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On Davenport expansions, Popov's formula and Fine's query

Abstract. We establish an explicit connection between a Davenport expansion and the Popov sum. Asymptotic analysis follows as a result of these formulas. New solutions to a query of N. J. Fine are offered, and a proof of Davenport expansions is detailed.

Keywords. Davenport expansions; Riemann zeta function; von Mangoldt function.

Mathematics Subject Classification:: 11L20, 11M06.

1 - Introduction and Main formulas

Let $\Lambda(n)$ denote the von Mangoldt function, $\zeta(s)$ the Riemann zeta function, and ρ the non-trivial (complex) zeros of the Riemann zeta function [**5**, p. 43]. In a recent paper [**11**] we established a proof of Popov's formula [**12**]:

$$\begin{aligned}\sum_{n>x}\frac{\Lambda(n)}{n^2}\left(\{\frac{n}{x}\}-\{\frac{n}{x}\}^2\right)=&\frac{2-\log(2\pi)}{x}+\sum_{\rho}\frac{x^{\rho-2}}{\rho(\rho-1)}\\&+\sum_{k\geq 1}\frac{k+1-2k\zeta(2k+1)}{2k(k+1)(2k+1)}x^{-2k-2},\end{aligned}\tag{1.1}$$

for $x>1$. Here $\{x\}$ is the fractional part of x, sometimes written as $\{x\}=x-\lfloor x\rfloor$, where $\lfloor x\rfloor$ is the floor function. The proof relied on the Mellin transform formula

$$\frac{1}{2}\left(\{x\}^2-\{x\}\right)=\frac{1}{2\pi i}\int_{(b)}\left(\frac{s+1}{2s(s-1)}-\frac{\zeta(s)}{s}\right)\frac{x^{s+1}}{s+1}ds,\tag{1.2}$$

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which is valid for $x > 0$, $-1 < b < 0$. This can be observed by noting (1.2) is known [**15**, p. 14, eq. (2.1.4)] for $x > 1$, and by [10, pg.405] for $0 < x < 1$,

$$\frac{1}{2\pi i}\int_{(b)} \frac{x^s}{s(s-1)}ds = 0.$$

From (1.2) it is easy to see that for $1 < c < 2$,

$$\sum_{n>x} \frac{\Lambda(n)}{n^2}\left(\{\frac{n}{x}\} - \{\frac{n}{x}\}^2\right) = \frac{1}{2\pi i}\int_{(c)} \left(\frac{2-s}{2s(s-1)} - \frac{\zeta(1-s)}{1-s}\right)\frac{\zeta'(s)}{\zeta(s)}\frac{x^{s-2}}{2-s}ds.$$

We apply the Cauchy residue theorem to a rectangle with vertices at with vertices at $(c, \pm iT)$, $(-N, \pm iT)$. The horizontals tend to 0 and the vertical one also tends to 0 as $N \to \infty$. Hence the integral on the right side is the sum of residues at poles in the strip $-N < \rho < c$. At non-trivial zeros ρ, the denominator $\zeta(s)$ is of the form $\zeta'(\rho)(s-\rho) + \cdots$, whence the convergent sum $\sum_\rho \frac{x^{\rho-2}}{\rho(\rho-1)}$ arises. The sum of residues at trivial zeros gives rise to the third term.

Let $f(n)$ be a suitable arithmetic function such that $L(s) = \sum_{n\geq 1} f(n)n^{-s}$ is absolutely convergent, whence analytic for $\Re(s) > 1$, and let $S(x) = \sum_{n\leq x} f(n)$. Examining the right hand side of (1.2), it is not difficult to see that

$$\begin{aligned}\frac{1}{2}\sum_{n>x} \frac{f(n)}{n^2}\left(\{\frac{n}{x}\} - \{\frac{n}{x}\}^2\right) =& \frac{1}{2\pi i}\int_{(b)} \frac{x^{-s-1}}{2s(s-1)}L(1-s)ds \\ & - \frac{1}{2\pi i}\int_{(b)} \frac{x^{-s-1}\zeta(s)}{s(s+1)}L(1-s)ds.\end{aligned} \tag{1.3}$$

If we replace s by $1-s$ in the first integral in (1.3) and apply Fubini's theorem to interchange the integrals, we see that this integral is equal to

$$\frac{1}{2x}\int_0^x \left(\frac{1}{2\pi i}\int_{(1-b)} \frac{y^{s-2}}{s}L(s)ds\right)dy = \frac{1}{2x}\int_0^x \frac{S(y)dy}{y^2}, \tag{1.4}$$

by the Mellin-Perron formula. If we generalize the formula of Segal [**14**, eq. (5)], we can see that the second integral on the right side of (1.3) is

$$\begin{aligned}\int_0^{\frac{1}{x}} \left(\sum_{n\geq 1} \frac{f(n)}{n}(\{ny\} - \frac{1}{2})\right)dy &= \int_0^{\frac{1}{x}} \left(-\frac{1}{\pi}\sum_{n\geq 1} \frac{F(n)}{n}\sin(2\pi ny)\right)dy \\ &= \frac{1}{2\pi^2}\sum_{n\geq 1} \frac{F(n)}{n^2}\left(\cos\left(\frac{2\pi n}{x}\right) - 1\right),\end{aligned} \tag{1.5}$$

assuming uniform convergence where $F(n) = \sum_{d|n} f(d)$. Collecting (1.3), (1.4), and (1.5), we obtain the following theorem upon noting that Davenport's proof of uniform convergence is dependent on a special estimate [**4**].

Theorem 1.1. *Let $f(n)$ be chosen such that $L(s)$ is analytic for $\Re(s) > 1$, and that $\sum_{n\le N} f(n)e^{2\pi inx} = O(N(\log(N))^{-h})$, for any fixed h. We have for $x > 1$,*

$$\frac{1}{2}\sum_{n>x}\frac{f(n)}{n^2}\left(\{\frac{n}{x}\}-\{\frac{n}{x}\}^2\right) = \frac{1}{2x}\int_0^x S(y)\frac{dy}{y^2} + \frac{1}{2\pi^2}\sum_{n\ge 1}\frac{F(n)}{n^2}\left(\cos\left(\frac{2\pi n}{x}\right)-1\right).$$

We have therefore proven the connection between Popov's formula and the integral $\int_0^x \frac{S(y)dy}{y^2}$ alluded to in [**11**] (see also [**4**, p. 69]). Perhaps even more fascinating is the connection to Davenport expansions [**3**, **8**] through the sum on the far right hand side of (1.5). This Fourier series is known to be connected to the periodic Bernoulli polynomial $B_2(\{x\}) - B_2 = (\{x\}^2 - \{x\})$. For relevant material on Davenport expansions connected to Bernoulli polynomials, see [**2**, **9**].

Recall that $h(x) \sim g(x)$ means that $\lim_{x\to\infty}\frac{h(x)}{g(x)} = 1$. Letting $x \to \infty$ and applying L'Hôpital's rule to Theorem 1, and the Residue Theorem to (1.3)–(1.4), we have the following.

Corollary 1.1.1. *Let $f(n)$ be chosen such that $L(s)$ is analytic for $\Re(s) > 1$. Suppose that $S(x) \sim \Delta(x)$ as $x \to \infty$. Then*

$$\frac{1}{2}\sum_{n>x}\frac{f(n)}{n^2}\left(\{\frac{n}{x}\}-\{\frac{n}{x}\}^2\right) - \frac{1}{2\pi^2}\sum_{n\ge 1}\frac{F(n)}{n^2}\left(\cos\left(\frac{2\pi n}{x}\right)-1\right) \sim \frac{\Delta(x)}{x^2},$$

as $x \to \infty$.

Notice that the sum on the left hand side of (1.1) is $\sim \frac{1}{x}$, which corresponds to the Prime Number theorem $\sum_{n\le x}\Lambda(n) \sim x$ when coupled with our corollary.

The integral $\int_0^x \frac{S(y)dy}{y^2}$ has appeared in many recent works in the analytic theory of numbers. Namely, in the case of the von Mangoldt function $S(x) = \psi(x) = \sum_{n\le x}\Lambda(n)$, see [**13**], where we find a study of the function

$$\sum_{n\le x}\frac{\Lambda(n)}{n} - \frac{\sum_{n\le x}\Lambda(n)}{x} = \int_1^x \frac{\psi(y)dy}{y^2}.$$

For the case of the Möbius function (i.e. $S(x) = M(x)$ the Mertens function [**15**, p. 370]), Inoue [**7**, Corollary 3, $k = 2$] gave, under the assumption of the weak

Mertens Hypothesis,

$$\int_1^x \frac{M(y)dy}{y^2} = x^{-\frac{1}{2}} \sum_{\rho} \frac{x^{i\gamma}}{\zeta'(\rho)\rho(\rho-1)} + A(2) + O(x^{-1}).$$

Here $A(2)$ is a constant, and $g(x) = O(h(x))$ means $|g(x)| \le c_1 h(x)$, $c_1 > 0$ a constant.

We mention there is another form of the Fourier series on the far right side of Theorem 1.1. Note that [**15**, p. 14, eq. (2.1.5)]

$$\{x\} = -\frac{1}{2\pi i}\int_{(c)} \frac{\zeta(s)}{s} x^s ds, \tag{1.6}$$

where $x > 0$, and $0 < c < 1$. Integrating, we have that

$$\frac{1}{2}\left(\{x\}^2 + \lfloor x \rfloor\right) = -\frac{1}{2\pi i}\int_{(c)} \frac{\zeta(s)}{s(s+1)} x^{s+1} ds. \tag{1.7}$$

We apply the Cauchy residue theorem to the rectangle with vertices at $(c, \pm iT)$, $(M, \pm iT)$, $M, T > 0$, to shift the line of integration, whereby by the known estimates, the horizontal integrals go to 0 as $T \to \infty$. Dividing by x, computing the residue at the pole $s = 0$, and inverting the desired series in (1.7), we have

$$\sum_{n\ge1} \frac{f(n)}{n}\left(\frac{1}{x2n}\left(\{nx\}^2 + \lfloor nx \rfloor\right) - \frac{1}{2}\right) = -\frac{1}{2\pi i}\int_{(c-1)} \frac{\zeta(s)}{s(s+1)} x^s L(1-s) ds. \tag{1.8}$$

Here we used the fact that $\zeta(0) = -\frac{1}{2}$. Hence, after comparing with our computation (1.5), we have proven the following result.

Theorem 1.2. *For $x > 0$,*

$$\sum_{n\ge1} \frac{f(n)}{n}\left(\frac{1}{x2n}\left(\{nx\}^2 + \lfloor nx \rfloor\right) - \frac{1}{2}\right) = \frac{1}{2x\pi^2}\sum_{n\ge1} \frac{F(n)}{n^2}(\cos(2\pi nx) - 1).$$

2 - Solution to The N. J. Fine query

In [**1**], a positive answer was presented to a query of N. J. Fine, who asked for a continuous function $\varphi(x)$ on $\mathbb{R}$, with period 1, $\varphi(x) \ne -\varphi(-x)$, and

$$\sum_{N\ge k\ge1} \varphi\left(\frac{k}{N}\right) = 0. \tag{2.1}$$

Namely, they gave the solutions

$$\sum_{n\geq 1}\frac{f(n)}{n}\cos(2\pi nx), \tag{2.2}$$

where $f(n)$ is chosen as the Möbius function $\mu(n)$ and the Liouville function $\lambda(n)$, [**14**]. Their proof utilizes a Ramanujan sum [**15**, p. 10]

$$\sum_{N\geq k\geq 1}\cos(\frac{2\pi kn}{N}), \tag{2.3}$$

which is N if $n \equiv 0 \pmod N$ and 0 otherwise. It is also dependent on $\sum_{n\geq 1} f(n)/n = 0$. In fact, it is possible to further generalize their result using these properties, which we offer in the following.

Example. *Suppose $f(n)$ is a multiplicative arithmetic function chosen such that $\sum_{n\geq 1} f(n)/n = 0$. Then*

$$\sum_{n\geq 1}\frac{f(n)}{n}\cos^m(\pi nx), \tag{2.4}$$

and

$$\sum_{n\geq 1}\frac{f(n)}{n}\sin^{2m}(\pi nx), \tag{2.5}$$

for each positive integer m satisfy the properties in Fine's query.

Proof. We will use [**6**, p. 31, section 1.320, no. 5 and 7] for (2.4) and (2.5) to obtain our $\varphi(x)$. Namely

$$\cos^{2m}(x) = \frac{1}{2^{2m}}\left(\sum_{m-1\geq k\geq 0} 2\binom{2m}{k}\cos(2(m-k)x) + \binom{2m}{m}\right), \tag{2.6}$$

$$\cos^{2m-1}(x) = \frac{1}{2^{2m-2}}\sum_{m-1\geq k\geq 0}\binom{2m}{k}\cos((2m-2k-1)x). \tag{2.7}$$

Putting $x = \frac{2\pi ln}{N}$, and summing over $N \geq l \geq 1$ we see that the sum is a linear combination of zeros and N's depending on weather $n(m-k)|N$. In the case where $n(m-k)|N$, we are left with a linear combination of terms which are independent of n. The last term is simply $N\frac{1}{2^{2m}}\binom{2m}{m}$. Therefore, summing over n gives the result upon invoking $\sum_{n\geq 1} f(n)/n = 0$. Similar arguments apply to (2.7). Since (2.5) follows in the same way using another formula from [**6**, p. 31, section 1.320, no. 1], we leave the details to the reader. $\square$

We were interested finding more solutions to Fine's query by constructing a special arithmetic function with the possible property $\sum_{n\geq 1} f(n)/n \neq 0$. Define

$$\chi^{\pm}_{m,l}(n) := \begin{cases} \pm(m^l \mp 1), & \text{if } n = 0 \pmod{m}, \\ 1, & \text{if } n \neq 0 \pmod{m}. \end{cases} \tag{2.8}$$

If $f(n)$ is completely multiplicative, this tells us that

$$\sum_{n\geq 1} \frac{\chi^{-}_{m,l}(n) f(n)}{n^s} = L(s) - m^l \sum_{n\equiv 0 \pmod{m}} \frac{f(n)}{n^s} = (1 - f(m)m^{l-s})L(s).$$

Definition. A function is said to be of the class $\aleph$ if: (i) it is continuous on $\mathbb{R}$, (ii) is 1-periodic (iii) is not odd, and (iv) satisfies (2.1) for each N coprime to m.

Theorem 2.1. *Suppose that $L(s)$ is analytic for $\Re(s) > 1$. For natural numbers $m > 1$, $l > 1$, and N is coprime to m, we have $D_1(x) \in \aleph$ where*

$$D_1(x) = \sum_{n\geq 1} \frac{\chi^{+}_{m,l}(n) f(n)}{n^l} \cos(2\pi n x), \tag{2.9}$$

for a completely multiplicative function with the property $f(m) = -1$, and $D_2(x) \in \aleph$ where

$$D_2(x) = \sum_{n\geq 1} \frac{\chi^{-}_{m,l}(n) f(n)}{n^l} \cos(2\pi n x), \tag{2.10}$$

for a completely multiplicative function with the property $f(m) = 1$.

Proof. First we consider (2.9). Note that because $\chi^{\pm}_{m,l}(n)$ is not completely multiplicative, we need to restrict N to be coprime to m, since then $\chi^{\pm}_{m,l}(Nn) = \chi^{\pm}_{m,l}(n)$. That is to say that $Nn \equiv 0 \pmod{m}$ is solved by $n \equiv 0 \pmod{m}$ provided that N is coprime to m. The same argument applies in the case $Nn \not\equiv 0 \pmod{m}$. Using (2.3) and the method applied in [**1**] we compute that

$$\begin{aligned} \sum_{N\geq k\geq 1} \sum_{n\geq 1} \frac{\chi^{+}_{m,l}(n) f(n)}{n^l} \cos(\frac{2\pi n k}{N}) &= N \sum_{n\equiv 0 \mod N} \frac{\chi^{+}_{m,l}(n) f(n)}{n^l} \\ &= \frac{f(N)}{N^{l-1}} \lim_{s\to l}(1 - m^{l-s})L(s) = 0. \end{aligned}$$

The computation for (2.10) is similar, and so we leave the details for the reader. □

An example for $D_2(x)$ is if $f(n) = \lambda(n)$, and $m = 4$, $l > 1$, since $\lambda(4) = 1$. One for $D_1(x)$ is if $f(n) = \mu(n)$, and $m = 5$, $l > 1$, since $\mu(5) = -1$.

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