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$p(x)$-biharmonic problem with Navier boundary conditions

Abstract. In this article, we study the following $p(x)$-biharmonic problem with Navier boundary conditions

$$\begin{cases} -\Delta^2_{p(x)} u = \lambda |u|^{p(x)-2} u + f(x,u), & x \in \Omega, \\ u = \Delta u = 0, & x \in \partial\Omega, \end{cases}$$

where f is a Carathéodory function satisfying only a growth condition. Using the Berkovits degree theory, we establish the existence of at least one weak solution of this problem.

Keywords. Navier boundary conditions, variable exponent spaces, Topological degree.

Mathematics Subject Classification: 35G30, 46E35, 47H11.

1 - Introduction

The study of higher order elliptic equations where variable exponents appear is a new and interesting topic. This is motivated by appearances in many applications such as: micro-electro-mechanical systems, thin film theory, surface diffusion on solids, interface dynamics, flow in Hele-Shaw cells and phase field models of multiphase systems (see [**10**,**13**]) and references therein. These types of equations also appear in the mathematical modelling of non-Newtonian fluids (electro-rheological fluids) whose viscosity is not constant and changes rapidly with the electric field, chemical reaction-diffusion equations, etc. For more information, the reader can refer to [**11**, **14**, **16**]. The appropriate functional framework for the study of his problems is that of the Lebegue and Sobolev spaces with variable exponents.

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Many authors have studied problems involving the $p(x)-$Laplacian operator (see for example [**4**, **8**, **15**]). In this paper, we will introduce another important operator that appears in many equations, the $p(x)-$biharmonic operator, for $p(x) > 1$, denoted $\Delta^2_{p(x)}$ defined as

$$\Delta^2_{p(x)} := \Delta(|\Delta u|^{p(x)-2}\Delta u).$$

Consider the following $p(x)$-biharmonic problem with Navier boundary conditions

$$\begin{cases} -\Delta^2_{p(x)} u = \lambda |u|^{p(x)-2}u + f(x,u), & x \in \Omega, \\ u = \Delta u = 0, & x \in \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega, N \geq 1, \lambda \leq 0$, $p : \overline{\Omega} \longrightarrow (1, +\infty)$ is a bounded continuous function and $f : \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ is a Carathéodory function satisgying a growth condition.

Using the Mountain Pass Theorem, the authors in [**7**] establish the existence of at least one solution of this problem. In this paper, we will study the problem (1.1) using another method based on the theory of topological degree, in particular the Berkovits degree for operators of class (S_+). We refer the reader to [**1**, **2**, **3**, **4**, **5**, **15**] and the references therein for more information on this theory and its use.

This document is organised as follows. In section 2, we give a brief overview of Berkovits topological degree, some definitions and fundamental properties of the generalized Lebesgue and Sobolev spaces. Section 3 is reserved for properties of the $p(x)$-Biharmonic operator and technical lemmas. In Section 4, we establish the existence of at least one weak solution to the problem (1.1) using as a main tool the Berkovits degree.

2 - Preliminaries results

2.1 - *Berkovits degree*

Let us start with a brief recapitulation of some essential operator classes to introduce the Berkovits topological degree.

Let X be a real separable reflexive Banach space with dual X^* and with continuous pairing $\langle . \ , \ . \rangle$ and let Ω be a nonempty subset of X. The symbol $\rightarrow$ ($\rightharpoonup$) stands for strong (weak) convergence.

Let Y be a real Banach space. We recall that a mapping $F : \Omega \subset X \to Y$ is *bounded,* if it takes any bounded set into a bounded set. F is said to be *demicontinuous,* if for any $(u_n) \subset \Omega$, $u_n \to u$ implies $F(u_n) \rightharpoonup F(u)$. F is said to be *compact* if it is continuous and the image of any bounded set is relatively compact. A mapping $F : \Omega \subset X \to X^*$ is said to be *of class* (S_+), if for any $(u_n) \subset \Omega$ with $u_n \rightharpoonup u$ and $limsup\langle Fu_n, u_n - u\rangle \leq 0$, it follows that $u_n \to u$. F is said to be *quasimonotone* , if for any $(u_n) \subset \Omega$ with $u_n \rightharpoonup u$, it follows that $limsup\langle Fu_n, u_n - u\rangle \geq 0$.

For any operator $F : \Omega \subset X \to X$ and any bounded operator $T : \Omega_1 \subset X \to X^*$ such that $\Omega \subset \Omega_1$, we say that F satisfies condition $(S_+)_T$, if for any $(u_n) \subset \Omega$ with $u_n \rightharpoonup u$, $y_n := Tu_n \rightharpoonup y$ and $limsup\langle Fu_n, y_n - y\rangle \leq 0$, we have $u_n \to u$.

Let $\mathcal{O}$ be the collection of all bounded open set in X. For any $\Omega \subset X$, we consider the following classes of operators:

$$\begin{aligned}
\mathcal{F}_1(\Omega) &:= \{F : \Omega \to X^* \mid F \text{ is bounded, demicontinuous and} \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{satisfies condition } (S_+)\}, \\
\mathcal{F}_{T,B}(\Omega) &:= \{F : \Omega \to X \mid F \text{ is bounded, demicontinuous and} \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{satisfies condition } (S_+)_T\}, \\
\mathcal{F}_T(\Omega) &:= \{F : \Omega \to X \mid F \text{ is demicontinuous and satisfies condition } (S_+)_T\}, \\
\mathcal{F}_B(X) &:= \{F \in \mathcal{F}_{T,B}(\bar{G}) \mid G \in \mathcal{O}, \mathrm{T} \in \mathcal{F}_1(\bar{\mathrm{G}})\}.
\end{aligned}$$

Here, $T \in \mathcal{F}_1(\bar{G})$ is called an *essential inner map* to F.

Lemma 2.1. [**5**, Lemma 2.2 and 2.4] *Suppose that $T \in \mathcal{F}_1(\bar{G})$ is continuous and $S : D_S \subset X^* \to X$ is demicontinuous such that $T(\bar{G}) \subset D_s$, where G is a bounded open set in a real reflexive Banach space X. Then the following statement are true:*

(i) *If S is quasimonotone, then $I + SoT \in \mathcal{F}_T(\bar{G})$, where I denotes the identity operator.*

(ii) *If S is of class (S_+), then $SoT \in \mathcal{F}_T(\bar{G})$.*

Definition 2.2. Let G be a bounded open subset of a real reflexive Banach space X, $T \in \mathcal{F}_1(\bar{G})$ be continuous and let $F, S \in \mathcal{F}_T(\bar{G})$. The affine homotopy $H : [0,1] \times \bar{G} \to X$ defined by

$$H(t, u) := (1 - t)Fu + tSu \text{ for } (t, u) \in [0,1] \times \bar{G}$$

is called an admissible affine homotopy with the common continuous essential inner map T.

Remark 2.3. [**5**] The above affine homotopy satisfies condition $(S_+)_T$.

We introduce the topological degree for the class $\mathcal{F}_B(X)$ due to Berkovits [**5**].

Theorem 2.4. *There exists a unique degree function*

$$d : \{(F, G, h) | G \in \mathcal{O}, T \in \mathcal{F}_1(\bar{G}), F \in \mathcal{F}_{T,B}(\bar{G}), h \notin F(\partial G)\} \to \mathbb{Z}$$

that satisfies the following properties

1. *(Existence) if $d(F, G, h) \neq 0$, then the equation $Fu = h$ has a solution in G.*
2. *(Additivity) Let $F \in \mathcal{F}_{T,B}(\bar{G})$. If G_1 and G_2 are two disjoint open subset of G such that $h \notin F(\bar{G} \setminus (G_1 \cup G_2))$, then we have*

$$d(F, G, h) = d(F, G_1, h) + d(F, G_2, h).$$

3. *(Homotopy invariance) If $H : [0, 1] \times \bar{G} \to X$ is a bounded admissible affine homotopy with a common continuous essential inner map and $h : [0, 1] \to X$ is a continuous path in X such that $h(t) \notin H(t, \partial G)$ for all $t \in [0, 1]$,then the value of $d(H(t, .), G, h(t))$ is constant for all $t \in [0, 1]$.*
4. *(Normalization) For any $h \in G$, we have $d(I, G, h) = 1$.*

Proposition 2.5. *Let $S : X \to X^*$ and $T : X^* \to X$ be two operators bounded and continuous such that S is quasimonotone and T is an homeomorphism, strictly monotone and of class (S_+). If*

$$\Lambda := \{v \in X^* | v + tSoTv = 0 \text{ for some } t \in [0, 1]\}$$

is bounded in X^*, then the equation

$$v + SoTv = 0$$

admits at lest one solution in X^*.

Proof. Since Λ is bounded in X^*, there exists $R > 0$ such that

$$\|v\|_{X^*} < R \text{ for all } v \in \Lambda.$$

This says that

$$v + tSoTv \neq 0 \text{ for all } v \in \partial B_R(0) \text{ and all } t \in [0, 1]$$

where $B_R(0)$ is the ball of center 0 and radius R in X^*.

Thanks to the Minty-Browder Theorem [**17**, Theorem 26A], the inverse operator $L := T^{-1}$ is bounded, continuous and of type (S_+).

From Lemma 2.7 it follows that

$$I + SoT \in \mathcal{F}_T(\overline{B_R(0)}) \text{ and } I = LoT \in \mathcal{F}_T(\overline{B_R(0)}).$$

Since the operators I, S and T are bounded, $I + SoT$ is also bounded. We conclude that

$$I + SoT \in \mathcal{F}_{T,B}(\overline{B_R(0)}) \text{ and } I \in \mathcal{F}_{T,B}(\overline{B_R(0)}).$$

Consider a homotopy $H : [0,1] \times \overline{B_R(0)} \to X^*$ given by

$$H(t, v) := v + tSoTv \text{ for } (t, v) \in [0,1] \times \overline{B_R(0)}.$$

Let us apply the homotopy invariance and normalization property of the Berkovits degree introduced in Theorem 2.4, we get

$$d(I + SoT, B_R(0), 0) = d(I, B_R(0), 0) = 1,$$

and hence there exists a point $v \in B_R(0)$ such that

$$v + SoTv = 0.$$

□

To study the problem (1.1), using the Berkovits degree, we need some basic properties of variable exponent Lebesgue and Sobolev spaces and of $p(x)-$Biharmonic operator.

2.2 - *Variable exponent Lebesgue and Sobolev spaces*

In this subsection, we recall some useful properties of variable exponent spaces. For more details we refer the reader to [**6, 9, 12**], and the references therein. Denote by $\mathcal{M}(\Omega)$ the set of all measurable real functions on Ω and Consider the set

$$C_+(\overline{\Omega}) = \left\{ p \in C(\overline{\Omega}) : p(x) > 1 \text{ for all } x \in \overline{\Omega} \right\}.$$

For all $p \in C_+(\overline{\Omega})$, we denote

$$p^+ = \sup_{x \in \overline{\Omega}} p(x) \quad \text{and} \quad p^- = \inf_{x \in \overline{\Omega}} p(x).$$

and for all $x \in \overline{\Omega}$ and $k \geq 1$,

$$p^*(x) := \begin{cases} \frac{Np(x)}{N-p(x)} & \text{if } p(x) < N, \\ +\infty & \text{if } p(x) \geq N, \end{cases}$$

$$p_k^*(x) := \begin{cases} \frac{Np(x)}{N-kp(x)} & \text{if } kp(x) < N, \\ +\infty & \text{if } kp(x) \geq N. \end{cases}$$

For any $p \in C_+(\overline{\Omega})$, we define the variable exponent Lebesgue space as

$$L^{p(\cdot)}(\Omega) = \Big\{ u \in \mathcal{M}(\Omega) : \int_\Omega |u(x)|^{p(x)} dx < +\infty \Big\}.$$

endowed with the *Luxemburg norm*, which is defined by

$$\|u\|_{p(\cdot)} = \inf \Big\{ \lambda > 0 : \int_\Omega \Big| \frac{u(x)}{\lambda} \Big|^{p(x)} dx \leq 1 \Big\}.$$

In this paper, we suppose that $p \in C_+(\overline{\Omega})$ such that

$$1 < p^- \leq p(x) \leq p^+ < +\infty. \tag{2.1}$$

From Theorems 1.6 and 1.10 in [**9**], we obtain the following proposition:

Proposition 2.6. *Suppose that* (2.1) *is satisfied. If* Ω *is a bounded open domain, then* $(L^{p(\cdot)}(\Omega), \|u\|_{p(\cdot)})$ *is a reflexive uniformly convex and separable Banach space.*

If $p(\cdot),\ q(\cdot) \in C_+(\overline{\Omega}),\ p(\cdot) \leq q(\cdot)$ a.e. in Ω then there exists the continuous embedding $L^{q(\cdot)}(\Omega) \to L^{q(\cdot)}(\Omega)$.

Let $p'(\cdot) \in C_+(\overline{\Omega})$ be the conjugate exponent of $p(\cdot)$, that is, $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$. Then we have the following Hölder-type inequality

Lemma 2.7. (Hölder inequality). *If* $u \in L^{p(\cdot)}(\Omega)$ *and* $v \in L^{p'(\cdot)}(\Omega)$, *then*

$$\Big| \int_\Omega uv dx \Big| \leq \Big(\frac{1}{p^-} + \frac{1}{p'^-} \Big) \|u\|_{p(\cdot)} \|v\|_{p'(\cdot)} \leq 2 \|u\|_{p(\cdot)} \|v\|_{p'(\cdot)}.$$

A very important role in manipulating the generalized Lebesgue spaces with variable exponent is played by the modular of the $L^{p(\cdot)}(\Omega)$ space, which defined by

$$\begin{aligned} \rho_{p(.)} : L^{p(\cdot)}(\Omega) &\longrightarrow \mathbb{R} \\ u &\longmapsto \rho_{p(.)}(u) = \textstyle\int_\Omega |u(x)|^{p(x)} dx. \end{aligned}$$

Proposition 2.8. *Let $u \in L^{p(\cdot)}(\Omega)$, then we have*

(i) $\|u\|_{p(\cdot)} < 1$ (*resp.* $= 1, > 1$) $\Leftrightarrow \rho_{p(.)}(u) < 1$ (*resp.* $= 1, > 1$),

(ii) $\|u\|_{p(\cdot)} < 1 \Rightarrow \|u\|_{p(\cdot)}^{p+} \leq \rho_{p(.)}(u) \leq \|u\|_{p(\cdot)}^{p-}$,

(iii) $\|u\|_{p(\cdot)} > 1 \Rightarrow \|u\|_{p(\cdot)}^{p-} \leq \rho_{p(.)}(u) \leq \|u\|_{p(\cdot)}^{p+}$.

Corollary 2.9. *Let $u \in L^{p(\cdot)}(\Omega)$, then we have*

(i) $\|u\|_{p(\cdot)} \leq \rho_{p(.)}(u) + 1$,

(ii) $\rho_{p(.)}(u) \leq \|u\|_{p(\cdot)}^{p-} + \|u\|_{p(\cdot)}^{p+}$.

Proposition 2.10. *If $u, u_k \in L^{p(\cdot)}(\Omega)$ and $k \in \mathbb{N}$, then the following assertions are equivalent*

(i) $\lim\limits_{k \to +\infty} \|u_k - u\|_{p(\cdot)} = 0$,

(ii) $\lim\limits_{k \to +\infty} \rho_{p(.)}(u_k - u) = 0$,

(iii) $u_k \longrightarrow u$ *in measure in* Ω *and* $\lim\limits_{k \to +\infty} \rho_{p(.)}(u_k) = \rho_{p(.)}(u)$.

The Sobolev space with variable exponent $W^{k,p(\cdot)}(\Omega)$ is defined as

$$W^{k,p(\cdot)}(\Omega) = \{u \in L^{p(\cdot)}(\Omega) : D^\alpha u \in L^{p(\cdot)}(\Omega), |\alpha| \leq k\},$$

where $D^\alpha u = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \cdots \partial x_N^{\alpha_N}} u$, (the derivation in distributions sense) with $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_N)$ is a multi-index and $|\alpha| = \sum_{i=1}^N \alpha_i$. The space $W^{k,p(\cdot)}(\Omega)$, equipped with the norm

$$\|u\|_{k,p(\cdot)} := \sum_{|\alpha| \leq k} \|D^\alpha u\|_{p(\cdot)},$$

also becomes a Banach, separable and reflexive space.

Proposition 2.11. [**9**] *For $p, r \in C_+(\overline{\Omega})$ such that $r(x) \leq p_k^*(x)$ for all $x \in \overline{\Omega}$, there is a continuous and compact embedding*

$$W^{k,p(\cdot)}(\Omega) \hookrightarrow L^{r(\cdot)}(\Omega).$$

We denote by $W_0^{k,p(\cdot)}(\Omega))$ the closure of $C_0^\infty(\Omega)$ in $W^{k,p(\cdot)}(\Omega)$.

3 - Properties of $p(x)-$Biharmonic operator and technical lemmas

By the form of the problem (1.1), the appropriate functional space for its weak solutions is the following generalized Sobolev space

$$X := W^{2,p(\cdot)}(\Omega) \cap W_0^{1,p(\cdot)}(\Omega)$$

equipped with the norm

$$\|u\| = \inf\{\alpha > 0 : \int_\Omega (|\frac{\Delta u(x)}{\alpha}|^{p(x)} - \lambda|\frac{u(x)}{\alpha}|^{p(x)})dx \leq 1\}.$$

This norm is equivalent to the norms $\|\Delta.\|_{p(\cdot)}$ and $\|\cdot\|_{2,p(\cdot)}$ (see [**7**]), so, taking into account proposition 2.11, there is a continuous and compact embedding of X into $L^{q(x)}(\Omega)$, where $q(x) < p_2^*(x)$ for all $x \in \overline{\Omega}$.

We consider the functional

$$J(u) = \int_\Omega (|\Delta u|^{p(x)} - \lambda|u|^{p(x)})dx.$$

By a proof similar to that of [**9**, Theorem 1.3], we have the following fundamental proposition

Proposition 3.1. *For all $u, u_n \in X$ we have*

(*i*) $\|u\| < 1$ (*resp.* $= 1, > 1$) $\Leftrightarrow J(u) < 1$ (*resp.* $= 1, > 1$),

(*ii*) $\|u\| < 1 \Rightarrow \|u\|^{p+} \leq J(u) \leq \|u\|^{p-}$,

(*iii*) $\|u\| > 1 \Rightarrow \|u\|^{p-} \leq J(u) \leq \|u\|^{p+}$,

(*iv*) $\lim_{n\to+\infty} \|u_n - u\| = 0 \Leftrightarrow \lim_{n\to+\infty} J(u_n - u) = 0$,

(*v*) $\lim_{n\to+\infty} \|u_n\| = \infty \Leftrightarrow \lim_{n\to+\infty} J(u_n) = \infty$.

The energy functional associated to (1.1) is defined by

$$\Psi(u) = \Phi(u) - \Gamma(u),$$

where

$$\Phi(u) = \int_\Omega \frac{1}{p(x)}(|\Delta u|^{p(x)} - \lambda|u|^{p(x)})dx,$$

$$\Gamma(u) = \int_\Omega F(x,u)dx \text{ and } F(x,s) = \int_0^s f(x,t)dt.$$

It is well known that Φ is well defined, even and C^1 in X.

To prove the existence of weak solutions for problem (1.1), one needs some technical lemmas and this growth condition on the Carathéodory function $f : \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$:

$$f(x,s)| \le a(x) + b|s|^{\alpha(x)-1} \tag{3.1}$$

for a.e. $x \in \Omega$ and all $s \in \mathbb{R}$, with $a(x) \ge 0$, b is a positive constant, $a(x) \in L^{\alpha'(\cdot)}(\Omega)$, $\alpha \in C_+(\overline{\Omega})$ and $\alpha^+ \le p^-$.

Lemma 3.2. [**7**, Proposition 3.3] *If f satisfies* (3.1) *then*

(*i*) $\Gamma \in C^1(X,\mathbb{R})$ *and for u, v in X, we have*

$$\langle \Gamma'(u), v\rangle = \int_\Omega f(x,u)v dx.$$

(*ii*) *The operator $\Gamma' : X \to X^*$ is completely continuous.*

Consequently, the weak solutions of (1.1) are the critical points of the functional Ψ. Moreover, the operator $L := \Phi' : X \to X^*$ defined as

$$\langle L(u), v\rangle = \int_\Omega (|\Delta u|^{p(x)-2}\Delta u \Delta v - \lambda |u|^{p(x)-2} uv) dx \ \ \forall u, v \in X$$

satisfies the assertions of the following lemma

Lemma 3.3. [**7**, Theorem 3.4]

(*i*) *L is continuous, bounded and strictly monotone,*

(*ii*) *L is of (S_+) type,*

(*iii*) *L is a homeomorphism.*

4 - Main results

As we have seen before, the weak solutions of (1.1) are the critical points of the functional Ψ. Therefore, we can consider this definition

Definition 4.1. We say that $u \in X$ is a weak solution of problem (1.1) if

$$\int_\Omega (|\Delta u|^{p(x)-2}\Delta u \Delta v - \lambda |u|^{p(x)-2} uv) dx - \int_\Omega f(x,u) v dx = 0 \ \ \forall v \in X.$$

The main result of this paper is given by the following theorem

Theorem 4.2. *Let Ω be a Lipschitz bounded domain in $\mathbb{R}^N$, $p \in C_+(\overline{\Omega})$ satisfies* (2.1) *and f satisfies* (3.1). *Then, for all $\lambda \leq 0$, problem* (1.1) *admits at least one weak solution.*

Proof. $u \in X$ is a weak solution of (1.1) if and only if

$$Lu - \Gamma' u = 0. \tag{4.1}$$

Thanks to the properties of operator L seen in Lemma 3.3 and in view of Minty-Browder Theorem [**17**, Theorem 26A], the inverse operator $T := L^{-1} : X^* \to X$ is bounded, continuous and of type (S_+). Moreover, note that, according to Lemma 3.2 and the reflexivity of X, the operator Γ' is bounded, continuous and quasimonotone. Therefore, equation (4.1) is equivalent to

$$u = Tv \text{ and } v + SoTv = 0, \tag{4.2}$$

where $S = -\Gamma'$. To solve equation (4.2), we will apply the Proposition 2.5. It is sufficient to show that the set

$$\Lambda := \{v \in X^* | v + tSoTv = 0 \text{ for some } t \in [0,1]\}$$

is bounded. Indeed, let $v \in \Lambda$. Set $u := Tv$, then $\|Tv\| = \|u\|$.

If $\|u\| \leq 1$, then $\|Tv\|$ is bounded.

If $\|u\| > 1$, then we get by the implication (iii) in Proposition 3.1 the estimate

$$\begin{aligned}
\|Tv\|_w^{p^-} &= \|u\|_w^{p-} \\
&\leq J(u) \\
&= \langle Lu, u\rangle \\
&= \langle v, Tv\rangle \\
&= -t\langle SoTv, Tv\rangle \\
&= t\int_\Omega f(x,u)u\,dx \\
&\leq \int_\Omega (a(x) + b|u|^{\alpha(x)-1})u\,dx.
\end{aligned}$$

Since $X \subset L^{\alpha(\cdot)}(\Omega)$ and in virtu of the Hölder inequality and (ii) of Corollary 2.9, we have

$$\begin{aligned}
\|Tv\|_w^{p^-} &\leq \|a\|_{\alpha'(\cdot)}\|u\|_{\alpha(\cdot)} + b\|u\|_{\alpha(\cdot)}^{\alpha^-} + b\|u\|_{\alpha(\cdot)}^{\alpha^+} \\
&\leq const(\|u\|_{\alpha(\cdot)} + \|u\|_{\alpha(\cdot)}^{\alpha^-} + \|u\|_{\alpha(\cdot)}^{\alpha^+}).
\end{aligned}$$

From the the compact embedding $X \hookrightarrow L^{\alpha(\cdot)}(\Omega)$ (because $\alpha(x) < p^- \leq p(x) < p_2^*$), we can deduct the estimate

$$\|Tv\|^{p^-} \leq const(\|Tv\| + \|Tv\|^{\alpha^+}).$$

It follows that $\{Tv|v \in \Lambda :\}$ is bounded.

Since the operator S is bounded, it is obvious from (4.2) that the set $\Lambda :$ is bounded in X^*. Hence, in virtu of Proposition 2.5, the equation $v+SoTv$ have at lest one solution $\bar{v}$ in X^*. We conclude that $\bar{u} = T\bar{v}$ is a weak solution of (1.1). □

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References

[**1**] M. Ait Hammou and E. Azroul, *Construction of a topological degree theory in generalized Sobolev spaces*, Recent advances in intuitionistic fuzzy logic systems, Stud. Fuzziness Soft Comput., **372**, Springer, Cham, 2019, 1–18.

[**2**] M. Ait Hammou and E. Azroul, *Existence of weak solutions for a nonlinear parabolic equation by topological degree*, Adv. Theory Nonlinear Anal. Appl. **4** (2020), no. 4, 292–298.

[**3**] M. Ait Hammou and E. Azroul, *Existence result for a nonlinear elliptic problem by topological degree in Sobolev spaces with variable exponent*, Moroccan J. of Pure and Appl. Anal. **7** (2020), no. 1, 50–65.

[**4**] M. Ait Hammou and B. Lahmi, *Weak solutions for a quasilinear elliptic equation involving the $p(x)-$Laplacian*, Rend. Semin. Mat. Univ. Politec. Torino **79** (2021), no. 1, 33–42.

[**5**] J. Berkovits, *Extension of the Leray-Schauder degree for abstract Hammerstein type mappings*, J. Differential Equations **234** (2007), no. 1, 289–310.

[**6**] L. Diening, P. Harjulehto, P. Hästö and M. Růžička, *Lebesgue and Sobolev spaces with variable exponents*, Lecture Notes in Math., **2017**, Springer, Heidelberg, 2011.

[**7**] A. El Amrouss, F. Moradi and M. Moussaoui, *Existence of solutions for fourth-order PDEs with variable exponents*, Electron. J. Differential Equations **2009** (2009), no. 153, 13 pp.

[**8**] X.-L. Fan and Q.-H. Zhang, *Existence of solutions for $p(x)$-Laplacian Dirichlet problem*, Nonlinear Anal. **52** (2003), no. 8, 1843–1852.

[**9**] X. Fan and D. Zhao, *On the spaces $L^{p(x)}(\Omega)$ and $W^{m,p(x)}(\Omega)$*, J. Math. Anal. Appl. **263** (2001), no. 2, 424–446.

[**10**] A. Ferrero and G. Warnault, *On solutions of second and fourth order elliptic equations with power-type nonlinearities*, Nonlinear Anal. **70** (2009), no. 8, 2889–2902.

[**11**] T. C. Halsey, *Electrorheological fluids*, Science **258** (1992), no. 5083, 761–766.

[**12**] O. Kováčik and J. Rákosník, *On spaces $L^{p(x)}$ and $W^{k,p(x)}$*, Czechoslovak Math. Jour. **41(116)** (1991), no. 4, 592–618.

[**13**] T. G. Myers, *Thin films with high surface tension*, SIAM Rev. **40** (1998), no. 3, 441–462.

[**14**] M. Ružička, *Electrorheological fluids: modeling and mathematical theory*, Lecture Notes in Math., **1748**, Springer-Verlag, Berlin, 2000.

[**15**] B.-S. Wang, G.-L. Hou and B. Ge, *Existence and uniqueness of solutions for the $p(x)-$Laplacian equation with convection term*, Mathematics **8** (2020), no. 10, paper n. 1768.

[**16**] W. Wang and E. Canessa, *Biharmonic pattern selection*, Phys. Rev. E **47** (1993), no. 2, 1243–1248.

[**17**] E. Zeidler, *Nonlinear functional analysis and its applications. II/B: Nonlinear monotone operators*, Springer-Verlag, New York, 1990.

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