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Typical labels of real forms

Abstract. Let $X(\mathbb{C}) \subset \mathbb{P}^r(\mathbb{C})$ be an integral projective variety defined over $\mathbb{R}$. Let σ denote the complex conjugation. A point $q \in \mathbb{P}^r(\mathbb{R})$ is said to have $(a, b) \in \mathbb{N}^2$ as a label if there is $S \subset X(\mathbb{C})$ such that $\sigma(S) = S$, S spans q, $\#S = 2a + b$ and $\#(S \cap X(\mathbb{R})) = b$. We say that (a, b) has weight $2a + b$. A label-weight t is typical for the k-secant variety $\sigma_k(X(\mathbb{C}))$ of $X(\mathbb{C})$ if there is a non-empty euclidean open subset V of $\sigma_k(X(\mathbb{C}))(\mathbb{R})$ such that all $q \in V$ have a label of weight t and no label of weight $< t$. The integer k is always the minimal label-weight of $\sigma_k(X(\mathbb{C}))(\mathbb{R})$ if $\sigma_{k-1}(X(\mathbb{C})) \neq \mathbb{P}^r(\mathbb{C})$. In this paper $X(\mathbb{C}) = X_{n,d}(\mathbb{C})$ is the order d Veronese embedding of $\mathbb{P}^n(\mathbb{C})$. We prove that k and $k+1$ are the typical label-weights of $\sigma_k(X(\mathbb{C}))(\mathbb{R})$ if $(n, d, k) \in \{(2, 6, 9), (3, 4, 8), (5, 3, 9), (2, 4, 5), (4, 3, 7)\}$. These examples are important, because the first 3 are the ones in which generic uniqueness for proper secant varieties fails for the k-secant variety (a theorem by Chiantini, Ottaviani and Vannieuwenhoven), the fourth is in the Mukai list (fano 3-fold V_{22}) and the last one appears in the Alexander-Hirschowitz list of exceptional secant varieties of Veronese embeddings.

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1 - Introduction

Let $X(\mathbb{C}) \subset \mathbb{P}^r(\mathbb{C})$ be an integral and non-degenerate projective variety. For any $q \in \mathbb{P}^r(\mathbb{C})$ the $X(\mathbb{C})$-*rank* $r_{X(\mathbb{C})}(q)$ of q is the minimal cardinality of a set $S \subset \mathbb{P}^r(\mathbb{C})$ whose linear span contains q. Now assume that both $X(\mathbb{C})$ and the embedding of $X(\mathbb{C})$ in $\mathbb{P}^r(\mathbb{C})$ are defined over $\mathbb{R}$ and that $X(\mathbb{R})$ is Zariski

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dense in $X(\mathbb{C})$. In this case the set $X(\mathbb{R})$ spans $\mathbb{P}^r(\mathbb{R})$. For each $q \in \mathbb{P}^r(\mathbb{R})$ the $X(\mathbb{R})$*-rank* $r_{X(\mathbb{R})}(q)$ of q is the minimal cardinality of a subset of $X(\mathbb{R})$ spanning q. Obviously $r_{X(\mathbb{C})}(q) \le r_{X(\mathbb{R})}(q)$ for all $q \in \mathbb{P}^n(\mathbb{R})$. There is a Zariski dense subset U of $\mathbb{P}^r(\mathbb{C})$ such that all $q \in U$ have the same $X(\mathbb{C})$-rank, called the *generic* $X(\mathbb{C})$*-rank*. Note that $\mathbb{P}^r(\mathbb{R})$ minus a hypersurface may have several connected components for the euclidean topology. Thus there may be several integers, each of them the $X(\mathbb{R})$-rank of a non-empty euclidean open of $\mathbb{P}^r(\mathbb{R})$ ([**4**,**5**,**9**,**10**,**11**,**12**,**17**,**18**]). These integers are called the *typical ranks of* $X(\mathbb{R})$. The minimal typical rank is the generic rank. The set of all typical ranks of $X(\mathbb{R})$ is connected, i.e. if $a < b$ are typical ranks of $X(\mathbb{R})$, then all integers between a and b are typical ([**9**, Theorem 1.1]). Typical ranks may be quite large. For instance, if $X(\mathbb{R})$ and $X(\mathbb{C})$ are the rational normal curve of the r-dimensional space, then the generic rank is $\lfloor (r+2)/2 \rfloor$, while all integers between $\lfloor (r+2)/2 \rfloor$ and r are typical ([**10**]).

A different definition of ranks for points in $\mathbb{P}^r(\mathbb{R})$ was studied in [**6**, **7**, **8**]. On any variety $Y(\mathbb{C})$ defined over $\mathbb{R}$ let $\sigma : Y(\mathbb{C}) \to Y(\mathbb{C})$ denote the complex conjugation. Note that $\sigma : Y(\mathbb{C}) \to Y(\mathbb{C})$ is an anti-holomorphic involution with $Y(\mathbb{R})$ as the set of its fixed point. A finite set $S \subset X(\mathbb{C})$ is said to have a *label* if $\sigma(S) = S$. We say that (a, b) is the label of S if $b = \#(S \cap X(\mathbb{R}))$ and $\#S = b + 2a$. The integer $2a + b$ is the *weight* of (a, b). For any $q \in \mathbb{P}^r$ we say that the *admissible rank* $\ell_{X(\mathbb{C})}(q)$ of q is the minimal weight of a set $S \subset X(\mathbb{C})$ with a label and spanning q ([**8**, Definition 2.2]). Obviously

$$r_{X(\mathbb{C})}(q) \le \ell_{X(\mathbb{C})}(q) \le r_{X(\mathbb{R})}(q)$$

for all $q \in \mathbb{P}^r(\mathbb{R})$. The pair (a, b) is a *typical label* if there is a non-empty euclidean open subset V of $\mathbb{P}^r(\mathbb{R})$ such that all $q \in V$ have (a, b) as a label and no label of lower weight. The *typical weights* or *typical label-weights* of $\mathbb{P}^r(\mathbb{R})$ with respect to $X(\mathbb{C})$ are the weights of all typical labels. The set of all typical weights of $X(\mathbb{C})$ is connected (see [**8**, Corollaries 3.9 and 3.11] for more general results.) Under a weak assumption (not always satisfied) we prove that no integer $\ge k+2$ is a typical label-weight (Lemma 2.2). There is an important (but very particular) example in which $k+1$ is not a typical label-weight, the rational normal curve ([**8**, Corollary 4.2]) and the smooth space curves of degree d and genus g with $(d-1)(d-2)/2 - g$ odd ([**8**, Theorem 3.5]). There is an example in which $k+1$ is a typical label-weight, the linearly normal elliptic curve ([**8**, Theorem 3.4]).

Let $\sigma_k(X(\mathbb{C})) \subseteq \mathbb{P}^r(\mathbb{C})$ denote the k-th secant variety of $X(\mathbb{C})$, i.e. the closure of the union of all linear subspaces of $\mathbb{P}^r(\mathbb{C})$ spanned by k points of $X(\mathbb{C})$. Let $\sigma_k(X(\mathbb{C}))(\mathbb{R}) := \sigma_k(X(\mathbb{C})) \cap \mathbb{P}^r(\mathbb{R})$ denote the real part of $\sigma_k(X(\mathbb{C}))$. A label (a, b) is typical for $\sigma_k(X(\mathbb{C}))(\mathbb{R})$ if there is a non-empty euclidean open subset V of $\sigma_k(X(\mathbb{C}))(\mathbb{R})$ such that all $q \in V$ have (a, b) as a label and no label of weight $< 2a + b$.

In this note we study the typical labels associated to the Veronese order d embeddings $\nu_d : \mathbb{P}^n(\mathbb{C}) \to \mathbb{P}^r(\mathbb{C})$, $r = \binom{n+d}{n} - 1$. Set $X_{n,d} := \nu_d(X)$. We prove that the typical label-weight for $\sigma_k(X_{n,d}(\mathbb{C}))(\mathbb{R})$ are k and $k+1$ in the following cases:

(1) $n = 2$, $d = 6$, $k = 9$;

(2) $n = 3$, $d = 4$, $k = 8$;

(3) $n = 5$, $d = 3$, $k = 9$;

(4) $n = 2$, $d = 4$, $k = 5$;

(5) $n = 4$, $d = 3$, $k = 7$.

We looked at cases (1), (2) and (3) (Theorem 2.5), because L. Chiantini, G. Ottaviani proved that they are the only cases (with $d \geq 3$) of generic non-uniqueness for subgeneric secant varieties of Veronese varieties ([**13**]). Case (4) is the last proper secant variety for one case in the Alexander-Hirschowitz list of defective secant varieties of Veronese varieties ([**2**, **3**]). Case (5) is the cubic case in the Alexander-Hirschowitz list of defective secant varieties of Veronese varieties, i.e. $\sigma_7(X_{4,3})$ is a hypersurface of $\mathbb{P}^{35}$ ([**2**, **3**]).

To prove case (4) (Theorem 2.6) we use the following concepts for varieties defined over an arbitrary algebraically closed field. Fix $q \in \mathbb{P}^r(\mathbb{C})$. Let $\mathcal{S}(X(\mathbb{C}), q)$ denote the set of all $S \subset X(\mathbb{C})$ such that $\#S = r_{X(\mathbb{C})}(q)$ and q is contained in the linear span $\langle S \rangle_{\mathbb{C}}$ of S. For any $q \in \mathbb{P}^r(\mathbb{C})$ set $E(X(\mathbb{C}), q) := \cup_{S \in \mathcal{S}(X(\mathbb{C}), q)} S \subseteq X(\mathbb{C})$ and let $F(X(\mathbb{C}), q)$ be the closure of $E(X(\mathbb{C}), q)$ in $X(\mathbb{C})$. Since $\mathcal{S}(X(\mathbb{C}), q)$ is a constructible subset of $X(\mathbb{C})$, a theorem of Chevalley gives that $E(X(\mathbb{C}), q)$ is a constructible subset of $X(\mathbb{C})$ for the Zariski topology ([**16**, Ex. II.3.18 and Ex. II.3.19]). Thus the closure of $E(X(\mathbb{C}), q)$ in $X(\mathbb{C})$ is the same if we take the closure for the Zariski topology or the euclidean topology and $F(X(\mathbb{C}), q)$ is a projective variety, possibly reducible. Seldom the algebraic set $F(X(\mathbb{C}), q)$ uniquely determines q, but in the proof of Theorem 2.6 the set $F(X(\mathbb{C}), q)$ is a smooth conic and we were able to see all $q' \in \mathbb{P}^r(\mathbb{C})$ such that $F(X(\mathbb{C}), q') = F(X(\mathbb{C}), q)$.

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2 - The proofs

On any variety $Y(\mathbb{C})$ defined over $\mathbb{R}$ let $\sigma : Y(\mathbb{C}) \to Y(\mathbb{C})$ denote the complex conjugation.

For any $S \subset \mathbb{P}^r(\mathbb{C})$ let $\langle S\rangle_{\mathbb{C}}$ denote the linear span of S. If $\sigma(S) = S$, then $\langle S\rangle_{\mathbb{C}} \cap \mathbb{P}^r(\mathbb{R})$ is an $\mathbb{R}$-projective subspace of $\mathbb{P}^r(\mathbb{R})$ of dimension $\dim_{\mathbb{C}}\langle S\rangle_{\mathbb{C}}$.

Remark 2.1. Fix $q \in \mathbb{P}^r(\mathbb{R})$ and set $x := r_{X(\mathbb{C})}(q)$ Since $\sigma(q) = q$, σ acts on the constructible set $\mathcal{S}(X(\mathbb{C}), q)$. The point q has a label of weight x if and only if this action of σ has at least one fixed point. Let $X(\mathbb{C}) \subset \mathbb{P}^r$ be a smooth curve defined over $\mathbb{R}$ and take r odd. Set $k := (r + 1)/2$. Set $d := \deg(X(\mathbb{C}))$ and $g := p_a(X(\mathbb{C}))$. Thus k is the generic complex rank of $X(\mathbb{C})$ ([**1**, Remark 1.6]). Fix a general $q \in \mathbb{P}^{2k+1}$. Since $\mathcal{S}(X(\mathbb{C}), q)$ is a finite set, this action of σ has a fixed point if $\#\mathcal{S}(X(\mathbb{C}), q)$ is odd. Now assume $d \geq 2g + 2k + 1$ and that $X(\mathbb{C})$ is a general linear projection of a linearly normal degree d embedding of $X(\mathbb{C})$. Under these assumptions the integer $\#\mathcal{S}(X(\mathbb{C}), q)$ is the integer $\deg(\Sigma_k)$ given in [**15**, part (1) of Prop. 5.10]. For instance if $g = 0$ we get $\#\mathcal{S}(X(\mathbb{C}), q) = \binom{d-k}{k+1}$. If $k = 2$ we get that $\#\mathcal{S}(X(\mathbb{C}), q)$ is odd if and only if $d \equiv 5 \pmod 4$.

The proof of the following lemma mimics the proof of [**6**, Theorem 1.4].

Lemma 2.2. *Fix an integer $k \geq 2$ such that $\sigma_k(X(\mathbb{C})) \neq \mathbb{P}^r(\mathbb{C})$ and generic uniqueness holds for $\sigma_{k-1}(X(\mathbb{C}))$. Then $k + 2$ is not a label-weight of X.*

Proof. Set $n := \dim X(\mathbb{C})$. Since generic uniqueness holds for $\sigma_{k-1}(X(\mathbb{C}))$, the variety $\sigma_{k-1}(X(\mathbb{C}))$ has the expected dimension $(n + 1)(k - 1) - 1$. Let $\mathcal{U}$ be the set of all $q \in \mathbb{P}^r(\mathbb{C})$ with rank $k - 1$ and such that $\#\mathcal{S}(X(\mathbb{C}), q) = 1$. By a theorem of Chevalley the set $\mathcal{U}$ is constructible ([**16**, Ex. II.3.18 and Ex. II.3.19]). By assumption the constructible set $\mathcal{U}$ contains a non-empty Zariski open subset of $\sigma_{k-1}(X(\mathbb{C}))$ and hence $\dim \sigma_{k-1}(X(\mathbb{C})) \setminus \mathcal{U} \leq (n+1)(k-1)-2$. Since the embedding of $X(\mathbb{C})$ is defined over $\mathbb{R}$, $\sigma(\mathcal{U}) = \mathcal{U}$, i.e. $\mathcal{U}$ is defined over $\mathbb{R}$. If $q \in \mathcal{U} \cap \mathbb{P}^r(\mathbb{R})$ and $S \in \mathcal{S}(X(\mathbb{C}), q)$, then $\sigma(S) = S$, because $\sigma(q) = q$ and $\mathcal{S}(X(\mathbb{C}), q) = \{S\}$. Set $\mathcal{V} := \mathcal{U} \cap \mathbb{P}^r(\mathbb{R})$. Since $\mathcal{U}$ is constructible, the semialgebraic set $\sigma_{k-1}(X(\mathbb{C}))(\mathbb{R}) \setminus \mathcal{V}$ has real dimension $\leq (n+1)(k-1)-2$. A Zariski dense constructible subset Γ of $\sigma_k(X(\mathbb{C}))$ is obtained taking the union of all complex lines $\langle\{x, y\}\rangle_{\mathbb{C}}$ with $x \in \mathcal{V}$ and $y \in X(\mathbb{C}) \setminus X(\mathbb{R})$ ([**6**, Claims 1 and 2 of Remark 2.2]). Since Γ is constructible and Zariski dense in $\sigma_k(X(\mathbb{C}))$, it contains a non-empty Zariski open subset of $\sigma_k(X(\mathbb{C}))$. Thus we get a Zariski dense open subset of $\sigma_k(X(\mathbb{C}))(\mathbb{R})$ taking the intersection with $\sigma_k(X(\mathbb{C}))(\mathbb{R})$ of the union of all planes $\langle\{x, y, \sigma(y)\rangle_{\mathbb{C}}$ with $x \in \mathcal{V}$ and $y \in X(\mathbb{C}) \setminus X(\mathbb{R})$. The complex plane $\langle\{x, y, \sigma(y)\rangle_{\mathbb{C}}$ is defined over $\mathbb{R}$ and hence $\langle\{x, y, \sigma(y)\rangle_{\mathbb{C}} \cap \mathbb{P}^r(\mathbb{R})$ is a real plane. Any $q \in \langle\{x, y, \sigma(y)\rangle_{\mathbb{C}} \cap \mathbb{P}^r(\mathbb{R})$ has a label of weight $k + 1$. □

Remark 2.3. Take $n = 2$ and $d = 7$. A general $q \in \sigma_{12}(X_{2,7}(\mathbb{C}))$ satisfies $\#\mathcal{S}(X_{2,7}(\mathbb{C}), q) = 5$ ([**14**], [**20**, Theorem 3.1]). Since 5 is odd, we get that

$X_{2,7}(\mathbb{C})$ has only typical labels of weight 12 and obviously all (a,b) with $2a+b=12$ are typical labels ([**8**, Proposition 3.2]).

Remark 2.4. Fix $n\geq 2$, $d\geq 3$ such that (n,d) is not in the Alexander-Hirschowitz list and take a positive integer k such that $k(n+1)<\binom{n+d}{n}$. Set $r:=\binom{n+n}{n}-1$ and $X(\mathbb{C}):=X_{n,d}(\mathbb{C})$. Uniqueness holds for a general $q\in\sigma_k(X(\mathbb{C}))$ unless (n,d,k) is in this list:

(1) $n=2$, $d=6$, $k=9$;

(2) $n=3$, $d=4$, $k=8$;

(3) $n=5$, $d=3$, $k=9$.

In each of these 3 exceptional cases $\#\mathcal{S}(X(\mathbb{C}),q)=2$ for a general $q\in\sigma_k(X(\mathbb{C}))$ ([**13**, Theorem 1.1]).

Theorem 2.5. *Take (n,d,k) as in one of the 3 cases of Remark 2.4. Set $X(\mathbb{C}):=X_{n,d}(\mathbb{C})$. The typical weights of $\sigma_k(X(\mathbb{C}))(\mathbb{R})$ are k and $k+1$.*

Proof. The integer k is always the minimal typical weight for a k-secant variety. Since uniqueness holds for a general element of $\sigma_{k-1}(X(\mathbb{C}))$, no typical label has weight $\geq k+2$ (Lemma 2.2). Thus it is sufficient to prove that we also need some weight $k+1$ label. We adapt the proof of [**8**, Theorem 3.4]. There is a non-empty Zariski open subset V of $\sigma_k(X(\mathbb{C}))$ such that $\#\mathcal{S}(X(\mathbb{C}),q)=2$ for all $q\in V$ and (since $\sigma_k(X(\mathbb{C})(\mathbb{R}))$ has the expected dimension $k(n+1)-1$) $\langle A\rangle_{\mathbb{C}}\cap\langle B\rangle_{\mathbb{C}}=\{q\}$, for all $q\in V$, where $\{A,B\}=\mathcal{S}(X(\mathbb{C}),q)$. Since $\mathbb{P}^r(\mathbb{R})$ is Zariski dense in $\mathbb{P}^r(\mathbb{C})$, there exists $q\in V\cap\mathbb{P}^r(\mathbb{R})$ and the set of all such q is Zariski dense in $\mathbb{P}^r(\mathbb{C})$. Write $\mathcal{S}(X(\mathbb{C}),q)=\{A,B\}$. Since $\sigma(q)=q$, $\sigma(\mathcal{S}(X(\mathbb{C}),q))=\mathcal{S}(X(\mathbb{C}),q)$. Thus either $\sigma(A)=A$ or $\sigma(A)=B$. Take q such that $\sigma(A)=B$ and hence $\sigma(B)=A$. Therefore q has no label of weight k. Therefore it is sufficient to prove that the set of all q such that $\sigma(A)=B$ contains a non-empty euclidean open subset of $\mathbb{P}^r(\mathbb{R})$. A general $S\subset X(\mathbb{C})$ with cardinality k satisfies $\sigma(S)\cap S=\emptyset$.

Claim 1: For a general S the linear space $\langle S\rangle_{\mathbb{C}}\cap\langle\sigma(S)\rangle_{\mathbb{C}}$ is a single point, q_S.

Proof of Claim 1: Claim 1 is equivalent to $\langle S\cup\sigma(S)\rangle_{\mathbb{C}}=\mathbb{P}^r(\mathbb{C})$. For a general $p\in X(\mathbb{C})$ we have $\sigma(p)\neq p$ and hence $\langle\{p,\sigma(p)\rangle_{\mathbb{C}}$ is a line. Assume $\langle S\cup\sigma(S)\rangle_{\mathbb{C}}\neq\mathbb{P}^r(\mathbb{C})$ and call s the maximal integer $<k$ such that $\dim\langle S'\cup\sigma(S')\rangle_{\mathbb{C}}=2s-1$ for a general $S'\subset X(\mathbb{C})$ with $\#S'=s$. Since $X(\mathbb{C})$ spans $\mathbb{P}^r(\mathbb{C})$, a general $p\in X(\mathbb{C})$ is not contained in the complex linear space $W:=\langle S'\cup\sigma(S')\rangle_{\mathbb{C}}$. The complex linear space W is defined over $\mathbb{R}$. By the

definition of s for a general $p \in X(\mathbb{C})$ the line $\langle\{p, \sigma(p)\}\rangle_{\mathbb{C}}$ meets W. Thus the rational map u_W from $X(\mathbb{C})$ to $\mathbb{P}^{r-2s}(\mathbb{C})$ satisfies $u_W(p) = u_W(\sigma(p))$ for a general $p \in X(\mathbb{C})$. Set $U_W(\mathbb{C}) := X(\mathbb{C}) \setminus X(\mathbb{C}) \cap W$. Since $X(\mathbb{C})$ is non-degenerate, $U_W(\mathbb{C})$ is a non-empty Zariski open subset of $X(\mathbb{C})$ defined over $\mathbb{R}$. The rational map u_W induces a morphism on U_W. Let $X_W(\mathbb{C})$ denote the closure of $u_W(U_W(\mathbb{C}))$ in $\mathbb{P}^{r-2s}(\mathbb{C})$. Since $X(\mathbb{C})$ is non-degenerate, $X_W(\mathbb{C})$ spans $\mathbb{P}^{r-2s}$. Since $r > 2s$, $\dim X_W(\mathbb{C}) > 0$. Since $u_W(p) = u_W(\sigma(p))$ for a general $p \in X(\mathbb{C})$, we have $X_W(\mathbb{R}) = X_W(\mathbb{C})$, absurd.

Note that $q_S \in \mathbb{P}^r(\mathbb{R})$ and that by Claim 1 and the fact that $\#\mathcal{S}(X(\mathbb{C}), q_S) = 2$ for a general S ([**13**, Theorem 1.1]) we have $\mathcal{S}(X(\mathbb{C}), q) = \{S, \sigma(S)\}$. $\square$

Theorem 2.6. *Take $n = 2$, $d = 4$ and $k = 5$.*

(i) *The typical label-weights of $\sigma_5(X_{2,4}(\mathbb{C}))(\mathbb{R})$ are 5 and 6.*

(ii) *All labels of weight 5 are typical for $\sigma_5(X_{2,4}(\mathbb{C}))(\mathbb{R})$.*

(iii) $(3, 0)$ *is a typical label of weight 6 for $\sigma_5(X_{2,4}(\mathbb{C}))(\mathbb{R})$.*

Proof. We have $r = 14$.

Part (ii) is true for all varieties $X(\mathbb{C})$ [**8**, Proposition 3.2].

We first check the existence of a non-empty euclidean open subset of $\mathbb{P}^r(\mathbb{R})$ with no label of weight 5. Fix a general $S \subset X(\mathbb{C})$ such that $\#S = 5$ and take any $q \in \mathbb{P}^r(\mathbb{C})$ such that $S \in \mathcal{S}(X(\mathbb{C}), q)$. Then S is contained in a unique conic, C_S, and this conic is smooth. A referee observed (with a full proof) that Claim 1 has a 2-line proof using the Apolarity Lemma.

Claim 1: The curve C_S is the one-dimensional part of $F(X(\mathbb{C}), q)$.

Proof of Claim 1: We know that $\dim \mathcal{S}(X(\mathbb{C}), q) = 1$ and that its closure is isomorphic to $\mathbb{P}^1$. Take any $A \in \mathcal{S}(X(\mathbb{C}), q)$ such that $A \neq S$. To prove Claim 1 it is sufficient to prove that $A \subset C_S$. Assume for the moment that no 3 of the points of A are collinear. Thus A is contained in a unique smooth conic C. Since $A, S \in \mathcal{S}(X(\mathbb{C}), q)$ and $A \neq S$, $h^1(\mathcal{I}_{A\cup S}(4)) > 0$. Assume $C \neq C_S$. Consider the residual exact sequence of C

$$0 \to \mathcal{I}_{S\setminus C\cap S}(2) \to \mathcal{I}_{A\cup S}(4) \to \mathcal{I}_{C\cap(A\cup S),C}(4) \to 0. \tag{1}$$

By assumption $\#(C \cap (A \cup S)) \leq 9$. Since C is a smooth conic, $h^1(C, \mathcal{I}_{C\cap(A\cup S),C}(4)) = 0$. Thus (1) gives $S \cap C = \emptyset$ and $h^1(\mathcal{I}_{A\cup S}(4)) = 1$, i.e. q is the only point of $\langle\nu_d(A)\rangle_{\mathbb{C}} \cap \langle\nu_d(S)\rangle_{\mathbb{C}}$. Recall that $r_{X(\mathbb{C})}(q) = 5$. Take $E \in |\mathcal{O}_{\mathbb{P}^2}(3)|$ containing A and at least 2 points of S and set $G := E \cap (A \cup S)$ and $G' := A \cup S \setminus G$. Since $\#G' \leq 3$ and no 3 of the point of S are collinear, $h^1(\mathcal{I}_{G'}(1)) = 0$. Thus the residual exact sequence of E gives $h^1(E, \mathcal{I}_{G,E}(4)) = 1$

and hence $h^1(\mathcal{I}_G(4)) > 0$. Since $r_{X(\mathbb{C})}(q) = 5$, no proper subset of $\nu_d(A)$ or $\nu_d(S)$ spans q. Thus $G = A \cup S$. Fix $A' \subset A$ such that $\#A' = 3$. Since $S \cap C = \emptyset$ and A' is not collinear, no 5 of the points of $A' \cup S$ are collinear and $S \cup A'$ is not contained in a conic. Thus $h^1(\mathcal{I}_{A' \cup S}(3)) = 0$. Thus $G \neq A \cup S$, a contradiction. At this point we have proved that $F(X(\mathbb{C}), q)$ is the union of C_S and all $A \in \mathcal{S}(X, q)$ containing at least 3 collinear points, concluding the proof of Claim 1.

Observation 1: If q is real, then $F(X(\mathbb{C}), q)$ is defined over $\mathbb{R}$. Thus Claim 1 implies that if q is real, then C_S is real even if $\sigma(S) \neq S$.

Now we prove that $(3, 0)$ is a typical label. It is sufficient to find a non-empty euclidean open subset of $\sigma_5(X(\mathbb{C}))$ with no label (x, y) with $y > 0$ and $(3, 0)$ as one of its labels. Let $\mathcal{E}$ be the set of all real smooth conics C such that $C(\mathbb{R}) = \emptyset$. It is a non-empty open subset of $|\mathcal{O}_{\mathbb{P}^2}(2)|(\mathbb{R})$. Fix $C \in \mathcal{E}$ and take a general $S \subset C(\mathbb{C})$ such that $\#S = 5$. We have $S \cap \sigma(S) = \emptyset$ and hence $\#(S \cup \sigma(S)) = 10$. Since $C(\mathbb{C})$ has genus 0, $h^1(C(\mathbb{C}), \mathcal{I}_{S \cup \sigma(C), C(\mathbb{C})}(4)) = 1$. Since conics are projectively normal, $h^1(\mathbb{P}^2(\mathbb{C}), \mathcal{I}_{S \cup \sigma(C)}(4)) = 1$. Since $S \cap \sigma(S) = \emptyset$, the Grassmann's formula gives that $\langle \nu_d(S) \rangle_{\mathbb{C}} \cap \langle \nu_d(\sigma(S)) \rangle_{\mathbb{C}}$ is a single point, q. Since $\{q\} = \langle \nu_d(S) \rangle_{\mathbb{C}} \cap \langle \nu_d(\sigma(S)) \rangle_{\mathbb{C}}$, we get $q \in \sigma_5(X(\mathbb{C}))(\mathbb{R})$. Claim 1 gives that C is uniquely determined by q. Varying $C \in \mathcal{E}$ and S we get a subset of $\sigma_5(X(\mathbb{C}))(\mathbb{R})$ of real dimension 14 with no label (x, y) with $y > 0$, because $C(\mathbb{R}) = \emptyset$. Taking σ-invariant subsets of $C(\mathbb{C})$ with cardinality 6 we get that $(3, 0)$ is a typical label.

To conclude the proof of part (i) it is sufficient to prove that no label of weight ≥ 7 is typical for $\sigma_5(X(\mathbb{C}))$. Since generic uniqueness holds for $\sigma_4(X(\mathbb{C}))$ ([**13**, Theorem 1.1]), it is sufficient to quote [**6**, Theorem 1.7]. □

For $n = 2$, $d = 4$ and $k = 6$ (the generic rank) the closures of $\mathcal{S}(X(\mathbb{C}), q)$ for a general $q \in \mathbb{P}^{14}(\mathbb{C})$ are exactly the Fano 3-folds V_{22} discovered by S. Mukai ([**19**, **20**]).

Theorem 2.7. *Take $n = 4$ and $d = 3$ and hence $r = 34$. Let $\sigma_7^0(X_{4,3}(\mathbb{C})) \subset \mathbb{P}^{34}$ be the set all of rank 7 forms. Its typical label-weights are 7 and 8.*

Proof. All labels with weight 7 are typical ([**8**, Proposition 3.2]). Since generic uniqueness holds for $\sigma_6(X_{4,3}(\mathbb{C}))$ ([**13**]), no integer ≥ 9 is a label-weight (Lemma 2.2)). Thus it is sufficient to prove that 8 is a label weight. Take only complex forms f which are sums of 7 linear forms which are in linear general position. These are the form f such that $T_f := \{f = 0\}$ is the secant variety of a rational normal curve C_f. Note that C_f is the singular locus of T_f. Thus is f is real, then C_f is defined over $\mathbb{R}$. Since 4 is even, over $\mathbb{R}$ there are up to real isomorphism 2 real rational normal curves C, the one with $C(\mathbb{R}) \neq \emptyset$ (and hence with $C(\mathbb{R})$ topologically a circle) and the ones with $C(\mathbb{R}) = \emptyset$. Since 7 is odd, no T_f with $C_f(\mathbb{R}) = \emptyset$ has a σ-invariant solution. $\square$

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