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Contractibility and countable projective limits

Abstract. We discuss contractibility in the framework of topological algebras given by countable projective limits of DF-spaces. Full characterization of Köthe-type PLB-algebras is provided.

Keywords. Projective/inductive limit, DF-space, topological algebra, contractible algebra.

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1 - Introduction

The present work is motivated by [**18**] where the author studied contractibility properties of DF-algebras. In this paper those results are extended onto algebras which (topologically) are countable projective limits of DF-spaces.

The paper is organized as follows. Section 2 contains a preliminary material and some technical results of structural nature. Section 3 focuses on abstract characterizations of general PDF-algebras. The next section deals with specific PDF-spaces, namely Köthe-type PLB spaces and is thought of as a short introduction into the final section which provides a characterization of contractible Köthe-type PLB-algebras in terms of the defining Köthe matrix.

General references are: [**19**] for projective limits of countable spectra, [**10**, **12**] for tensor products of locally convex spaces, [**3**] for Banach (and topological) algebra theory and [**8**, **9**] for homological algebra.

2 - Preliminaries

All the vector spaces are assumed over $\mathbb{C}$ but analogous results remain true if one considers them over $\mathbb{R}$. Specific categories with their symbols are as follows:

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B – Banach spaces/algebras, F - Fréchet spaces/algebras, DF - complete DF-spaces/algebras, PDF - complete PDF-spaces/algebras, V - vector spaces. It will be clear from the context whether those symbols are used to indicate the category of spaces or algebras.

If $a = (a_j)_{j\in\mathbb{N}}$ and $b = (b_j)_{j\in\mathbb{N}}$ are two sequences of non-negative numbers then $a/b \in \ell_\infty$ means that there is a constant $C > 0$ such that $a_j \leqslant Cb_j$ for all $j \in \mathbb{N}$. This notation will be used regularly in the sequel.

The main objects in the sequel are algebras which arise as countable projective limits of some locally convex spaces. This construction can be found e.g. in [**19**, Definition 3.1.3] or [**10**, Section I.2.6] but we recall it below for convenience of the reader. Our construction is restricted to the category of Hausdorff locally convex spaces but it is also possible in more general categories, e.g. topological vector spaces, abelian topological groups, etc. A *projective spectrum* $\mathcal{X}$ is a sequence $(X_n)_{n\in\mathbb{N}}$ of Hausdorff locally convex spaces together with continuous linear mappings $(\pi^n_m\colon X_n \to X_m)_{m\leqslant n}$ which satisfy:

(i) $\pi^n_n = \mathrm{id}_{X_n} \quad (n \in \mathbb{N})$,

(ii) $\pi^n_k \circ \pi^k_m = \pi^n_m \quad (m \leqslant k \leqslant n)$.

We then write $\mathcal{X} = (X_n, \pi^n_m)$. The subspace

$$X := \Big\{(x_n)_{n\in\mathbb{N}} \in \prod_{n\in\mathbb{N}} X_n\colon\ x_m = \pi^n_m x_n \quad \text{for all} \quad m, n \in \mathbb{N},\, m \leqslant n\Big\},$$

denoted by $X = \mathrm{proj}_n X_n$ is called the *projective limit* of $\mathcal{X}$. If $P_n\colon \prod_k X_k \to X_n$, $n \in \mathbb{N}$ is the sequence of canonical projections then their restrictions $\pi_n := P_n|_{X_n}$ satisfy the relation

$$\pi_m = \pi^n_m \circ \pi_n \quad (m \leqslant n). \tag{1}$$

The *projective limit topology* on X is the weakest locally convex topology for which all the mappings $\pi_n\colon X \to X_n$, $n \in \mathbb{N}$ are continuous. We will consider projective limits always with their projective limit topologies. A projective limit is *reduced* if $\pi_n(X)$ is dense in X_n for every $n \in \mathbb{N}$. Without loss of generality we may (and will) assume that projective limits are always reduced – [**10**, 2.6, Proposition 2]. From [**10**, 3.4, Proposition 6] it follows that if $X = \mathrm{proj}_n X_n$ then $\widetilde{X} = \mathrm{proj}_n \widetilde{X_n}$, where $\widetilde{\ }$ denotes the completion. Therefore we may (and always will) assume that a complete projective limit arises from a projective spectrum of complete spaces. A locally convex space is called a *PDF-space* if it is topologically isomorphic to a projective limit of a sequence of some DF-spaces (for the definition of a DF-space see [**16**, p. 297]). In general we do not assume completeness. Below we list some examples of such objects.

Example 2.1. The following are PDF-spaces:

1. Banach spaces, Fréchet spaces, DF-spaces and PLB-spaces,
2. the space $\mathscr{A}(\Omega)$ of real analytic functions on some open subset $\Omega \subset \mathbb{R}^d$,
3. the space $\mathscr{D}'(\Omega)$ of distributions on some open subset $\Omega \subset \mathbb{R}^d$,
4. the multiplier algebra of the noncommutative Schwartz space.

There is an extensive literature concerning PDF-spaces, especially concerning $\mathscr{A}(\Omega)$ and $\mathscr{D}'(\Omega)$ – see the work of Bonet, Domański, Vogt and others. The multiplier algebra of the noncommutative Schwartz space is considered in [**2**] and for more on PLB-spaces (in particular, PLS-spaces and PLN-spaces) – see [**5**] and references therein.

Before introducing PDF-algebras, we collect some hereditary properties of PDF-spaces.

Proposition 2.2. *If X and Y are PDF-spaces then $X\widehat{\otimes}Y$ and $X \oplus \mathbb{C}$ are PDF-spaces as well.*

Proof. Let $X = \operatorname{proj}_n X_n$ and $Y = \operatorname{proj}_n Y_n$ with all X_n's and Y_n's being DF-spaces. From [**10**, 15.4, Theorem 2] it follows that

$$X\widehat{\otimes}Y = \operatorname{proj}_n(X_n\widehat{\otimes}Y_n),$$

and from [**10**, 15.6, Theorem 2] it follows that $X_n\widehat{\otimes}Y_n$ is a DF-space for every $n \in \mathbb{N}$. Therefore $X\widehat{\otimes}Y$ is a PDF-space. Moreover,

$$X \oplus \mathbb{C} = \operatorname{proj}_n(X_n \oplus \mathbb{C})$$

and $X_n \oplus \mathbb{C}$ is a DF-space by [**10**, 12.4, Theorem 8]. The conclusion follows. □

Proposition 2.3. *If Y is a complemented subspace of a PDF-space X then Y is also a PDF-space.*

Proof. Let $X = \operatorname{proj}_n(X_n, \pi^n_m)$ be a PDF-space and assume that $X = Y \oplus Z$ topologically. Let $P\colon X \to X$ be a projection onto Y and denote

$$Y_n := \overline{\pi_n(Y)}^{X_n}, \quad Z_n := \overline{\pi_n(Z)}^{X_n} \qquad (n \in \mathbb{N}).$$

Clearly, $Y = \operatorname{im} P$ and $Z = \ker P$. Let us now consider the projective spectra

$$\left(X_n/Z_n, \rho^n_m\right) \quad \text{and} \quad (Y_n, \iota^n_m),$$

where the linking maps are defined as

$$\rho_m^n\colon X_n/Z_n \to X_m/Z_m, \qquad \iota_m^n\colon Y_n \to Y_m,$$
$$\rho_m^n(x+Z_n) := \pi_m^n x + Z_m, \qquad \iota_m^n y := \pi_m^n y.$$

Clearly, the projective limits $\operatorname{proj}_n X_n/Z_n$ and $\operatorname{proj}_n Y_n$ exist – recall that the maps $(\rho_m^n)_{m\leqslant n}$ are well-defined due to (1). We will now show that

$$\operatorname{proj}_n X_n/Z_n \simeq \operatorname{proj}_n Y_n \qquad \text{(topologically)}. \tag{2}$$

To this end, observe that from [**4**, Lemma 4] (the original proof works in our case as well) it follows that for every $m \in \mathbb{N}$ there is $n \geqslant m$ such that there is a continuous linear mapping $P_{m,n}\colon X_n \to X_m$ and the diagram

$$\begin{array}{ccc} X & \xrightarrow{P} & X \\ \big\downarrow{\scriptstyle \pi_n} & & \big\downarrow{\scriptstyle \pi_m} \\ X_n & \xrightarrow{P_{m,n}} & X_m \end{array}$$

commutes. Therefore one may define continuous and linear maps

$$X_n/Z_n \xrightarrow{f_m} Y_m \xrightarrow{g_m} X_m/Z_m,$$

where the mappings f_m, g_m, $m \in \mathbb{N}$ are defined as

$$f_m(x+Z_n) := P_{m,n}x, \quad g_m y := y + Z_m \qquad (m \in \mathbb{N}).$$

Straightforward computation shows that the above maps are injective and satisfy the relations

$$g_m \circ f_m = \rho_m^n, \qquad f_n \circ g_n = \iota_m^n \qquad (m \leqslant n).$$

This proves (2). From the proof of [**7**, Proposition 1.2] it follows that $Y = \operatorname{proj}_n Y_n$ whereas from [**10**, 12.4, Theorem 8] it follows that $\operatorname{proj}_n X_n/Z_n$ is a PDF-space. Consequently, Y is a PDF-space as well and the proof is thereby complete. □

Let now A be a locally convex space in which one can define a multiplication $m\colon A \times A \to A$. We say that A is a *topological algebra* if m is jointly continuous – see [**3**, Definition 2.2.5]. Again we do not assume completeness. Some authors call A a topological algebra if m is only separately continuous – see e.g. [**14**]. Then algebras with jointly continuous multiplication are called $\widehat{\otimes}$-algebras – see e.g. [**13**, **17**]. A *PDF-algebra* is a topological algebra for which the underlying locally convex space is a PDF-space. If A is a PDF-algebra then the assignment $a \otimes b \mapsto ab$ extends to a continuous linear map

$$\pi_A\colon A \widehat{\otimes} A \to A,$$

where $A\widehat{\otimes}A$ denotes the completed projective tensor product of A with itself. This map is called the *product map.* We will omit the subscript since it will always be clear to which algebra the product map is referred to. A unit in a unital algebra will be denoted by $\mathbf{1}$. If A is not unital then by A_1 we denote its *unitization*, i.e.

$$A_1 := A\oplus\mathbb{C} \quad \text{and} \quad (a,\alpha)\cdot(b,\beta) := (ab+\beta a+\alpha b, \alpha\beta) \qquad (a,b\in A,\ \alpha,\beta\in\mathbb{C}).$$

By $A^{\#}$ we will denote the *conditional unitization* of the algebra A, i.e. $A^{\#}$ is A if the algebra in question is unital and A_1 elsewhere. From Proposition 2.2 it follows that if A is a PDF-algebra then so is $A^{\#}$.

If A is a topological algebra then a locally convex space is an A-*bimodule* if it is an algebraic bimodule and the bimodule operations are jointly continuous. We then write $X\in A\text{-}\mathsf{mod}\text{-}A$. If A and X belong to some category C then we write $X\in A\text{-}\mathsf{mod}\text{-}A(\mathsf{C})$. Similar notation applies to left and right modules. If A is a PDF-algebra then by a PDF-A-*bimodule* we will mean an A-bimodule which – topologically – is a PDF-space. Recall also that a *derivation* is a linear (not necessarily continuous) map $\delta\colon A\to X$ from an algebra A into an A-bimodule X satisfying the so-called ‘derivation rule’, i.e.

$$\delta(ab) = a\cdot\delta(b)+\delta(a)\cdot b \qquad (a,b\in A).$$

If there is an element $x\in X$ such that

$$\delta(a) = a\cdot x - x\cdot a \qquad (a\in A)$$

then δ is an *inner derivation* and we use the notation $\delta = \mathrm{ad}_x$.

3 - Contractible PDF-algebras

In this section we will show that several definitions of contractibility are equivalent in PDF. We start with the language of homological algebra. All the definitions and results are formulated for bimodules. Obvious changes lead to respective definitions and results for left and right modules. Let A be a PDF-algebra. Following Helemskiĭ [**8**, Ch. III, Definition 1.13] we say that a PDF-A-bimodule P is *projective* if for every admissible short exact sequence in $A\text{-}\mathsf{mod}\text{-}A(\mathsf{PDF})$

$$0 \longrightarrow X \xrightarrow{\ i\ } Y \xrightarrow{\ q\ } Z \longrightarrow 0$$

the sequence

$$0 \longrightarrow {}_AL_A(P,X) \xrightarrow{\ I\ } {}_AL_A(P,Y) \xrightarrow{\ Q\ } {}_AL_A(P,Z) \longrightarrow 0$$

is exact in V.

Remark 3.1. 1. *The adjective 'admissible' in the above definition means that i and q are bimodule maps, q is open and has a right inverse.*
2. *By ${}_AL_A(P,X)$ we denote the space of all bimodule maps from P to X.*

Before characterizing projective PDF-bimodules let us recall that by an *admissible epimorphism* $T \in {}_AL_A(X,Y)$ between PDF-A-bimodules X,Y we mean a bimodule map which is an open surjection and has a continuous right inverse $Q \in L(X,Y)$. We emphasise the assumption that the map T is open. This follows from the fact that there is no Open Mapping Theorem in PDF.

The following result is proved along the same lines as in B therefore we omit it.

Proposition 3.2 ([**8**, Ch. III, Proposition 1.14] (cf. [**3**, Definition 2.8.35])). *Let A be a* PDF*-algebra. A* PDF*-A-bimodule P is projective if and only if for any $Y,Z \in A$-mod-$A(\mathsf{PDF})$, every admissible epimorphism $T \in {}_AL_A(Y,Z)$ and every $S \in {}_AL_A(P,Z)$ there exists $R \in {}_AL_A(P,Y)$ such that $T \circ R = S$, i.e. the diagram*

$$\begin{tikzcd} & P \arrow[dl, "R"'] \arrow[d, "S"] \\ Y \arrow[r, "T"] & Z \end{tikzcd}$$

is commutative.

From [**10**, 3.4, Proposition 6] and [**10**, 12.4, Theorem 8] it follows that completions of PDF-spaces are again PDF. Moreover, if A is a PDF-algebra and X is a PDF-A-bimodule then $\widetilde{X}$ becomes a PDF-A-bimodule in the obvious way. Therefore we get the following result.

Proposition 3.3. *Let A be a* PDF*-algebra and let P be a* PDF*-A-bimodule. Then P is projective if and only if $\widetilde{P}$ is projective.*

Following [**8**, Ch. IV, Definition 5.1] we say that a PDF-algebra A is *biprojective* if A is a projective PDF-A-bimodule.

Proposition 3.4. *A complete* PDF*-algebra A is biprojective if and only if the product map is a retraction, i.e. there exists a bimodule map $\sigma\colon A \to A\widehat{\otimes}A$ such that $\pi \circ \sigma = \mathrm{id}_A$.*

Proof. This follows from Proposition 2.2 and the proof of [**3**, Proposition 2.8.41]. $\square$

Corollary 3.5. *If $A \in \mathsf{PDF}$ is biprojective then the product map is an open surjection.*

Proposition 3.6. *If $A \in \mathsf{PDF}$ is unital then* $\ker \pi$ *is complemented in* $A\widehat{\otimes}A$ *and hence a* PDF*-space.*

Proof. The operator

$$\rho\colon A\widehat{\otimes}A \to A\widehat{\otimes}A, \qquad \rho(u) := u - \pi(u) \otimes \mathbf{1}$$

is a projection onto $\ker \pi$. The conclusion follows from Proposition 2.3. □

Recall that if $A \in \{\mathsf{B}, \mathsf{F}, \mathsf{DF}\}$ then biprojectivity of $A^{\#}$ can be rephrased in the language of derivations. The following result extends it onto the category PDF.

Theorem 3.7. *Let A be a complete* PDF*-algebra. The following conditions are equivalent:*

(*i*) *every continuous derivation from A into any complete* PDF*-A-bimodule is inner,*

(*ii*) *A is unital and biprojective,*

(*iii*) *A is unital and has a projective diagonal in $A\widehat{\otimes}A$, i.e. an element $d \in A\widehat{\otimes}A$ such that*

$$\pi(d) = \mathbf{1}, \qquad\qquad a \cdot d = d \cdot a \quad (a \in A),$$

(*iv*) *$A^{\#}$ is biprojective.*

Proof. (*i*) ⇒ (*ii*): From Proposition 3.6 we know that $\ker \pi$ is a PDF-A-bimodule and we define a derivation

$$\delta\colon A \to \ker \pi, \quad \delta(a) := a \otimes \mathbf{1} - \mathbf{1} \otimes a \qquad (a \in A)$$

to conclude that π is a retraction. By Proposition 3.4 A is biprojective. That A is unital is proved along the same lines as in the proof of [**3**, Theorem 1.9.21].
(*ii*) ⇒ (*iii*): If $\sigma\colon A \to A\widehat{\otimes}A$ is a right bimodule inverse to the product map then $\sigma(\mathbf{1})$ is a projective diagonal.
(*iii*) ⇒ (*i*): Let $\delta\colon A \to X$ be a continuous derivation into some $X \in A\text{-}\mathsf{mod}\text{-}A(\mathsf{PDF})$. Then the assignment $a \otimes b \mapsto a \cdot \delta(b)$ gives rise to a linear and continuous map $D\colon A\widehat{\otimes}A \to X$. Additionally, the maps

$$D_a\colon A\widehat{\otimes}A \to X, \quad D_a(v) := D(v \cdot a) - D(v) \cdot a - \pi(v) \cdot \delta(a) \qquad (a \in A)$$

are also continuous. If now $d \in A\widehat{\otimes}A$ denotes a projective diagonal then for any $a \in A, u \in A \otimes A$ we get

$$a \cdot D(d) - D(d) \cdot a - \delta(a) = D_a(d - u). \tag{3}$$

Let $\mathscr{U}$ be a zero neighbourhood basis in X. For any $U \in \mathscr{U}$ we choose another zero neighbourhood $V \subset A \widehat{\otimes} A$ such that $D_a(V) \subset U$ and an element $u \in A \otimes A$ so that $d - u \in V$. From (3) we get that $a \cdot D(d) - D(d) \cdot a - \delta(a) \in U$. Consequently,

$$a \cdot D(d) - D(d) \cdot a - \delta(a) \in \bigcap_{U \in \mathscr{U}} U = \overline{\{0\}} = \{0\},$$

since X is a Hausdorff space. As $a \in A$ was arbitrary, we get

$$\delta(a) = a \cdot D(d) - D(d) \cdot a \qquad (a \in A)$$

which shows that δ is inner.
(ii) $\Leftrightarrow$ *(iv)*: this is proved as in the Banach algebra case. □

Remark 3.8. The proof of *(iii)* $\Rightarrow$ *(i)* cannot rely on the representations of arbitrary $u \in A \widehat{\otimes} A$ of the form $u = \sum_n \lambda_n (x_n \otimes y_n)$, where $(\lambda_n)_n \in \ell_1$ and $(x_n)_n, (y_n)_n$ are bounded sequences in A. In the category PDF such a decomposition is not possible in general. It is not even possible in the category DF – see [**15**, Proposition 4].

We now define contractibility in the category PDF. Following [**9**, Definition VII.1.59] we say that a complete PDF-algebra A is *contractible* if $A^{\#}$ is biprojective. A PDF-algebra is *contractible* if its completion is a contractible PDF-algebra.

Corollary 3.9. *An algebra $A \in \mathsf{PDF}$ is contractible if and only if $A^{\#}$ is contractible.*

Proof. Follows from Theorem 3.7 and Proposition 3.3. □

Using Theorem 3.7 and mimicking the proof of [**3**, Theorem 2.8.64] we can derive another consequence.

Corollary 3.10. *Let $A, B \in \mathsf{PDF}$ be algebras and let $\theta \colon A \to B$ be a dense range algebra homomorphism. If A is contractible then so is B.*

From Example 2.1 it follows that $\mathsf{B}, \mathsf{F}, \mathsf{DF} \subset \mathsf{PDF}$. Therefore Theorem 3.7 allows us to derive the following consequence.

Corollary 3.11. *Let A be any of the categories B, F or DF and let $A \in \mathsf{A}$ be an algebra. Then A is contractible in PDF if and only if it is contractible in A.*

In case A is a PDF-algebra with some additional topological properties we can slightly simplify condition (i) of Theorem 3.7. To this end, let us introduce three subcategories of DF:

- DFM – duals of Fréchet-Montel spaces,
- DFS – duals of Fréchet-Schwartz spaces,
- DFN – duals of nuclear Fréchet spaces.

and three subcategories of PDF:

- PDFM – countable projective limits of elements from DFM,
- PDFS – countable projective limits of elements from DFS,
- PDFN – countable projective limits of elements from DFN.

Observe that the objects in all the above categories are complete spaces and $\mathsf{DFN} \subset \mathsf{DFS} \subset \mathsf{DFM}$.

Corollary 3.12. *Let* $\mathsf{A} \in \{\mathsf{PDFM}, \mathsf{PDFS}, \mathsf{PDFN}\}$ *and let* $A \in \mathsf{A}$ *be an algebra. Then* A *is contractible if and only if every continuous derivation into any bimodule* $X \in A$-mod-$A(\mathsf{A})$ *is inner.*

Proof. The 'if' part is clear. To get the 'only if' part it is enough to repeat the proof of $(i) \Rightarrow (ii)$ of Theorem 3.7. To guarantee this we will show that $\ker \pi \in \mathsf{A}$ whenever $A \in \mathsf{A}$. First of all we observe that from the assumption it follows that A is unital (this is proved exactly as in the Banach algebra case – see e.g. [**3**, Theorem 2.8.48]) and hence by Proposition 3.6 $\ker \pi$ is complemented in $A \widehat{\otimes} A$. If $\mathbf{1}$ is the unit in A and if we denote $A = \mathrm{proj}_n(A_n, \iota_m^n) \in \mathsf{PDF}$ then from the proof of Proposition 2.3 it follows that

$$\ker \pi = \mathrm{proj}_n (A_n \widehat{\otimes} A_n)/L_n, \quad \text{where} \quad L_n = \overline{\iota_n(\mathrm{im}\, \pi \otimes \mathbf{1})} \qquad (n \in \mathbb{N}).$$

Moreover, from [**12**, §45.3 (7)] it follows that

$$A_n \widehat{\otimes} A_n \in \mathsf{DFM} \qquad (n \in \mathbb{N})$$

and from [**11**, §29.5 (1)] it follows that

$$((A_n \widehat{\otimes} A_n)/L_n)' = L_n^{\perp} \qquad (n \in \mathbb{N}) \tag{4}$$

is a Fréchet-Montel space hence reflexive by [**16**, Remark 24.24]. We now distinguish three cases:

- If $A \in \mathsf{PDFM}$ then $L_n^{\perp}$ is a closed subspace of the Fréchet-Montel space

$(A_n \widehat{\otimes} A_n)'$. Therefore Fréchet-Montel by [**16**, Proposition 23.23] and [**10**, 11.5, Proposition 4].
• If $A \in \mathsf{PDFS}$ then $A_n \widehat{\otimes} A_n \in \mathsf{DFS}$ by [**10**, 16.4, Corollary 3] and $L_n^\perp$ is a closed subspace of the Fréchet-Schwartz space $(A_n \widehat{\otimes} A_n)'$. Therefore Fréchet-Schwartz by [**16**, Proposition 24.18].
• If $A \in \mathsf{PDFN}$ then $A_n \widehat{\otimes} A_n \in \mathsf{DFN}$ by [**10**, 18.1, Theorem 8 and 21.2, Theorem 1] (recall that nuclear spaces have the approximation property) and $L_n^\perp$ is a closed subspace of the nuclear Fréchet space $(A_n \widehat{\otimes} A_n)'$. Therefore a nuclear Fréchet space by [**16**, Proposition 28.6].
From (4) it now follows that $(A_n \widehat{\otimes} A_n)/L_n \in \mathsf{A}$, whenever $A \in \mathsf{A}$. Consequently, $\ker \pi \in \mathsf{A}$ whenever $A \in \mathsf{A}$. □

Let PBDF denote the category of countable projective limits of complete barrelled DF-spaces and recall from [**10**, 11.3, Proposition 1 and 12.4, Theorem 8] that quotients of barrelled DF-spaces are again barrelled DF-spaces. We therefore may derive another consequence of Theorem 3.7.

Corollary 3.13. *Let $A \in \mathsf{PBDF}$ be an algebra. Then A is contractible if and only if every continuous derivation into any bimodule $X \in A\text{-}\mathsf{mod}\text{-}A(\mathsf{PBDF})$ is inner.*

We end this section with an instructive example of a natural PDF-algebra which is not contractible.

Example 3.14. Let $\Omega \subset \mathbb{R}^d$ be open and let $\mathscr{A}(\Omega)$ be the space of real analytic functions. It becomes a PDF-algebra with multiplication defined as

$$(f \cdot g)(t) := f(t)g(t) \qquad (t \in \Omega, f, g \in \mathscr{A}(\Omega)). \tag{5}$$

To see it is jointly continuous, recall that

$$\mathscr{A}(\Omega) = \operatorname{proj}_N \operatorname{ind}_n \mathscr{H}^\infty(U_{N,n}),$$

where $(U_{N,n})_{n\in\mathbb{N}}$ is a fundamental sequence of neighbourhoods of I_N in $\mathbb{C}^d$, $(I_N)_{N\in\mathbb{N}}$ is a compact exhaustion of Ω and $\mathscr{H}^\infty(U_{N,n})$ is a Banach space of bounded holomorphic functions on $U_{N,n}$ with the sup-norm – see [**6**] for details. $\mathscr{H}^\infty(U_{N,n})$ is also a Banach algebra since

$$\|fg\|_{\infty,N,n} := \sup_{z\in U_{N,n}} |(fg)(z)| \leqslant \|f\|_{\infty,N,n}\|g\|_{\infty,N,n} \qquad (f, g \in \mathscr{H}^\infty(U_{N,n})).$$

Consequently, the pointwise multiplication (5) is jointly continuous. Although the algebra $\mathscr{A}(\Omega)$ is unital with the constant one function as the unit, it is not contractible since the derivation

$$\delta\colon \mathscr{A}(\Omega) \to \mathscr{A}(\Omega), \qquad \delta(f) := f'$$

is not inner (recall that by commutativity all inner derivations are trivial).

4 - Köthe-type PLB-spaces

The best source of information on PLB-spaces is a survey article [**5**] which contains also a comprehensive list of references on this topic. We therefore restrict ourselves to recalling only basic definitions and facts. A *PLB-space* is a projective limit of a sequence of LB-spaces. This means that every PLB-space X can be viewed as

$$X = \mathrm{proj}_{N\in\mathbb{N}}\, \mathrm{ind}_{n\in\mathbb{N}}\, X_{N,n},$$

where all the $X_{N,n}$'s are Banach spaces and all the linking maps $\iota_{N,n}^{N,n+1}\colon X_{N,n} \to X_{N,n+1}$ are continuous. If, in addition, all these linking maps are compact (nuclear) then X is called a *PLS-space* (*PLN-space*). We also define $X_N := \mathrm{proj}_{n\in\mathbb{N}}\, X_{N,n}$ and by $\iota_N\colon X \to X_N$ we denote the canonical projection. Since LB-spaces are DF-spaces, PLB-spaces are PDF-spaces. A special case of the above general procedure realizes in the framework of sequence spaces. A sequence $a := (a_j)_{j\in\mathbb{N}}$ of non-negative numbers is called a *weight*. A matrix $A = (a_{N,n})_{N,n\in\mathbb{N}}$ of weights is called a *Köthe-type PLB-matrix* if the following conditions are satisfied:

- $\forall\, N \in \mathbb{N}\ \exists\, n \in \mathbb{N}\ \forall\, j \in \mathbb{N}\colon\ a_{N,n,j} > 0$,
- $\forall\, N, n, j \in \mathbb{N}\colon\ a_{N,n,j} \leqslant a_{N+1,n,j}$,
- $\forall\, N, n, j \in \mathbb{N}\colon\ a_{N,n+1,j} \leqslant a_{N,n,j}$.

If $1 \leqslant p \leqslant \infty$ or $p = 0$ and a is a weight then the weighted ℓ_p-space is defined as

$$\ell_p(a) := \{x = (x_j)_{j\in\mathbb{N}}\colon\ (x_j a_j)_{j\in\mathbb{N}} \in \ell_p\}$$

with the Banach space norm

$$\|x\| := \|(x_j a_j)\|_p \qquad (x \in \ell_p(a)).$$

For the case $p = 0$ we clearly consider the respective weighted c_0-spaces. The sequence space

$$\Lambda_p(A) := \big\{x = (x_j)_{j\in\mathbb{N}} \in \mathbb{C}^{\mathbb{N}} \mid \quad \forall\, N \in \mathbb{N}\ \exists\, n \in \mathbb{N}\colon\ x \in \ell_p(a_{N,n})\big\}$$

is called the *Köthe-type PLB-space associated with* A. We consider it canonically with a locally convex topology τ_p such that

$$(\Lambda_p(A), \tau_p) = \mathrm{proj}_N\, \mathrm{ind}_n\, \ell_p(a_{N,n}) \qquad (1 \leqslant p \leqslant \infty)$$

and

$$(\Lambda_0(A), \tau_0) = \mathrm{proj}_N\, \mathrm{ind}_n\, c_0(a_{N,n}).$$

It is clear from the definition that all the linking maps $\iota_{N,n}^{N,n+1}$, $N, n \in \mathbb{N}$ are formal inclusions. If $1 \leqslant p < \infty$ or $p = 0$ then the standard unit vectors $e_j := (\delta_{ij})_{i\in\mathbb{N}}$, $j \in \mathbb{N}$ form a Schauder basis in $\Lambda_p(A)$.

Using the notation from [**1**] we may write $\Lambda_p(A) = \operatorname{proj}_N k_p(A_N)$, where $k_p(A_N)$ is the so-called *Köthe co-echelon space* associated with the sequence $A_N := (a_{N,n})_{n\in\mathbb{N}}$ of weights. It is known that $\Lambda_p(A)$ is a PLS-space if and only if

$$\forall\, N \in \mathbb{N}\ \exists\, M \in \mathbb{N}\ \forall\, m \in \mathbb{N}\ \exists\, n \in \mathbb{N}\colon \quad \lim_{j\to\infty} \frac{a_{N,n,j}}{a_{M,m,j}} = 0 \tag{6}$$

and a PLN-space if

$$\forall\, N \in \mathbb{N}\ \exists\, M \in \mathbb{N}\ \forall\, m \in \mathbb{N}\ \exists\, n \in \mathbb{N}\colon \quad \sum_{j=1}^{\infty} \frac{a_{N,n,j}}{a_{M,m,j}} < \infty. \tag{7}$$

If a Köthe-type PLB-matrix A is defined as

$$a_{N,n,j} := e^{r_N \alpha_j - s_n \beta_j} \qquad (N, n, j \in \mathbb{N})$$

where

$$r, s \in \mathbb{R} \cup \{\infty\}, \qquad r_N \nearrow r, \qquad s_n \nearrow s, \qquad \lim_j \alpha_j = \lim_j \beta_j = \infty$$

then the Köthe-type PLB-space $\Lambda_p(A)$ is called a *power series-type PLB-space* and is denoted by $\Lambda^p_{r,s}(\alpha, \beta)$. It is straightforward to observe that (in the category of locally convex spaces) one only needs to consider four pairwise non-isomorphic cases, namely $r, s \in \{0, \infty\}$.

Remark 4.1. From [**10**, 3.4, Proposition 6] and [**1**, Theorem 2.3 and Corollary 2.8] it follows that if $1 \leqslant p \leqslant \infty$ then $\Lambda_p(A)$ is always complete whereas $\Lambda_0(A)$ may be incomplete.

5 - Köthe-type PLB-algebras

Given a Köthe-type PLB-matrix A one can consider the point-wise multiplication in $\Lambda_p(A)$. If it is jointly continuous then $\Lambda_p(A)$ becomes a PDF-algebra. It turns out that this property is equivalent to a specific condition one has to impose on the matrix A.

Proposition 5.1. *Let A be a Köthe-type PLB-matrix and let $1 \leqslant p \leqslant \infty$ or $p = 0$. The space $\Lambda_p(A)$ is a PDF-algebra if and only if*

$$\forall\, N \in \mathbb{N}\ \exists\, M \in \mathbb{N}\ \forall\, m \in \mathbb{N}\ \exists\, n \in \mathbb{N}\colon \quad a_{N,n}/(a_{M,m})^2 \in \ell_\infty. \tag{8}$$

Proof. *Necessity.* If $\Lambda_p(A)$ is a PDF-algebra then the product map π is continuous therefore for every $N \in \mathbb{N}$ there is $M \in \mathbb{N}$ such that for every $m \in \mathbb{N}$ there is $n \in \mathbb{N}$ such that the bilinear mapping

$$\mu\colon \ell_p(a_{M,m}) \times \ell_p(a_{M,m}) \to \ell_p(a_{N,n}), \quad \mu(x,y) := (x_j y_j)_{j\in\mathbb{N}} \tag{9}$$

is continuous. Therefore there exists a constant $C > 0$ such that

$$\|\mu(x,y)\|_{\ell_p(a_{N,n})} \leqslant C\|x\|_{\ell_p(a_{M,m})}\|y\|_{\ell_p(a_{M,m})}.$$

Applying the above inequality to the unit basis vectors e_j, $j \in \mathbb{N}$ we get the condition (8).

Sufficiency. Condition (8) implies that the mapping (9) is well-defined (with all the indices having the same meaning). Therefore the product map $\pi\colon \Lambda_p(A)\widehat{\otimes}\Lambda_p(A) \to \Lambda_p(A)$ is continuous implying that $\Lambda_p(A)$ is a PDF-algebra. □

Corollary 5.2. *A power series-type PLB-space $\Lambda^p_{r,s}(\alpha,\beta)$ is a PDF-algebra if and only if (simultaneously) $r \geqslant 0$ (in particular $r = \infty$) and $s \leqslant 0$ or $s = \infty$.*

Recall that a unit in a Köthe-type PLB-algebra is the constant one sequence $\mathbf{1} = (1,1,\ldots)$. Unital PLB-algebras have an additional structural feature which can be observed through the following result.

Proposition 5.3. *Let $1 \leqslant p < \infty$ and let $\Lambda_p(A)$ be a PDF-algebra. The following assertions are equivalent:*

(*i*) *$\Lambda_p(A)$ is unital, i.e. $\mathbf{1} \in \Lambda_0(A)$,*

(*ii*) *For every $N \in \mathbb{N}$ there is $n \in \mathbb{N}$ such that $a_{N,n} \in \ell_p$,*

(*iii*) *For every $N \in \mathbb{N}$ there is $n \in \mathbb{N}$ such that $a_{N,n} \in \ell_1$,*

(*iv*) *For every $N \in \mathbb{N}$ there is $n \in \mathbb{N}$ such that $a_{N,n} \in \ell_\infty$ and $\Lambda_p(A)$ is a PLN-space.*

Proof. (*i*) $\Leftrightarrow$ (*ii*) and (*iii*) $\Rightarrow$ (*ii*): Clear.

(*ii*) $\Rightarrow$ (*iii*): Let $2^k > p$. Applying condition (8) k times we find for every $N \in \mathbb{N}$ a constant $C > 0$ and $M, n \in \mathbb{N}$ such that $a_{N,n} \leqslant C(a_{M,m})^{2^k}$ and $a_{M,m} \in \ell_{2^k}$. Therefore $a_{N,n} \in \ell_1$.

(*iii*) $\Rightarrow$ (*iv*): For a fixed $N \in \mathbb{N}$ we find $M \in \mathbb{N}$ such that (8) holds. For this $M \in \mathbb{N}$ we find by assumption $m \in \mathbb{N}$ such that $a_{M,m} \in \ell_1$. Consequently,

$$\sum_{j=1}^{\infty} \frac{a_{N,n,j}}{a_{M.m,j}} \leqslant C \sum_{j=1}^{\infty} a_{M,m,j} < \infty.$$

From (7) it now follows that $\Lambda_p(A)$ is a PLN-space.

$(iv) \Rightarrow (iii)$: By assumption

$$\sum_{j=1}^{\infty} \frac{a_{N,n,j}}{a_{M.m,j}} < \infty \qquad \text{and} \qquad a_{M,m} \in \ell_\infty,$$

where all the indices have the meaning from (7). Therefore

$$\sum_{j=1}^{\infty} a_{N,n,j} = \sum_{j=1}^{\infty} a_{M,m,j} \frac{a_{N,n,j}}{a_{M.m,j}} \leqslant C \sum_{j=1}^{\infty} \frac{a_{N,n,j}}{a_{M.m,j}} < \infty.$$

Consequently, $a_{N,n} \in \ell_1$. □

A similar proof applies to the following result therefore we omit it.

Proposition 5.4. *Let $\Lambda_0(A)$ be a PDF-algebra. The following assertions are equivalent:*

(i) *$\Lambda_0(A)$ is unital, i.e. $\mathbf{1} \in \Lambda_0(A)$,*

(ii) *For every $N \in \mathbb{N}$ there is $n \in \mathbb{N}$ such that $a_{N,n} \in c_0$,*

(iii) *For every $N \in \mathbb{N}$ there is $n \in \mathbb{N}$ such that $a_{N,n} \in \ell_\infty$, and $\Lambda_0(A)$ is a PLS-space.*

In particular, unital Köthe-type PLB-algebras $\Lambda_0(A)$ are complete.

We now proceed to the main characterization results.

Theorem 5.5. *Let $1 \leqslant p < \infty$ and let $\Lambda_p(A)$ be a PDF-algebra. Then it is contractible if and only if it is unital.*

Proof. From Remark 4.1 it follows that $\Lambda_p(A)$ is complete therefore only sufficiency needs to be proved. The result will be a consequence of Theorem 3.7 once we find a projective diagonal. Since $\Lambda_p(A)$ is unital, it follows from Proposition 5.3 that for every $N \in \mathbb{N}$ there is $n \in \mathbb{N}$ such that $a_{N,n} \in \ell_p$. We will now show that $d := \sum_{j=1}^{\infty} e_j \otimes e_j \in \Lambda_p(A) \widehat{\otimes} \Lambda_p(A)$ and it is a projective diagonal. To this end, let $(r_j(t))_{j\in\mathbb{N}}$ be the sequence of Rademacher functions on the interval $[0,1]$. Then for $k \leqslant m$ we have

$$\sum_{j=k}^{m} e_j \otimes e_j = \int_0^1 \Big(\sum_{j=k}^{m} r_j(t) e_j\Big) \otimes \Big(\sum_{j=k}^{m} r_j(t) e_j\Big)\, dt.$$

Therefore

$$\Big\|\Big(\sum_{j=k}^{m} e_j \otimes e_j\Big)\Big\|_{\ell_p(a_{N,n})\widehat{\otimes}\ell_p(a_{N,n})} \leqslant \sup_{0\leqslant t\leqslant 1} \Big\|\sum_{j=k}^{m} r_j(t)e_j\Big\|^2_{\ell_p(a_{N,n})} \leqslant \Big(\sum_{j=k}^{m}(a_{N,n,j})^p\Big)^{\frac{2}{p}}. \tag{10}$$

In particular the sequence

$$\Big(\sum_{j=1}^{m} e_j \otimes e_j\Big)_{m\in\mathbb{N}}$$

is Cauchy in $\ell_p(a_{N,n})\widehat{\otimes}\ell_p(a_{N,n})$ and hence convergent to d. We have thus shown that for all $1 \leqslant p < \infty$ and all $N \in \mathbb{N}$ we have $d \in \mathrm{ind}_n(\ell_p(a_{N,n})\widehat{\otimes}\ell_p(a_{N,n}))$. From [**15**, Theorem 7] it now follows that

$$k_p(A_N)\widehat{\otimes}k_p(A_N) = \mathrm{ind}_n(\ell_p(a_{N,n})\widehat{\otimes}\ell_p(a_{N,n})).$$

This implies that $d \in k_p(A_N)\widehat{\otimes}k_p(A_N)$ for all $N \in \mathbb{N}$ whence $d \in \Lambda_p(A)\widehat{\otimes}\Lambda_p(A)$. It only remains to show that d is a projective diagonal but this is straightforward. □

Theorem 5.6. *A PDF-algebra $\Lambda_0(A)$ is contractible if and only if its completion $\widetilde{\Lambda_0(A)}$ is a unital PDF-algebra.*

Proof. *Necessity.* Follows from Theorem 3.7.

Sufficiency. In view of Theorem 3.7 it is enough to show that $\widetilde{\Lambda_0(A)}$ is biprojective. To this end, recall that $(e_j)_{j\in\mathbb{N}}$ is a Schauder basis in $\Lambda_0(A)$. Therefore, one may define a mapping

$$\sigma\colon \Lambda_0(A) \to \Lambda_0(A)\widehat{\otimes}\Lambda_0(A), \qquad \sigma(x) := \sum_{j=1}^{\infty} x_j e_j \otimes e_j,$$

where $x = \sum_{j\in\mathbb{N}}^{\infty} x_j e_j$. Using (10) for every $N \in \mathbb{N}$ we obtain $n \in \mathbb{N}$ such that

$$\|\sigma(x)\|_{c_0(a_{N,n})\widehat{\otimes}c_0(a_{N,n})} \leqslant \|\mathbf{1}\|_{\ell_\infty(a_{N,n})}\|x\|_{c_0(a_{N,n})}.$$

Recall that $\mathbf{1} \in \widetilde{\Lambda_0(A)}$ therefore $\|\mathbf{1}\|_{\ell_\infty(a_{N,n})}$ is finite. All this implies that the mapping σ is continuous and one can extend it to a continouous linear mapping

$$\widetilde{\sigma}\colon \widetilde{\Lambda_0(A)} \to \widetilde{\Lambda_0(A)}\widehat{\otimes}\widetilde{\Lambda_0(A)}.$$

It is now straightforward to check that $\widetilde{\sigma}$ is a bimodule map. Consequently, $\widetilde{\Lambda_0(A)}$ is contractible and the proof is thereby complete. □

The case of $p = \infty$ is missing from the above characterization. Therefore, we end the paper with the following conjecture.

Conjecture 5.7. *The PDF-algebra* $\Lambda_\infty(A)$ *is contractible if and only if* $\Lambda_\infty(A) = \widetilde{\Lambda_0(A)}$ *if and only if* $\mathbf{1} \in \widetilde{\Lambda_0(A)}$.

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