}^{(1)}, \dots, \mathcal{B}^{(h)}$ be linearly independent and skew-symmetric matrices in $\mathbb{R}^{m\times m}$, $E \subset \mathbb{R}^k$, and $L_i \geq 0$ for $i \in \{k+1, \dots, m+h\}$.

We say that a map $\phi = (\phi_{k+1}, \dots, \phi_{m+h}) : E \to \mathbb{R}^{m+h-k}$ is $(L_{k+1}, \dots, L_{m+h})$-*tame* if

(1) ϕ_i is Euclidean L_i-Lipschitz for $i = k+1, \dots, m$;

(2) It holds

$$\begin{aligned} &\Big|\phi_{m+\ell}(y) - \phi_{m+\ell}(x) + \frac{1}{2}\sum_{i,j=k+1}^{m} b_{ij}^{(\ell)}\phi_j(y)\phi_i(x) - \sum_{i=k+1}^{m}\sum_{j=1}^{k} b_{ji}^{(\ell)}\phi_i(y)(y_j - x_j)\Big| \\ &+ \Big|\phi_{m+\ell}(y) - \phi_{m+\ell}(x) + \frac{1}{2}\sum_{i,j=k+1}^{m} b_{ij}^{(\ell)}\phi_j(y)\phi_i(x) - \sum_{i=k+1}^{m}\sum_{j=1}^{k} b_{ji}^{(\ell)}\phi_i(x)(y_j - x_j)\Big| \\ &\leq L_{m+\ell}|y - x|^2, \end{aligned}$$

for $\ell = 1, \dots, h$ and for all $x = (x_1, \dots, x_k), y = (y_1, \dots, y_k) \in E$.

Remark 2.14. It is possible to replace the condition (2) in Definition 2.13 (with different constant $L_{m+\ell}$ which depends on $L_{k+1}, \dots, L_m, L_{m+\ell}, \mathcal{B}^{(1)}, \dots, \mathcal{B}^{(h)}$) by its one-sided version:

$$\begin{aligned} &\Big|\phi_{m+\ell}(y) - \phi_{m+\ell}(x) + \frac{1}{2}\sum_{i,j=k+1}^{m} b_{ij}^{(\ell)}\phi_j(y)\phi_i(x) - \sum_{i=k+1}^{m}\sum_{j=1}^{k} b_{ji}^{(\ell)}\phi_i(y)(y_j - x_j)\Big| \\ &\qquad\qquad \leq K_{m+\ell}|y - x|^2, \end{aligned} \tag{2.15}$$

for $\ell = 1, \dots, h$ and for all $x = (x_1, \dots, x_k), y = (y_1, \dots, y_k) \in E$. Indeed, it is trivial fact that the condition (2) in Definition 2.13 implies (2.15). Moreover, the other implication follows using the condition (1) in Definition 2.13 and recalling that the matrices $\mathcal{B}^{(1)}, \dots, \mathcal{B}^{(h)}$ are skew-symmetric, i.e.,

$$
\begin{aligned}
&\Big|\phi_{m+\ell}(y) - \phi_{m+\ell}(x) + \frac{1}{2}\sum_{i,j=k+1}^{m} b_{ij}^{(\ell)}\phi_j(y)\phi_i(x) - \sum_{i=k+1}^{m}\sum_{j=1}^{k} b_{ji}^{(\ell)}\phi_i(y)(y_j - x_j)\Big| \\
+ &\Big|\phi_{m+\ell}(y) - \phi_{m+\ell}(x) + \frac{1}{2}\sum_{i,j=k+1}^{m} b_{ij}^{(\ell)}\phi_j(y)\phi_i(x) - \sum_{i=k+1}^{m}\sum_{j=1}^{k} b_{ji}^{(\ell)}\phi_i(x)(y_j - x_j)\Big| \\
&\le L_{m+\ell}|y-x|^2 \\
&\qquad + \Big|\phi_{m+\ell}(y) - \phi_{m+\ell}(x) + \frac{1}{2}\sum_{i,j=k+1}^{m} b_{ij}^{(\ell)}\phi_j(y)\phi_i(x) \\
&\qquad - \sum_{i=k+1}^{m}\sum_{j=1}^{k} b_{ji}^{(\ell)}(\phi_i(x) \pm \phi_i(y))(y_j - x_j)\Big| \\
&\le 2L_{m+\ell}|y-x|^2 + \Big|\sum_{i=k+1}^{m}\sum_{j=1}^{k} b_{ji}^{(\ell)}(\phi_i(x) - \phi_i(y))(y_j - x_j)\Big| \\
&\le \left(2L_{m+\ell} + \max_{j,i=1,\dots,m} b_{ji}^{(\ell)} \sum_{i=k+1}^{m} L_i\right)|y-x|^2,
\end{aligned}
$$

as desired.

Remark 2.16. For all $1 \le k \le m$, Definition 2.13 implies that $\phi_{m+\ell}$ is locally Lipschitz. This fact follows noting that

$$
\begin{aligned}
&\sum_{i,j=k+1}^{m} b_{ij}^{(\ell)}\phi_j(y)\phi_i(x) \\
&\quad = \sum_{i=k+1}^{m}\sum_{\substack{j=k+1,\dots,m \\ j>i}} b_{ij}^{(\ell)}(\phi_j(x)\phi_i(y) - \phi_j(y)\phi_i(x)) \qquad (2.17)\\
&\quad = \sum_{i=k+1}^{m}\sum_{\substack{j=k+1,\dots,m \\ j>i}} b_{ij}^{(\ell)}\big(\phi_j(x)(\phi_i(y) - \phi_i(x)) - \phi_i(x)(\phi_j(y) - \phi_j(x))\big),
\end{aligned}
$$

where, in the first equality, we used the skew-symmetry of the matrices $\mathcal{B}^{(1)}, \dots, \mathcal{B}^{(h)}$. However, to require that $\phi_{m+1}, \dots, \phi_{m+h}$ is locally Lipschitz does not implies that $\phi = (\phi_{k+1}, \dots, \phi_{m+h})$ is a tame map because if E is an unbounded set and $\phi_{m+\ell}$ does not a globally Lipschitz map the condition (2) in Definition 2.13 does not work (you can choose any map $\phi_{m+\ell}$ with unlimited derivative).

3 - Link between tame maps and intrinsic lipschitz functions

In this section, we explore the connection between intrinsic Lipschitz functions (as in Definition 2.12) and tame maps (as in Definition 2.13). This connection is motivated by [**10, 12**] where the authors prove their equivalence in the context of Heisenberg groups. Throughout this section, we assume that $1 \le k \le m$, $1 \le h \le \frac{m(m-1)}{2}$ and $\mathbb{V}$ is a k-dimensional horizontal subgroup of $\mathbb{G}$ with complementary normal subgroup $\mathbb{W}$ with coordinate expressions as in (2.8). Slightly abusing notation, we identify a set $E \subset \mathbb{V}$ with $E \subset \mathbb{R}^k$, and $\phi : E \to \mathbb{W}$ with $\phi : E \to \mathbb{R}^{m+h-k}$.

Lemma 3.1. *A function $(\phi_{k+1}, \dots, \phi_{m+h}) : E \subset \mathbb{V} \to \mathbb{W}$ is intrinsic L-Lipschitz if and only if*

$$\|(0, \dots, 0, \phi_{k+1}(v') - \phi_{k+1}(v), \dots, \phi_m(v') - \phi_m(v), H(v, v'))\| \le L|v' - v|, \quad v, v' \in E,$$

where

$$H(v, v') := (H_{m+1}(v, v'), \dots, H_{m+h}(v, v'))$$

is defined as

$$H_{m+\ell}(v, v') := \phi_{m+\ell}(v') - \phi_{m+\ell}(v) - \frac{1}{2} \sum_{i,j=k+1}^{m} b_{ij}^{(\ell)} \phi_j(v')\phi_i(v) + \sum_{i=k+1}^{m} \sum_{j=1}^{k} b_{ji}^{(\ell)} \phi_i(v)(v'_j - v_j),$$

for $\ell = 1, \dots, h$ and $v = (v_1, \dots, v_k, 0, \dots, 0), v' = (v'_1, \dots, v'_k, 0, \dots, 0) \in E$.

Proof. We recall from Definition 2.12 that ϕ is intrinsic L-Lipschitz if and only if

$$\|\mathbf{P}_{\mathbb{W}}(\Phi(v)^{-1}\Phi(v'))\| \le L\|\mathbf{P}_{\mathbb{V}}(\Phi(v)^{-1}\Phi(v'))\|, \quad v, v' \in E. \tag{3.2}$$

Moreover, because $\mathbb{W}$ is a normal subgroup, we get

$$\begin{aligned}
\mathbf{P}_{\mathbb{V}}(\Phi(v)^{-1}\Phi(v')) &= \mathbf{P}_{\mathbb{V}}(\phi(v)^{-1}v^{-1}v'\phi(v')) = \mathbf{P}_{\mathbb{V}}(\phi(v)^{-1}v^{-1}v') \\
&= \mathbf{P}_{\mathbb{V}}((v^{-1}v')(v^{-1}v')^{-1}\phi(v)^{-1}v^{-1}v') \\
&= v^{-1}v', \\
\mathbf{P}_{\mathbb{W}}(\Phi(v)^{-1}\Phi(v')) &= \mathbf{P}_{\mathbb{W}}(\phi(v)^{-1}v^{-1}v'\phi(v')) = \mathbf{P}_{\mathbb{W}}(\phi(v)^{-1}v^{-1}v')\phi(v') \\
&= \mathbf{P}_{\mathbb{W}}((v^{-1}v')(v^{-1}v')^{-1}\phi(v)^{-1}v^{-1}v')\phi(v') \\
&= (v^{-1}v')^{-1}\phi(v)^{-1}v^{-1}v'\phi(v'),
\end{aligned} \tag{3.3}$$

where we used the simply fact $(v^{-1}v')^{-1}\phi(v')(v^{-1}v') \in \mathbb{W}$ (because $\mathbb{W}$ is normal) and $v^{-1}v' \in \mathbb{V}$.

Consequently, the right-hand side of (3.2) is equal to $L|v'-v|$. On the other hand, for $v = (v_1, \dots, v_k, 0, \dots, 0), v' = (v'_1, \dots, v'_k, 0, \dots, 0) \in E$, it follows

$$
\begin{aligned}
(v^{-1}v')^{-1}\phi(v)^{-1} = \Bigg(& v - v', -\phi_{k+1}(v), \dots, -\phi_m(v), \\
& - \phi_{m+1}(v) + \frac{1}{2}\sum_{i=k+1}^{m}\sum_{j=1}^{k} b_{ij}^{(1)}\phi_i(v)(v_j - v'_j), \dots, \\
& - \phi_{m+h}(v) + \frac{1}{2}\sum_{i=k+1}^{m}\sum_{j=1}^{k} b_{ij}^{(h)}\phi_i(v)(v_j - v'_j)\Bigg),
\end{aligned}
$$

and, recall that $(x,y)^{-1} = (-x,-y)$, we also obtain the explicit form of $v^{-1}v'\phi(v')$. Then, by an easy computation, we have that the left-hand side of (3.2) can be written as

$$
\begin{aligned}
&\|\mathbf{P}_{\mathbb{W}}(\Phi(v)^{-1}\Phi(v'))\| \\
&\qquad = \|(0, \dots, 0, \phi_{k+1}(v') - \phi_{k+1}(v), \dots, \phi_m(v') - \phi_m(v), H(v,v'))\|,
\end{aligned}
$$

for $v, v' \in E$, where $H(v,v')$ is defined as in the statement of the lemma. □

Lemma 3.1 provides a link between intrinsic Lipschitz and tame maps. We formulate this in two separate statements because, as we will see, the Lipschitz constants change.

Theorem 3.4. *If $\phi = (\phi_{k+1}, \dots, \phi_{m+h})\colon E \subset \mathbb{V} \to \mathbb{W}$ is intrinsic L-Lipschitz, then $(\phi_{k+1}, \dots, \phi_m, -\phi_{m+1}, \dots, -\phi_{m+h})$ is an $(L_{k+1}, \dots, L_{m+h})$-tame map from $E \subset \mathbb{R}^k$ to $\mathbb{R}^{m+h-k}$ with*

$$
L_i = \begin{cases} L, & \text{for } i = k+1, \dots, m, \\ \dfrac{2}{\varepsilon^2}L^2, & \text{for } i = m+1, \dots, m+h, \end{cases} \tag{3.5}
$$

where $\varepsilon \in (0,1]$ is given by (2.4).

Proof. Let ϕ be an intrinsic L-Lipschitz function. According to Lemma 3.1 this means that

$$
\begin{aligned}
&\|(0, \dots, 0, \phi_{k+1}(v') - \phi_{k+1}(v), \dots, \phi_m(v') - \phi_m(v), H(v,v'))\| \\
&\qquad\qquad \le L|v'-v|, \quad v, v' \in E,
\end{aligned} \tag{3.6}
$$

where $H(v,v') := (H_{m+1}(v,v'),\dots,H_{m+h}(v,v'))$ is defined as

$$H_{m+\ell}(v,v') := \phi_{m+\ell}(v') - \phi_{m+\ell}(v) - \frac{1}{2}\sum_{i,j=k+1}^{m} b_{ij}^{(\ell)}\phi_j(v')\phi_i(v) + \sum_{i=k+1}^{m}\sum_{j=1}^{k} b_{ji}^{(\ell)}\phi_i(v)(v'_j - v_j),$$

for $\ell = 1,\dots,h$ and $v,v' \in E$.

Recalling that $\|(x,y)\| = \max\{|x|, \varepsilon\sqrt{|y|}\}$ for $(x,y) \in \mathbb{R}^m \times \mathbb{R}^h$ and $\varepsilon \in (0,1]$, inequality (3.6) implies first that ϕ_i is a Euclidean L-Lipschitz function for $i = k+1,\dots,m$, which is part (1) of the tameness condition in Definition 2.13. Second, we deduce from (3.6) that

$$\left|\phi_{m+\ell}(v') - \phi_{m+\ell}(v) - \frac{1}{2}\sum_{i,j=k+1}^{m} b_{ij}^{(\ell)}\phi_j(v')\phi_i(v) + \sum_{i=k+1}^{m}\sum_{j=1}^{k} b_{ji}^{(\ell)}\phi_i(v)(v'_j - v_j)\right|^{1/2} \leq \frac{L}{\varepsilon}|v'-v|,$$

for $\ell = 1,\dots,h$ and $v = (v_1,\dots,v_k,0,\dots,0), v = (v'_1,\dots,v'_k,0,\dots,0) \in E$.

Hence, recall Remark 2.14, $(\phi_2,\dots,\phi_m,-\phi_{m+1},\dots,-\phi_{m+h})$ is $(L,\dots L, \frac{2}{\varepsilon^2}L^2,\dots,\frac{2}{\varepsilon^2}L^2)$-tame in both cases and the proof of the statement is complete. $\square$

We now consider the converse implication.

Theorem 3.7. *If* $(\phi_{k+1},\dots,\phi_m,-\phi_{m+1},\dots,-\phi_{m+h})\colon E \subset \mathbb{R}^k \to \mathbb{R}^{m+h-k}$ *is an* $(L_{k+1},\dots,L_{m+h})$*-tame map, then* $\phi = (\phi_{k+1},\dots,\phi_{m+h}) : E \subset \mathbb{V} \to \mathbb{W}$ *is intrinsic* L*-Lipschitz with*

$$L := \max\left\{|(L_{k+1},\dots,L_m)|, \varepsilon\sqrt{|(L_{m+1},\dots,L_{m+h})|}\right\},$$

where $\varepsilon \in (0,1]$ *is given by* (2.4).

Proof. If $(\phi_{k+1},\dots,\phi_m,-\phi_{m+1},\dots,-\phi_{m+h})$ is $(L_{k+1},\dots,L_{m+h})$-tame, we find by the first condition in Definition 2.13 that for $i = k+1,\dots,m$, the function ϕ_i is Euclidean L_i-Lipschitz on E. Moreover, the second condition in the tameness definition for

$(\phi_{k+1},\dots,\phi_m,-\phi_{m+1},\dots,-\phi_{m+h})$ reads as follows:

$$\Big|\phi_{m+\ell}(v')-\phi_{m+\ell}(v)+\frac{1}{2}\sum_{i,j=k+1}^{m} b_{ij}^{(\ell)}\phi_j(v')\phi_i(v)-\sum_{i=k+1}^{m}\sum_{j=1}^{k} b_{ji}^{(\ell)}\phi_i(v')(v_j'-v_j)\Big|$$
$$+\Big|\phi_{m+\ell}(v')-\phi_{m+\ell}(x)+\frac{1}{2}\sum_{i,j=k+1}^{m} b_{ij}^{(\ell)}\phi_j(v')\phi_i(v)-\sum_{i=k+1}^{m}\sum_{j=1}^{k} b_{ji}^{(\ell)}\phi_i(v)(v_j'-v_j)\Big|$$
$$\le L_{m+\ell}|v'-v|^2,$$

for every $\ell=1,\dots,h$ and for all $v=(v_1,\dots,v_k,0,\dots,0), v=(v_1',\dots,v_k',0,\dots,0)\in E$.

Using Lemma 3.1 and in particular (3.3), we conclude that $\phi:=(\phi_{k+1},\dots,\phi_{m+h}):E\subset\mathbb{V}\to\mathbb{W}$ is an intrinsic L-Lipschitz function since its graph map satisfies

$$\begin{aligned}\|\mathbf{P}_{\mathbb{W}}(\Phi(v')^{-1}\Phi(v))\| &\le \max\left\{|(L_{k+1},\dots,L_m)|,\varepsilon\sqrt{|L_{m+1},\dots,L_{m+h}|}\right\}|v-v'|\\ &= L|v-v'|=L\|\mathbf{P}_{\mathbb{V}}(\Phi(v)^{-1}\Phi(v'))\|,\quad v,v'\in E,\end{aligned}$$

as desired. $\square$

4 - Infinitesimal condition for tame maps on open sets

There is an equivalent condition of tame maps in terms of an infinitesimal one where the "vertical" constants depend on the structure of the group (see (4.3)).

In this section, we consider the case $k=1$.

Proposition 4.1. *Let $I\subset\mathbb{R}$ be an open interval, and let $\phi=(\phi_2,\dots,\phi_{m+h}):I\to\mathbb{R}^{m+h-1}$.*

(1) *If ϕ is $(L_2,\dots,L_{m+h})$-tame, then ϕ_i is L_i-Lipschitz for $i=2,\dots,m$, and $\phi_{m+1},\dots,\phi_{m+h}$ are differentiable almost everywhere on I, $\dot\phi_{m+\ell}\in\mathcal{L}^\infty_{loc}(I)$ for $\ell=1,\dots,h$, and*

$$\dot\phi_{m+\ell}=\sum_{i=2}^{m} b_{1i}^{(\ell)}\phi_i-\frac{1}{2}\sum_{i=2}^{m}\sum_{j>i} b_{ij}^{(\ell)}(\phi_j\dot\phi_i-\phi_i\dot\phi_j),\quad \text{a.e. on } I, \tag{4.2}$$

for $\ell=1,\dots,h$.

(2) *Conversely, if ϕ_i is L_i-Lipschitz for $i=2,\dots,m$, $\phi_{m+\ell}$ is locally Lipschitz for $\ell=1,\dots,h$, and (4.2) holds for every $\ell=1,\dots,h$, then ϕ is*

$(L'_2, \dots, L'_{m+h})$-*tame with*

$$
\begin{aligned}
&L'_i := L_i && \textit{for } i = 2, \dots, m, \\
(4.3) \qquad &L'_{m+\ell} := 2\mathcal{B}_M \left(\sum_{i=2}^{m} L_i + \sum_{i=2}^{m} \sum_{j>i} L_i L_j \right), && \textit{for } \ell = 1, \dots, h,
\end{aligned}
$$

where $\mathcal{B}_M = \max\{b_{ij}^{(\ell)} \, : \, i, j = 1, \dots, m, \ell = 1, \dots, h\}$.

Remark 4.4. We underline that $\mathcal{B}_M > 0$ because the matrices $\mathcal{B}^{(1)}, \dots, \mathcal{B}^{(h)}$ are skew-symmetric.

Proof. We assume first that ϕ is $(L_2, \dots, L_{m+h})$-tame, in particular, ϕ_i is a Lipschitz function on I for $i = 2, \dots, m$. Rademacher's theorem implies that ϕ_i is differentiable almost everywhere on I with bounded derivative. Condition (2) in Definition 2.13 reads

(4.5)

$$
\begin{aligned}
&\left| \frac{\phi_{m+\ell}(y) - \phi_{m+\ell}(x)}{y - x} - \sum_{i=2}^{m} b_{1i}^{(\ell)} \phi_i(y) + \frac{1}{2} \sum_{i=2}^{m} \sum_{j>i} b_{ij}^{(\ell)} \frac{\phi_i(y)\phi_j(x) - \phi_i(x)\phi_j(y)}{y - x} \right| \\
+ &\left| \frac{\phi_{m+\ell}(y) - \phi_{m+\ell}(x)}{y - x} - \sum_{i=2}^{m} b_{1i}^{(\ell)} \phi_i(x) + \frac{1}{2} \sum_{i=2}^{m} \sum_{j>i} b_{ij}^{(\ell)} \frac{\phi_i(y)\phi_j(x) - \phi_i(x)\phi_j(y)}{y - x} \right| \\
&\qquad \le L_{m+\ell} |y - x|,
\end{aligned}
$$

for $\ell = 1, \dots, h$ and for all $x, y \in I$ with $x \neq y$. By

$$
(4.6) \quad \phi_i(y)\phi_j(x) - \phi_i(x)\phi_j(y) = \phi_i(y)(\phi_j(x) - \phi_j(y)) - \phi_j(y)(\phi_i(x) - \phi_i(y)),
$$

it is easy to see that $\dot{\phi}_{m+\ell}$ exists almost everywhere on I for $\ell = 1, \dots, h$, and (4.2) holds. In particular, $\dot{\phi}_{m+\ell} \in \mathcal{L}^\infty_{loc}(I)$ for $\ell = 1, \dots, h$, as desired.

Conversely, assume that ϕ_i is an L_i–Lipschitz function for $i = 2, \dots, m$ and $\phi_{m+\ell}$ is a locally Lipschitz function satisfying (4.2) for every $\ell = 1, \dots, h$. Then, the corresponding one-sided version of (4.5) is satisfied for

$$
\text{“}2\mathcal{B}_M \left(\sum_{i=2}^{m} L_i + \sum_{i=2}^{m} \sum_{j>i} L_i L_j \right) \text{”} \qquad \text{instead of} \qquad \text{“}L_{m+\ell}\text{”}.
$$

Indeed, for $x, y \in I$ with $x < y$, the expression (4.6) can be rewritten as

$$\phi_i(y)\phi_j(x) - \phi_i(x)\phi_j(y) = -\phi_i(y)\int_x^y \dot{\phi}_j(s)\,ds + \phi_j(y)\int_x^y \dot{\phi}_i(s)\,ds,$$

and, according to Remark 2.14, it suffices to verify the one-sided version of (4.5) (without the constant "2"). Hence,

$$\begin{aligned}
&\Bigg|\phi_{m+\ell}(y) - \phi_{m+\ell}(x) - \sum_{i=2}^{m} b_{1i}^{(\ell)}\phi_i(y)(y-x) \\
&\qquad + \frac{1}{2}\sum_{i=2}^{m}\sum_{j>i} b_{ij}^{(\ell)}(\phi_i(y)\phi_j(x) - \phi_i(x)\phi_j(y))\Bigg| \\
&\overset{(4.6)}{=} \Bigg|\int_x^y \dot{\phi}_{m+\ell}(s)\,ds - \sum_{i=2}^{m} b_{1i}^{(\ell)}\int_x^y \phi_i(y)\,ds \\
&\qquad + \tfrac{1}{2}\sum_{i=2}^{m} b_{ij}^{(\ell)}\phi_j(y)\int_x^y \dot{\phi}_i(s)\,ds - b_{ij}^{(\ell)}\phi_i(y)\int_x^y \dot{\phi}_j(s)\,ds\Bigg| \\
&\overset{(4.2)}{=} \Bigg|\int_x^y \sum_{i=2}^{m} b_{1i}^{(\ell)}(\phi_i(s) - \phi_i(y)) \\
&\qquad + \tfrac{1}{2}\sum_{i=2}^{m}\sum_{j>i} b_{ij}^{(\ell)}\left(\dot{\phi}_i(s)[\phi_j(y) - \phi_j(s)] + \dot{\phi}_j(s)[\phi_i(s) - \phi_i(y)]\,ds\right)\Bigg| \\
&\le \mathcal{B}_M\left(\sum_{i=2}^{m} L_i + \sum_{i=2}^{m}\sum_{j>i} L_iL_j\right)|y-x|^2,
\end{aligned}$$

where in the last inequality we used the fact that ϕ_i is L_i-Lipschitz for every $i = 2, \dots, m$. As a consequence, the proof is complete. □

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