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Wijsman convergence of set sequences in asymmetric metric spaces

Abstract. In this study, we will give some convergence definitions for set sequences in an asymmetric metric space. Later, we will prove the theorems expressing the inclusion relations between these concepts.

Keywords. Asymmetric metric space, forward statistical convergence, backward statistical convergence, Cesàro mean, approximate metric axiom, almost convergence.

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1 - Introduction and Background

For a function $d : \mathbb{R} \times \mathbb{R} \to \mathbb{R}^+ \cup \{0\}$, if only the symmetry condition of metric conditions is not valid then d is called asymmetric metric.

For example, let the function d be defined by

$$d(u,v) = \begin{cases} v-u & \text{if } v \geq u \\ 1 & \text{if } v < u. \end{cases}$$

Then d is an asymmetric metric on $\mathbb{R}$. This asymmetric metric is called Sorgenfrey asymmetric metric. Asymmetric metric spaces are investigated in [**16**], [**21**], [**23**], [**29**]. In addition, asymmetric metric spaces have many applications in different fields of science (see e.g. [**7**], [**20**]). Unlike the metric case, there are actually two concepts corresponding to each concept, namely forward and backward concepts that arise for two natural topologies in the same space. Especially in [**7**], the concepts of convergence and Cauchy sequence in asymmetric metric space are examined. Recently, using the concept of ideal in [**9**],

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the definitions of convergence and Cauchy sequence in asymmetric metric space have been generalized. It was observed that some known classical results were not provided in the absence of symmetry.

Recall the following fundamental concepts of an asymmetric metric space from [**7**], [**23**], [**29**].

A function $d : U \times U \to [0, \infty)$ is an asymmetric metric and (U, d) is an asymmetric metric space if

(i) For every $u, v \in U$, $d(u, v) \geq 0$ and $d(u, v) = 0$ if and only if $u = v$.

(ii) For every $u, v, w \in U$, we have $d(u, w) \leq d(u, v) + d(v, w)$.

A sequence (u_n) is called to be forward convergent to $u \in U$, if $\lim_{n\to\infty} d(u, u_n) = 0$ and is denoted by $u_n \xrightarrow{\text{F}} u$. For this case we write $F - \lim_{n\to\infty} u_n = u$.

Similarly, a sequence (u_n) is called to be backward convergent to $u \in U$, if $\lim_{n\to\infty} d(u_n, u) = 0$ and is denoted by $u_n \xrightarrow{\text{B}} u$. For this case we write $B - \lim_{n\to\infty} u_n = u$.

A sequence (u_n) is convergent to the number u if and only if the sequence is both forward convergent and backward convergent to the number u.

For example, fix $a \in \mathbb{R}$ and let $u_n = a\left(1 + \frac{1}{n}\right)$. Then the sequence (u_n) is forward convergent to a but not backward convergent to a with respect to Sorgenfrey asymmetric metric. Similarly, let's take the sequence (v_n) as $v_n = a\left(1 - \frac{1}{n}\right)$ then, the sequence (v_n) is backward convergent to a but not forward convergent to a with respect to Sorgenfrey asymmetric metric.

Convergence of number sequences was generalized by many authors to set sequences. One such generalization that we take into consideration in this paper is the Wijsman convergence. The authors of [**22**] defined the concept of Wijsman statistical convergence for set sequences and proved some fundamental theorems. In this article, we will introduce the concepts of forward and backward Cesàro convergence, strong Cesàro convergence, statistical convergence, almost convergence, and strong almost convergence for sequences of sets in asymmetric metric spaces, and we will give some of their properties and some inclusion relations. Although the concept of set convergence was given by Painleve, it has a very long mathematical history. The concept was popularized by Kuratowski by his famous book Topologie [**17**] and therefore often referred to as Kuratowski convergence. In the last fifty years, it has regarded as an important implement for dealing with approximations in optimization, systems of equations and related subjects.

For asymmetric metric space (U, ϱ), non-empty subset B of U and $u \in U$, define the distance from u to B by

$$d(u, B) = \inf_{b\in B} \varrho(u, b)$$

and the distance from B to u by

$$d(B,u) = \inf_{b \in B} \varrho(b,u).$$

2 - Wijsman Convergence

$CL(U)$ will denote the set of nonempty closed subsets of U.

Definition 2.1. Let (U,ϱ) be an asymmetric metric space. For $A_k \in CL(U)$, if $\sup_k d(u,A_k) < \infty$ for each $u \in U$ then we say that the sequence (A_n) is *forward bounded.*

Similarly, if $\sup_k d(A_k,u) < \infty$ for each $u \in U$ then we say that the sequence (A_n) is *backward bounded.*

Definition 2.2. Let (U,ϱ) be an asymmetric metric space. For $A, A_k \in CL(U)$, we say that the sequence (A_k) is *forward Wijsman convergent* to A if

$$\lim_{k\to\infty} d(u,A_k) = d(u,A)$$

for each $u \in U$. In this case we write $FW - \lim A_k = A$. Similarly, we say that the sequence (A_k) is *backward Wijsman convergent* to A if

$$\lim_{k\to\infty} d(A_k,u) = d(A,u)$$

for each $u \in U$. For this case we write $BW - \lim A_k = A$.

A sequence (A_n) is Wijsman convergent to the set A if and only if the sequence is both forward Wijsman convergent and backward Wijsman convergent to the number A.

Definition 2.3. Let (U,ϱ) be an asymmetric metric space. For $A_k \in CL(U)$, we say that the sequence (A_k) is *forward Wijsman Cauchy* if for $\epsilon > 0$ and for each $u \in U$, there is a positive integer ℓ_1 such that for all $m, n > \ell_1$,

$$|d(u,A_n) - d(u,A_m)| < \epsilon.$$

Similarly, we say that the sequence (A_n) is *backward Wijsman Cauchy* if for $\epsilon > 0$ and for each $u \in U$, there is a positive integer ℓ_2 such that for all $m, n > \ell_2$,

$$|d(A_n,u) - d(A_m,u)| < \epsilon.$$

A sequence (A_n) is Wijsman Cauchy sequence if and only if the sequence is both forward Wijsman Cauchy sequence and backward Wijsman Cauchy sequence.

Definition 2.4. (U, ϱ) be an asymmetric metric space. For $A, A_n \in CL(U)$ the sequence (A_n) is called *forward Wijsman Cesàro summable* to the set A if

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} d(u, A_k) = d(u, A)$$

for each $u \in U$. For this case we write $FW(C,1) - \lim_{n\to\infty} A_n = A$.

Similarly, a sequence (A_n) is called to be *backward Wijsman Cesàro summable to* the set A if

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} d(A_k, u) = d(A, u)$$

for each $u \in U$ For this case we write $BW(C,1) - \lim_{n\to\infty} A_n = A$.

A sequence (A_n) is Wijsman Cesàro summable to the set A if and only if the sequence is both forward Wijsman Cesàro summable and backward Wijsman Cesàro summable to the number A.

Definition 2.5. (U, ϱ) be an asymmetric metric space. For $A, A_n \in CL(U)$, the sequence (A_n) is called to be *forward Wijsman strongly Cesàro summable* to the set$A$ if

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} |d(u, A_k) - d(u, A)| = 0$$

for each $u \in U$. For this case we write $FW[C,1] - \lim_{n\to\infty} A_n = A$.

Similarly, a sequence (A_n) is called to be *backward Wijsman strongly Cesàro summable* to the set A if

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} |d(A_k, u) - d(A, u)| = 0$$

for each $u \in U$. For this case we write $BW[C,1)] - \lim_{n\to\infty} A_n = A$.

A sequence (A_n) is *Wijsman strongly Cesàro summable* to the set A if and only if the sequence is both forward Wijsman strongly Cesàro summable and backward Wijsman strongly Cesàro summable to the number A.

Definition of statistical convergence of number sequences was given by Fast [**10**]. In [**24**] Schoenberg obtained some fundamental properties of statistical convergence and also examined the concept as a summability method.

If for every $\epsilon > 0$,

$$\lim_{n\to\infty} \frac{1}{n}|\{k \leq n : \quad |u_k - u| \geq \epsilon\}| = 0,$$

then we say that (u_n) is statistically convergent to u, here $|K|$ denotes the cardinality of the set K. For this case we write $st - \lim u_n = u$. $\lim u_n = u$ implies $st{-}\lim u_n = u$, therefore statistical convergence is a regular summability method.

Statistical convergence of set sequences was introduced and some properties was examined in [**22**].

Definition 2.6. Let (U, ϱ) be an asymmetric metric space. For $A, A_k \in CL(U)$, we say that the sequence (A_k) is *forward Wijsman statistically convergent* to A if for each $\epsilon > 0$ and for each $u \in U$,

$$\lim_{n\to\infty} \frac{1}{n} |\{k \le n : \quad |d(u, A_k) - d(u, A)| \ge \epsilon\}| = 0.$$

For this case we write $st - \lim_{FW} A_k = A$.

Similarly, we say that the sequence (A_k) is *backward Wijsman statistically convergent* to A if for each $\epsilon > 0$ and for each $u \in U$,

$$\lim_{n\to\infty} \frac{1}{n} |\{k \le n : \quad |d(A_k, u) - d(A, u)| \ge \epsilon\}| = 0.$$

In this case we write $st - \lim_{BW} A_k = A$.

It is clear that $FW - \lim A_k = A$ implies $st - \lim_{FW} A_k = A$ and $BW - \lim A_k = A$ implies $st - \lim_{BW} A_k = A$

Theorem 2.7. *Let (X, ϱ) be a asymmetric metric space. Then, for $A, A_k \in CL(U)$*

(a) (A_n) *is forward Wijsman statistically convergent to A if it is forward Wijsman strongly Cesàro summable to A,*

(b) *If (A_n) is forward Wijsman statistically convergent to A and forward bounded then it is forward Wijsman strongly Cesàro convergent to A.*

Proof. (a) For any (A_n), fix an $\epsilon > 0$. Then

$$\sum_{k=1}^{n} |d(u, A_k) - d(u, A)| \ge \epsilon |\{k \le n : |d(u, A_k) - d(x, A)|u \ge \epsilon\}|,$$

and it follows that if (A_n) is forward Wijsman strongly Cesàro summable to A then (A_n) is forward Wijsman statistically convergent to A.

(b) Let (A_n) be forward bounded and forward Wijsman statistically convergent to A. Since (A_n) is forward bounded, set $\sup_k |d(u, A_k)| + d(u, A) = \lambda$.

Let $\epsilon > 0$ be given. Since the sequence (A_n) is forward Wijsman statistically convergent to A, we can select N_ϵ such that

$$\frac{1}{n}|\{k \leq n : |d(u, A_k) - d(u, A)| \geq \frac{\epsilon}{2}\}| < \frac{\epsilon}{2\lambda}$$

for all $n > N_\epsilon$. Let's denote the set $\{k \leq n : |d(u, A_k) - d(u, A)| \geq \frac{\epsilon}{2}\}$ by $\mathcal{L}_n$.

$$\begin{aligned}
& \frac{1}{n}\sum_{k=1}^{n}|d(u, A_k) - d(u, A)| \\
= & \frac{1}{n}\left(\sum_{k\in\mathcal{L}_n}|d(u, A_k) - d(u, A)| + \sum_{k\leq n; k\notin\mathcal{L}_n}|d(u, A_k) - d(u, A)|\right) \\
< & \frac{1}{n}\frac{n\epsilon}{2\lambda}\lambda + \frac{1}{n}\frac{n\epsilon}{2} \\
= & \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.
\end{aligned}$$

Hence, (A_n) is forward Wijsman strongly Cesàro summable to A. □

3 - Almost Convergence

Definition of almost convergence of a sequence was first given by G. G. Lorentz [**18**]. Almost convergence is an issue with Banach limits.

Let ℓ_∞ be denote the space of real bounded sequences. If

1. $\phi(u_n) \geq 0$ when the sequence (u_n) has $u_n \geq 0$ for all n,
2. $\phi(e) = 1$ where $e = (1, 1, 1, ...)$ and
3. $\phi(u_{n+1}) = \phi(u_n)$ for all $(u_n) \in l_\infty$.

then we say that the continuous linear functional ϕ on ℓ_∞ is a *Banach limit.*

If all Banach limits of $(u_n) \in l_\infty$ is equal to u then (u_n) is almost convergent to u. It is easy to verify that if (u_n) is a convergent sequence, then $\phi(u_n) = \lim_{n\to\infty} u_n$ for any Banach limits ϕ. In other words, $\phi(u_n)$ takes the same value for any Banach limits ϕ.

Lorentz [**18**] gave the following characterization of the almost convergence.

A sequence (u_n) is said to be *almost convergent* to u if and only if

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}u_{m+k} = u$$

uniformly in m.

Definition 3.1. Let (U, ρ) be an asymmetric metric space. For $A, A_k \in CL(U)$, we say that the sequence (A_n) is *forward Wijsman almost convergent* to the set A if

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} d(u, A_{k+m}) = d(u, A)$$

uniformly in m.

Similarly, we say that the sequence (A_n) is *backward Wijsman almost convergent* to the set A if

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} d(A_{k+m}, u) = d(A, u)$$

uniformly in m.

These will be denoted FWĉ-$\lim_{n\to\infty} A_n = A$ and BWĉ-$\lim_{n\to\infty} A_n = A$ or $A_n \xrightarrow{\text{FW}\hat{\text{c}}} A$ and $A_n \xrightarrow{\text{BW}\hat{\text{c}}} A$ respectively.

A sequence (A_n) is Wijsman almost convergent to the set A if and only if the sequence is both forward Wijsman almost convergent and backward Wijsman almost convergent to the set A.

Definition 3.2. Let (U, ϱ) be an asymmetric metric space. For $A, A_k \in CL(U)$, we say that the sequence (A_n) is *forward Wijsman strongly almost convergent* to the set A if

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} |d(u, A_{k+m}) - d(u, A)| = 0$$

uniformly in m.

Similarly, we say that the sequence (A_n) is *backward Wijsman strongly almost convergent* to the set A if

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} |d(A_{k+m}, u) - d(A, u)| = 0$$

uniformly in m.

These will be denoted FW[ĉ]-$\lim_{n\to\infty} A_n = A$ and BW[ĉ]-$\lim_{n\to\infty} A_n = A$ or $A_n \xrightarrow{\text{FW}[\hat{\text{c}}]} A$ and $A_n \xrightarrow{\text{BW}[\hat{\text{c}}]} A$ respectively.

A sequence (A_n) is Wijsman strongly almost convergent to the set A if and only if the sequence is both forward Wijsman strongly almost convergent and backward Wijsman strongly almost convergent to the set A.Now we give a property of an asymmetric metric space which plays a very important role in the next part of the paper [**7**].

An asymmetric metric space (U, d) is said to satisfy **approximate metric axiom** if there exists a function $c : U \times U \to \mathbb{R}$ such that for any $u, v \in U$, $d(v, u) \leq c(u, v)d(u, v)$ where the function c is such that for any $u \in U$, there is a $\delta_x > 0$ with the property that $v \in B^+(u, \delta_x) \Rightarrow c(u, v) \leq C(u)$, where $C(u) > 0$ depends only on u and $B^+(u, \epsilon) = \{v \in U : d(u, v) < \epsilon \text{ for } u \in U, \epsilon > 0\}$.

If (U, d) is a metric space then clearly it satisfies approximate metric axiom with the function c defined by $c(u, v) = 1$ for all $u, v \in U$. However the condition approximate metric axiom is strictly weaker than the requirement of an asymmetric metric space to be a metric space.

For example, let $\beta > 0$ and the function $d : \mathbb{R} \times \mathbb{R} \to [0, \infty)$ be defined by

$$d(a, b) = \begin{cases} b - a & \text{if } b \geq a \\ \beta(a - b) & \text{if } b < a. \end{cases}$$

It is clear that d is an asymmetric metric on $\mathbb{R}$. This d satisfies approximate metric axiom with $C = \max\{\beta, \frac{1}{\beta}\}$.

The Sorgenfrey asymmetric metric, was defined above, does not satisfy the approximate metric axiom.

Theorem 3.3. *Let (U, ϱ) be an asymmetric metric space and $A_k \in CL(U)$. If (A_n) forward almost convergent to A and backward almost convergent to B then $A = B$.*

Proof. Fix $\epsilon > 0$. Let (A_n) be forward almost convergent to $A \in CL(U)$, then there exists $n_1 \in \mathbb{N}$ such that

$$\sup_{u \in U, m} \left| \frac{1}{n} \sum_{k=1}^{n} d(u, A_{k+m}) - d(u, A) \right| < \frac{\epsilon}{2}$$

for $n > n_1$. Also, let (A_n) be backward almost convergent to $B \in CL(U)$, then there exists $n_2 \in \mathbb{N}$ such that

$$\sup_{u \in U, m} \left| \frac{1}{n} \sum_{k=1}^{n} d(A_{k+m}, u) - d(B, u) \right| < \frac{\epsilon}{2}$$

for $n > n_2$. Therefore, for all $m \in \mathbb{N}$ and $n \geq \max\{n_1, n_2\}$, by using Hausdorff forward and backward distances $d(A, B) = \sup_{u \in U} |d(u, A) - d(u, B)|$ and $d(B, A) = \sup_{u \in U} |d(A, u) - d(B, u)|$, we can write

$$\begin{aligned} d(A, B) &\leq d(A, A_{k+m}) + d(A_{k+m}, B) \\ &= \sup_{u \in U, m} |d(u, A) - d(u, A_{k+m})| + \sup_{u \in U, m} |d(A_{k+m}, u) - d(B, u)|. \end{aligned}$$

$$\begin{aligned}\frac{1}{n}\sum_{k=1}^{n} d(A,B) &\leq \sup_{u\in U,m}\left|\frac{1}{n}\sum_{k=1}^{n} d(u,A_{k+m})-d(u,A)\right| \\ &+ \sup_{u\in U,m}\left|\frac{1}{n}\sum_{k=1}^{n} d(A_{k+m},u)-d(B,u)\right|.\end{aligned}$$

$$d(A,B)<\frac{\epsilon}{2}+\frac{\epsilon}{2}=\epsilon.$$

Since ϵ was arbitrary, we deduce that $d(u,A)=d(u,B)$ holds for every $u\in U$. It is implies the equality $A=B$ because of $A,B\in CL(U)$. □

Corollary 3.4. *If forward Wijsman almost convergence of a sequence implies backward Wijsman almost convergence, then the forward Wijsman almost limit is unique.*

Theorem 3.5. *If (U,ϱ) is an asymmetric metric space satisfying the approximate metric axiom property, then $FW\hat{c}$-$\lim_{n\to\infty} A_n = A$ implies $BW\hat{c}$-$\lim_{n\to\infty} A_n = A$ for any closed set sequence $(A_n)\in U$ and limits are unique.*

Proof. Since FWĉ-$\lim_{n\to\infty} A_n = A$,

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n} d(u,A_{k+m})=d(u,A)$$

uniformly in m. From the approximate metric axiom property, we can write

$$d(A_{k+m},u)\leq c(u,A_{k+m})d(u,A_{k+m})\leq C(u)d(u,A_{k+m})$$

and

$$d(A,u)\leq c(u,A)d(u,A)\leq C(u)d(u,A)$$

hence we have

$$d(A_{k+m},u)-d(A,u)\leq C(u)[d(u,A_{k+m})-d(u,A)]$$

for all $k,m\in\mathbb{N}$. Then

$$\begin{aligned}\frac{1}{n}\sum_{k=1}^{n} d(A_{k+m},u)-d(A,u) &\leq \frac{1}{n}\sum_{k=1}^{n} C(u)[d(u,A_{k+m})-d(u,A)] \\ &= C(u)\frac{1}{n}\sum_{k=1}^{n}[d(u,A_{k+m})-d(u,A)].\end{aligned}$$

Hence we have,

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n} d(A_{k+m},u) = d(A,u)$$

uniformly in m, that is, BWĉ-$\lim_{n\to\infty} A_n = A$.

By the Corollary 1 limits are unique. □

In [**7**] it was proved that if (U,ϱ) is forward sequentially compact then $u_n \xrightarrow{\mathrm{B}} u$ implies $u_n \xrightarrow{\mathrm{F}} u$. It is not clear whether this result remains valid if convergence is replaced by almost convergence.

Definition 3.6. A set $K \subset U$ is *forward almost sequentially compact* if every sequence in K has a forward almost convergent subsequence with limit in K.

Similarly, A set $K \subset U$ is *backward almost sequentially compact* if every sequence in K has a backward almost convergent subsequence with limit in K.

Theorem 3.7. *Let (U,ϱ) be an asymmetric metric space. If (U,ϱ) is forward almost sequentially compact and $A_n \xrightarrow{BW\hat{c}} A$ then $A_n \xrightarrow{FW\hat{c}} A$.*

Proof. Consider a sequence (A_n) such that $A_n \xrightarrow{\mathrm{BW\hat{c}}} A$ for some $A \in U$. By forward almost sequential compactness, every subsequence $(A_{n_k})_{k\in\mathbb{N}}$ of (A_n) has a forward almost convergent subsequence in U. Say $A_{n_{k_j}} \xrightarrow{\mathrm{FW\hat{c}}} B \in U$ as $j\to\infty$. Then $A = B$ by Theorem 3.3.

Suppose $A_n \xrightarrow{\mathrm{FW\hat{c}}} A$. Then there exists $\epsilon_0 > 0$ and a subsequence $(A_{n_k})_{k\in\mathbb{N}}$ with

$$\frac{1}{n}\sum_{k=1}^{n} d(u,A_{n_k+m}) - d(u,A) \ge \epsilon_0$$

for all n and m. But this subsequence has a subsequence $(A_{n_{k_j}})$ forward almost converging to A, so there exists $J \in \mathbb{N}$ such that for $j \ge J$, one has

$$\frac{1}{n}\sum_{j=1}^{n} d(u,A_{n_{k_j}+m}) - d(u,A) < \epsilon_0$$

for all m. This is a contradiction, so $A_n \xrightarrow{\mathrm{FW\hat{c}}} A$. □

Definition 3.8. Let (U,ϱ) be an asymmetric metric space. For $A, A_k \in CL(U)$, we say that the sequence (A_n) is *forward Wijsman almost statistically convergent* to A if for each $\epsilon > 0$ and for each $u \in U$,

$$\lim_{n\to\infty}\frac{1}{n}|\{k\le n: \quad |d(u,A_{k+m}) - d(u,A)| \ge \epsilon\}| = 0$$

uniformly in m. For this case, we write $st - \lim_{\hat{\mathrm{F}}\mathrm{W}} A_k = A$.

Similarly, we say that the sequence (A_n) is *backward Wijsman almost statistically convergent* to A if for each $\epsilon > 0$ and for each $u \in U$,

$$\lim_{n\to\infty} \frac{1}{n} |\{k \leq n : \quad |d(A_{k+m}, u) - d(A, u)| \geq \epsilon\}| = 0$$

uniformly in m. For this case we write $st - \lim_{\hat{\mathrm{B}}\mathrm{W}} A_k = A$.

The relationships between forward Wijsman strongly Cesàro and forward Wijsman statistical convergence also exist between forward Wijsman strongly almost and forward Wijsman almost statistically convergent sequences. To avoid repetition, these relationships are not given here.

4 - I and I*-Convergence

A non-empty subset of I of $P(\mathbb{N})$ is called an ideal on $\mathbb{N}$ if

(i) $B \in I$ whenever $B \subseteq A$ for some $A \in I$,

(ii) $A \cup B \in I$ whenever $A, B \in I$.

An ideal I is called proper if $\mathbb{N} \notin I$. An ideal I is called admissible if it is proper and contains all finite subsets. For any ideal I there is a filter $F(I)$ corresponding with I, given by $F(I) = \{K \subseteq \mathbb{N} : \quad \mathbb{N} \setminus K \in I\}$.

Let $I \subset P(\mathbb{N})$ be a proper ideal in $\mathbb{N}$. The sequence (u_n) is said to be I-convergent to u, if for $\epsilon > 0$ the set

$$\{n : \quad |u_n - u| \geq \epsilon\}$$

belongs to I. If (u_n) is I-convergent to u then we write $I - \lim u_n = u$.

A sequence (u_n) is said to be I^*-convergent to the number u if there exists a set $M = \{m_1 < m_2 < ...\} \in F(I)$ such that $\lim_{n\to\infty} u_{m_n} = u$. In this case we write $I^* - \lim u_n = u$ (see [**15**]).

Definition 4.1. Let (U, ϱ) be an asymmetric metric space. For $A, A_k \in CL(U)$, we say that the sequence (A_k) is *forward Wijsman I-convergent* to A if for each $\epsilon > 0$ and for each $u \in U$,

$$\{k : \quad |d(u, A_k) - d(u, A)| \geq \epsilon\} \in I.$$

For this case we write $I - \lim_{FW} A_k = A$.

Similarly, we say that the sequence (A_k) is *backward Wijsman I-convergent* to A if for each $\epsilon > 0$ and for each $u \in U$,

$$\{k : \quad |d(A_k, u) - d(A, u)| \geq \epsilon\} \in I.$$

In this case we write $I - \lim_{BW} A_k = A$.

Definition 4.2. Let (U, ϱ) be an asymmetric metric space. For $A, A_k \in CL(U)$, we say that the sequence (A_k) is *forward Wijsman I^*-convergent* to A if and only if there exists a set $M \in \mathcal{F}(I)$ i.e., $\mathbb{N} \setminus M \in I$, $M = \{m_1 < m_2 < m_3 < ... < m_k < ...\}$ such that $lim_{k\to\infty} d(A_{m_k}, u) = d(A, u)$ for each $u \in U$. For this case we write $I^* - \lim_{FW} A_k = A$.

Similarly, we say that the sequence (A_k) is *backward Wijsman I^*-convergent* to A if and only if there exists a set $M \in \mathcal{F}(I)$ i.e., $\mathbb{N} \setminus M \in I$, $M = \{m_1 < m_2 < m_3 < ... < m_k < ...\}$ such that $\lim_{k\to\infty} d(u, A_{m_k}) = d(u, A)$. For this case we write $I^* - \lim_{BW} A_k = A$.

Theorem 4.3. *If a sequence (A_k) in an asymmetric metric space (U, ϱ) is forward Wijsman I-convergent to A and backward Wijsman I-convergent to B then A=B.*

Proof. For each $\epsilon > 0$,

$$\left\{k : |d(A_k, u) - d(A, u)| < \frac{\epsilon}{2}\right\} \text{ and } \left\{k : |d(u, A_k) - d(u, B)| < \frac{\epsilon}{2}\right\} \in \mathcal{F}(I).$$

Since

$$\left\{k : |d(A_k, u) - d(A, u)| < \frac{\epsilon}{2}\right\} \cap \left\{k : |d(u, A_k) - d(u, B)| < \frac{\epsilon}{2}\right\}$$

belongs to $\mathcal{F}(I)$ and $\emptyset \notin \mathcal{F}(I)$, then

$$\sup_{k,u} |d(A_k, u) - d(A, u)| < \frac{\epsilon}{2}$$

and

$$\sup_{k,u} |d(u, A_k) - d(u, B)| < \frac{\epsilon}{2}$$

which implies

$$\begin{aligned} d(A, B) &\leq d(A, A_k) + d(A_k, B) \\ &= \sup_{k,u} |d(A, u) - d(A_k, u)| + \sup_{k,u} |d(u, A_k) - d(u, B)| < \epsilon. \end{aligned}$$

Since $\epsilon > 0$ is arbitrary, we have $A = B$. □

Theorem 4.4. *Let(U, ϱ) be an asymmetric metric space satisfying the property approximate metric axiom. Then forward Wijsman I-convergence of (A_k) implies the backward Wijsman I-convergence and the limits are same.*

Proof. Let (A_k) be forward Wijsman I-convergence to A. Then by the aproximate metric axiom, we can write

$$|d(A_k, u) - d(A, u)| \leq |C(u)[d(u, A_k) - d(u, A)]|.$$

Let $\epsilon > 0$ be given.

$$\begin{aligned} \{k: \quad |d(A_k, u) - d(A, u)| \geq \epsilon\} \quad &\subset \quad \{k: \quad |C(u)[d(u, A_k) - d(u, A)]| \geq \epsilon\} \\ &= \quad \{k: \quad |d(u, A) - d(u, A_k)| \geq \frac{\epsilon}{|C(u)|}\} \in I. \end{aligned}$$

Hence we have $I - \lim_{BW} A_k = A$. □

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