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Harmonic $(1,1)$-forms on compact almost Hermitian 4-manifolds

Abstract. We recall the most recent results concerning the spaces of Dolbeault and Bott-Chern harmonic $(1,1)$-forms on a compact almost Hermitian 4-manifold and compute the dimension of the space of Dolbeault harmonic $(1,1)$-forms in some explicit examples.

Keywords. Dolbeault Laplacian; Bott-Chern Laplacian; 4-manifold.

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1 - Introduction

Let (M,J) be an almost complex manifold of real dimension $2n$. Given, on (M,J), an almost Hermitian metric g, with fundamental form ω, the triple (M,J,ω) will be called an almost Hermitian manifold. The exterior derivative decomposes into

$$d=\mu+\partial+\overline{\partial}+\overline{\mu}.$$

By the map $*:A^{p,q}\to A^{n-q,n-p}$, we denote the $\mathbb{C}$-linear extension of the real Hodge $*$ operator. We set $\partial^*:=-*\overline{\partial}*$ and $\overline{\partial}^*:=-*\partial*$, which are the formal adjoint operators respectively of ∂ and $\overline{\partial}$. Recall that

$$\Delta_{\overline{\partial}}:=\overline{\partial}\,\overline{\partial}^*+\overline{\partial}^*\overline{\partial}$$

is the Dolbeault Laplacian, which is a formally self adjoint elliptic operator of order 2. We can also define the Bott-Chern Laplacian, as

$$\Delta_{BC}=\partial\overline{\partial}\,\overline{\partial}^*\partial^*+\overline{\partial}^*\partial^*\partial\overline{\partial}+\partial^*\overline{\partial}\,\overline{\partial}^*\partial+\overline{\partial}^*\partial\partial^*\overline{\partial}+\partial^*\partial+\overline{\partial}^*\overline{\partial},$$

which is a formally self adjoint elliptic operator of order 4. We set

$$\mathcal{H}^{p,q}_{\overline{\partial}}:=\ker\Delta_{\overline{\partial}}\cap A^{p,q},\quad \mathcal{H}^{p,q}_{BC}:=\ker\Delta_{BC}\cap A^{p,q}$$

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to be the spaces of Dolbeault and Bott-Chern harmonic (p,q)-forms. If M is compact, it is well known that the dimensions

$$h^{p,q}_{\overline{\partial}} := \dim_{\mathbb{C}} \mathcal{H}^{p,q}_{\overline{\partial}}, \quad h^{p,q}_{BC} := \dim_{\mathbb{C}} \mathcal{H}^{p,q}_{BC}$$

are finite. Note that, a priori, $\partial^*, \overline{\partial}^*, \Delta_{\overline{\partial}}, \mathcal{H}^{p,q}_{\overline{\partial}}, h^{p,q}_{\overline{\partial}}$ depend both on the almost complex structure J and on the almost Hermitian metric ω.

If (M,J) is a compact complex manifold, then by Hodge theory we know that the space of Dolbeault harmonic forms is isomorphic to the Dolbeault cohomology, and the space of Bott-Chern harmonic forms is isomorphic to the Bott-Chern cohomology,

$$\mathcal{H}^{p,q}_{\overline{\partial}} \cong H^{p,q}_{\overline{\partial}} := \frac{\ker \overline{\partial} \cap A^{p,q}}{\operatorname{im} \overline{\partial}}, \quad \mathcal{H}^{p,q}_{BC} \cong H^{p,q}_{BC} := \frac{\ker d \cap A^{p,q}}{\operatorname{im} \partial\overline{\partial}}.$$

Since the Dolbeault and Bott-Chern cohomologies are complex invariants, the numbers $h^{p,q}_{\overline{\partial}}$ and $h^{p,q}_{BC}$ do not depend on the choice of the metric in the integrable case.

Conversely, in the almost complex setting, Kodaira and Spencer asked the following question, which appeared as Problem 20 in Hirzebruch's 1954 problem list [**9**]: given a compact almost Hermitian manifold (M,J,ω), do the numbers $h^{p,q}_{\overline{\partial}}$ depend on the choice of the almost Hermitian metric ω? Recently in [**12**] Holt and Zhang solved this problem, proving that the numbers $h^{p,q}_{\overline{\partial}}$ indeed depend on the choice of the metric. They provided an explicit example, building a family of almost Hermitian structures on the Kodaira-Thurston manifold, which has real dimension 4. They also proved that on every 4-dimensional compact almost Hermitian manifold (M,J,ω), if the metric ω is almost Kähler, i.e., $d\omega = 0$, then $h^{1,1}_{\overline{\partial}} = b^- + 1$. Here b^- is the dimension of the space of anti self dual harmonic forms, which is a topological invariant (see, e.g., [**6**]).

This paper is devoted to the study of the spaces of Dolbeault and Bott-Chern harmonic $(1,1)$-forms, and in particular of their dimensions $h^{1,1}_{\overline{\partial}}$ and $h^{1,1}_{BC}$, on a given compact almost Hermitian 4-manifold.

After Holt and Zhang, the study of the number $h^{1,1}_{\overline{\partial}}$ on compact almost Hermitian 4-manifolds (M,J,ω) has been continued by Tardini and Tomassini in [**18**]. We say that ω is *strictly locally conformally almost Kähler* if

$$d\omega = \theta \wedge \omega,$$

and $\theta \in A^1$ is d-closed but non d-exact. Conversely, we say that ω is *globally conformally almost Kähler*, if

$$d\omega = \theta \wedge \omega,$$

and $\theta \in A^1$ is d-exact. Indeed, if $\theta = dh$, then the metric $e^{-h}\omega$ is almost Kähler. Also note that, for any given almost Hermitian metric ω, the 1-form θ such that $d\omega = \theta \wedge \omega$ is uniquely determined by the Lefschetz isomorphism. We will also say that ω is *locally conformally almost Kähler* if it is either strictly locally conformally almost Kähler or globally conformally almost Kähler.

Tardini and Tomassini proved that $h^{1,1}_{\bar\partial} = b^-$ on every compact almost complex 4-manifold with a strictly locally conformally almost Kähler metric. They also noted that $h^{1,1}_{\bar\partial}$ is a conformal invariant on almost Hermitian 4-manifolds, which implies that $h^{1,1}_{\bar\partial} = b^- + 1$ on every compact almost complex 4-manifold with a globally conformally almost Kähler metric, by the previous result by Holt and Zhang in [**12**]. Moreover, very recently, in [**10**, Theorem 3.1] Holt proved that $h^{1,1}_{\bar\partial} = b^- + 1$ and $h^{1,1}_{\bar\partial} = b^-$ are the only two possible options on a compact almost Hermitian 4-manifold.

Note that, in the integrable case, it is well known that a compact complex surface (M, J) admits a Kähler metric if and only if b^1 is even, and b^1 is even if and only if $h^{1,1}_{\bar\partial} = b^- + 1$, see e.g., [**1**]. However, in the non integrable case, it might happen that $h^{1,1}_{\bar\partial} = b^- + 1$ when the almost Hermitian metric is not globally conformally almost Kähler. Indeed, in [**17**], Tomassini and the author of the present paper proved that $h^{1,1}_{\bar\partial} = b^- + 1$ on a explicit example of a 4-dimensional compact almost complex manifold endowed with a non locally conformally almost Kähler metric.

In [**16**] Tomassini and the author studied Bott-Chern harmonic forms on compact almost Hermitian manifolds. They showed that Dolbeault and Bott-Chern harmonic forms do not coincide when the metric is almost Kähler, differently from what happens in the integrable case. They also proved that either $h^{1,1}_{BC} = b^-$ or $h^{1,1}_{BC} = b^- + 1$ holds on a compact almost Hermitian 4-manifold, and if the metric is almost Kähler then $h^{1,1}_{BC} = b^- + 1$. This result has been improved by Holt in [**10**], who proved that it always holds $h^{1,1}_{BC} = b^- + 1$.

All the results mentioned before which concern the numbers $h^{1,1}_{\bar\partial}$ and $h^{1,1}_{BC}$ are, indeed, generalizations of what is already known in the integrable case, i.e., in the case of compact complex surfaces. In particular, e.g., the fact that $h^{1,1}_{\bar\partial}$ is either b^- or $b^- + 1$ and that $h^{1,1}_{BC}$ is always $b^- + 1$ turns out not to depend on the integrability of a given almost complex structure.

See [**2**, **11**, **15**] for recent papers studying the relation between the primitive decomposition of forms and Dolbeault/Bott-Chern harmonic forms in higher dimension. See [**13**, **14**, **19**, **21**] for other interesting new results concerning Dolbeault harmonic forms on compact almost Hermitian manifolds. See also [**3**, **4**, **5**] for some proposals of cohomologies in the almost complex setting, which do not have harmonic counterparts.

This paper is divided in two main parts. The first part, consisting in sections 3, 4 and 5, is a systematic and self-contained collection of the previously mentioned results, which have already appeared in [**10**, **12**, **16**, **18**], concerning Dolbeault and Bott-Chern harmonic $(1,1)$-forms on compact almost Hermitian 4-manifolds. Some of these results were presented at the conference *Cohomology of Complex Manifolds and Special Structures, II* in July 2021. The second part, consisting in section 6, includes new examples of computations of Dolbeault harmonic $(1,1)$-forms on compact almost Hermitian 4-manifolds. These examples have been the object of further studies in [**14**].

2 - Preliminaries

Throughout this paper, we will only consider connected manifolds without boundary. Let (M,J) be an almost complex manifold of dimension $2n$, i.e., a $2n$-differentiable manifold endowed with an almost complex structure J, that is $J \in \mathrm{End}(TM)$ and $J^2 = -\mathrm{id}$. The complexified tangent bundle $T_{\mathbb{C}}M = TM \otimes \mathbb{C}$ decomposes into the two eigenspaces of J associated to the eigenvalues $i, -i$, which we denote respectively by $T^{1,0}M$ and $T^{0,1}M$, giving

$$T_{\mathbb{C}}M = T^{1,0}M \oplus T^{0,1}M.$$

Denoting by $\Lambda^{1,0}M$ and $\Lambda^{0,1}M$ the dual vector bundles of $T^{1,0}M$ and $T^{0,1}M$, respectively, we set

$$\Lambda^{p,q}M = \bigwedge^{p} \Lambda^{1,0}M \wedge \bigwedge^{q} \Lambda^{0,1}M$$

to be the vector bundle of (p,q)-forms, and let $A^{p,q} = \Gamma(M, \Lambda^{p,q}M)$ be the space of smooth sections of $\Lambda^{p,q}M$. We denote by $A^k = \Gamma(M, \Lambda^k M)$ the space of k-forms. Note that $\Lambda^k M \otimes \mathbb{C} = \bigoplus_{p+q=k} \Lambda^{p,q}M$.

Let $f \in \mathcal{C}^\infty(M,\mathbb{C})$ be a smooth function on M with complex values. Its differential df is contained in $A^1 \otimes \mathbb{C} = A^{1,0} \oplus A^{0,1}$. On complex 1-forms, the exterior derivative acts as

$$d : A^1 \otimes \mathbb{C} \to A^2 \otimes \mathbb{C} = A^{2,0} \oplus A^{1,1} \oplus A^{0,2}.$$

Therefore, it turns out that the exterior derivative operates on (p,q)-forms as

$$d : A^{p,q} \to A^{p+2,q-1} \oplus A^{p+1,q} \oplus A^{p,q+1} \oplus A^{p-1,q+2},$$

where we denote the four components of d by

$$d = \mu + \partial + \bar{\partial} + \bar{\mu}.$$

From the relation $d^2 = 0$, we derive

$$\begin{cases} \mu^2 = 0, \\ \mu\partial + \partial\mu = 0, \\ \partial^2 + \mu\overline{\partial} + \overline{\partial}\mu = 0, \\ \partial\overline{\partial} + \overline{\partial}\partial + \mu\overline{\mu} + \overline{\mu}\mu = 0, \\ \overline{\partial}^2 + \overline{\mu}\partial + \partial\overline{\mu} = 0, \\ \overline{\mu}\overline{\partial} + \overline{\partial}\overline{\mu} = 0, \\ \overline{\mu}^2 = 0. \end{cases}$$

We also define the operator $d^c := J^{-1}dJ$. It is a straightforward computation to show that

$$d^c = i(\mu - \partial + \overline{\partial} - \overline{\mu}).$$

Let (M, J) be an almost complex manifold. If the almost complex structure J is induced from a complex manifold structure on M, then J is called integrable. Recall that J being integrable is equivalent to the decomposition of the exterior derivative as $d = \partial + \overline{\partial}$.

A Riemannian metric on M for which J is an isometry is called almost Hermitian. Let g be an almost Hermitian metric, the 2-form ω such that

$$\omega(u, v) = g(Ju, v) \quad \forall u, v \in \Gamma(TM)$$

is called the fundamental form of g. We will call (M, J, ω) an almost Hermitian manifold. We denote by h the Hermitian extension of g on the complexified tangent bundle $T_{\mathbb{C}}M$, and by the same symbol g the $\mathbb{C}$-bilinear symmetric extension of g on $T_{\mathbb{C}}M$. Also denote by the same symbol ω the $\mathbb{C}$-bilinear extension of the fundamental form ω of g on $T_{\mathbb{C}}M$. Thanks to the elementary properties of the two extensions h and g, we may want to consider h as a Hermitian operator $T^{1,0}M \times T^{1,0}M \to \mathbb{C}$ and g as a $\mathbb{C}$-bilinear operator $T^{1,0}M \times T^{0,1}M \to \mathbb{C}$. Recall that it holds $h(u, v) = g(u, \bar{v})$ for all $u, v \in \Gamma(T^{1,0}M)$.

Let (M, J, ω) be an almost Hermitian manifold of real dimension $2n$. Extend h on (p, q)-forms and denote the Hermitian inner product by $\langle\cdot,\cdot\rangle$. Let $* : A^{p,q} \longrightarrow A^{n-q,n-p}$ the $\mathbb{C}$-linear extension of the standard Hodge $*$ operator on Riemannian manifolds with respect to the volume form $\mathrm{Vol} = \frac{\omega^n}{n!}$, i.e., $*$ is defined by the relation

$$\alpha \wedge *\overline{\beta} = \langle \alpha, \beta \rangle \,\mathrm{Vol} \quad \forall \alpha, \beta \in A^{p,q}.$$

Then the operators

$$d^* = - * d *, \quad \mu^* = - * \overline{\mu} *, \quad \partial^* = - * \overline{\partial} *, \quad \overline{\partial}^* = - * \partial *, \quad \overline{\mu}^* = - * \mu *,$$

are the formal adjoint operators respectively of $d, \mu, \partial, \overline{\partial}, \overline{\mu}$. Recall that

$$\Delta_d = dd^* + d^* d$$

is the Hodge Laplacian, and, as in the integrable case, set

$$\Delta_\partial = \partial\partial^* + \partial^*\partial, \quad \Delta_{\overline{\partial}} = \overline{\partial}\overline{\partial}^* + \overline{\partial}^*\overline{\partial},$$

respectively as the ∂ and $\overline{\partial}$ Laplacians. Again, as in the integrable case, set

$$\Delta_{BC} = \partial\overline{\partial}\overline{\partial}^*\partial^* + \overline{\partial}^*\partial^*\partial\overline{\partial} + \partial^*\overline{\partial}\overline{\partial}^*\partial + \overline{\partial}^*\partial\partial^*\overline{\partial} + \partial^*\partial + \overline{\partial}^*\overline{\partial},$$

and

$$\Delta_A = \partial\overline{\partial}\overline{\partial}^*\partial^* + \overline{\partial}^*\partial^*\partial\overline{\partial} + \partial\overline{\partial}^*\overline{\partial}\partial^* + \overline{\partial}\partial^*\partial\overline{\partial}^* + \partial\partial^* + \overline{\partial}\overline{\partial}^*,$$

to be respectively the Bott-Chern and the Aeppli Laplacians. Note that

$$*\Delta_{BC} = \Delta_A * \qquad \Delta_{BC}* = *\Delta_A.$$

If M is compact, then we easily deduce the following relations

$$(1) \qquad \begin{cases} \Delta_d = 0 & \iff d = 0,\ d* = 0, \\ \Delta_\partial = 0 & \iff \partial = 0,\ \overline{\partial}* = 0, \\ \Delta_{\overline{\partial}} = 0 & \iff \overline{\partial} = 0,\ \partial* = 0, \\ \Delta_{BC} = 0 & \iff \partial = 0,\ \overline{\partial} = 0,\ \partial\overline{\partial}* = 0, \\ \Delta_A = 0 & \iff \partial* = 0,\ \overline{\partial}* = 0,\ \partial\overline{\partial} = 0, \end{cases}$$

which characterize the spaces of harmonic forms

$$\mathcal{H}^k_d, \quad \mathcal{H}^{p,q}_\partial, \quad \mathcal{H}^{p,q}_{\overline{\partial}}, \quad \mathcal{H}^{p,q}_{BC}, \quad \mathcal{H}^{p,q}_A,$$

defined as the spaces of forms which are in the kernel of the associated Laplacians. All these Laplacians are elliptic operators on the almost Hermitian manifold (M, J, ω) (cf. [**9**], [**16**]), implying that all the spaces of harmonic forms are finite dimensional when the manifold is compact. Denote by $\mathcal{H}^{p,q}_d$ the space $\left(\mathcal{H}^{p+q}_d \otimes \mathbb{C}\right) \cap A^{p,q}$, and by

$$b^k, \quad h^{p,q}_d, \quad h^{p,q}_\partial, \quad h^{p,q}_{\overline{\partial}}, \quad h^{p,q}_{BC}, \quad h^{p,q}_A$$

respectively the real dimension of $\mathcal{H}^k_d$ and the complex dimensions of $\mathcal{H}^{p,q}_d$, $\mathcal{H}^{p,q}_\partial$, $\mathcal{H}^{p,q}_{\overline{\partial}}$, $\mathcal{H}^{p,q}_{BC}$, $\mathcal{H}^{p,q}_A$.

Let (M, J, ω) be a $2n$-dimensional almost Hermitian manifold. We denote with

$$L : \Lambda^k M \to \Lambda^{k+2} M, \quad \alpha \mapsto \omega \wedge \alpha$$

the Lefschetz operator and with

$$\Lambda : \Lambda^k M \to \Lambda^{k-2} M\,, \quad \Lambda = *^{-1} L *$$

its dual. A differential k-form α_k on M, for $k \le n$, is said to be *primitive* if $\Lambda\alpha_k = 0$, or equivalently $L^{n-k+1}\alpha_k = 0$. Then, the following vector bundle decomposition holds (see e.g., [**20**, p. 26, Théorème 3])

$$\Lambda^k M = \bigoplus_{r \ge \max(k-n,0)} L^r (P^{k-2r} M), \tag{2}$$

where we denoted by

$$P^s M := \ker\left(\Lambda : \Lambda^s M \to \Lambda^{s-2} M\right)$$

the bundle of primitive s-forms. Furthermore, the decomposition above is compatible with the bidegree decomposition on the bundle of complex k-forms $\Lambda^k_{\mathbb{C}} M$ induced by J, that is

$$P^k M \otimes \mathbb{C} = \bigoplus_{p+q=k} P^{p,q} M,$$

where

$$P^{p,q} M = \left(P^k M \otimes \mathbb{C}\right) \cap \Lambda^{p,q} M.$$

Let us set $P^s := \Gamma(M, P^s M)$ and $P^{p,q} := \Gamma(M, P^{p,q} M)$. We recall that the map $L^h : \Lambda^k M \to \Lambda^{k+2h} M$ is injective for $h + k \le n$ and is surjective for $h + k \ge n$. For $h + k = n$, the map is called the Lefschetz isomorphism.

3 - Fundamental lemmas

In this section, we prove some preliminary lemmas which will be used systematically in the next sections.

Let (M, g) be a compact oriented Riemannian manifold of real dimension 4, and set

$$\Lambda^- = \{\alpha \in \Lambda^2 M \,:\, *\alpha = -\alpha\}$$

to be the bundle of anti self dual 2-forms. Denote by $A^- = \Gamma(M, \Lambda^-)$ the space of smooth anti self dual 2-forms, and by

$$\mathcal{H}^- = \{\alpha \in A^- \,:\, \Delta_d \alpha = 0\},$$

the subspace of harmonic anti self dual 2-forms. Set $b^- = \dim_{\mathbb{R}} \mathcal{H}^-$. Note that b^- is metric independent: see [**6**, Chapter 1] for its topological meaning.

Let (M, J, ω) be an almost Hermitian manifold of real dimension 4. Note that the space of anti self dual complex valued 2-forms $A^- \otimes \mathbb{C}$ is indeed a subspace of $A^{1,1}$, which will be denoted by $A^-_{\mathbb{C}}$. Furthermore, the space $\mathcal{H}^- \otimes \mathbb{C}$ is indeed a subspace of $\mathcal{H}^{1,1}_d$, and will be denoted by $\mathcal{H}^-_{\mathbb{C}}$.

Let (M, J, ω) be an almost Hermitian manifold of real dimension 4. The primitive decomposition of 2-forms is, by (2),

$$\Lambda^2 M = L\big(P^0 M\big) \oplus P^2 M,$$

where $P^0 M = \Lambda^0 M$. Passing to the decomposition of complex $(1,1)$-forms, we derive

$$\Lambda^{1,1} M = L\big(\mathbb{C}\big) \oplus P^{1,1} M.$$

Now, choose a local coframe of $(1,0)$-forms φ^1, φ^2, such that the metric is locally given by

$$\omega = i(\varphi^{1\bar{1}} + \varphi^{2\bar{2}}).$$

We can locally write a form $\alpha \in \Lambda^{1,1} M$ as

$$\alpha = A\varphi^{1\bar{1}} + B\varphi^{1\bar{2}} + C\varphi^{2\bar{1}} + D\varphi^{2\bar{2}}.$$

Note that $L\alpha = \omega \wedge \alpha = 0$ is equivalent to $A + D = 0$. In the same way, $*\alpha = -\alpha$ is equivalent to $A + D = 0$. Therefore, we can characterize the space of primitive $(1,1)$-forms as

$$P^{1,1} M = \{\alpha \in \Lambda^{1,1} M \mid *\,\alpha = -\alpha\}.$$

The above discussion implies that the decomposition of the bundle of $(1,1)$-forms is given by

$$\Lambda^{1,1} M = \mathbb{C} < \omega > \oplus \big(\Lambda^- \otimes \mathbb{C}\big),$$

and, passing to the space of smooth sections, we have just proved the following

Lemma 3.1. *Let (M, J, ω) be an almost Hermitian 4-manifold. Then*

$$A^{1,1} = \{f\omega \mid f \in \mathcal{C}^\infty(M, \mathbb{C})\} \oplus A^-_{\mathbb{C}}.$$

Thanks to Lemma 3.1, we will often run into $(1,1)$-forms of the type $f\omega + \gamma$, with $f \in \mathcal{C}^\infty(M, \mathbb{C})$ and $*\gamma = -\gamma$. In many cases, we will use the next result to prove that indeed the function f is a complex constant. Recall that, on a given almost Hermitian manifold (M, J, ω) of dimension $2n$, the almost Hermitian metric is called Gauduchon if $\partial\bar{\partial}\omega^{n-1} = 0$ or, equivalently, if $dd^c\omega^{n-1} = 0$.

Lemma 3.2 ([**18**, Proposition 3.4], cf. [**16**, Theorem 4.3]). *Let (M, J, ω) be a compact almost Hermitian 4-manifold and let $f \in \mathcal{C}^\infty(M, \mathbb{C})$ be a complex valued smooth function. If ω is Gauduchon and $\partial\bar{\partial}(f\omega) = 0$, then f is a complex constant.*

Proof. Since ω is Gauduchon, then

$$0 = \partial\overline{\partial}(f\omega) = \partial\overline{\partial}f \wedge \omega - \overline{\partial}f \wedge \partial\omega + \partial f \wedge \overline{\partial}\omega.$$

Now, the differential operator $P : \mathcal{C}^\infty(M,\mathbb{C}) \to \mathcal{C}^\infty(M,\mathbb{C})$ defined by

$$P : f \mapsto -i * (\partial\overline{\partial}f \wedge \omega - \overline{\partial}f \wedge \partial\omega + \partial f \wedge \overline{\partial}\omega)$$

is strongly elliptic, since its principal part is given by

$$-i * (\partial\overline{\partial}f \wedge \omega),$$

and real, i.e., $P(\overline{f}) = \overline{P(f)}$. By the maximum principle applied to $\mathrm{Re}(f) \in \ker P$ and $\mathrm{Im}(f) \in \ker P$, it follows that f is a complex constant. See [**16**, Theorem 4.3] or [**18**, Proposition 3.4] for further technical details. □

To apply Lemma 3.2, we will need to work with Gauduchon metrics. Let us recall the following fundamental result by Gauduchon, in [**7**]: given a compact almost Hermitian $2n$-manifold $(M, J, \tilde{\omega})$, there always exists a Gauduchon metric $\omega = e^t\tilde{\omega}$ in the conformal class of $\tilde{\omega}$, with $t \in \mathcal{C}^\infty(M)$, which is unique up to homothety for $n > 1$. The following lemma tells us that the dimensions of the spaces of harmonic (p,q)-forms is a conformal invariant in some special bidegrees.

Lemma 3.3 ([**18**, Lemma 3.1]). *Let (M, J) be a compact almost complex $2n$-manifold. The numbers $h^{p,q}_d$, $h^{p,q}_\partial$, $h^{p,q}_{\overline{\partial}}$, $h^{p,q}_{BC}$ and $h^{p,q}_A$ are conformal invariants of almost Hermitian metrics for $p + q = n$.*

Proof. Let $\tilde{\omega}$, $\omega = e^t\tilde{\omega}$, with $t \in \mathcal{C}^\infty(M)$, be two conformal almost Hermitian metrics. The two Hodge star operators behave, on the space $A^{p,q}$, as

$$*_\omega = e^{t(n-p-q)} *_{\tilde{\omega}} .$$

Therefore, when $p + q = n$, the spaces of harmonic (p,q)-forms are conformal invariants of almost Hermitian metrics, thanks to their characterizations (1). In particular, when $p+q = n$, also their dimensions are conformal invariants. □

Thanks to Lemma 3.3, we will often be able to assume that our Hermitian metric is Gauduchon, with the purpose of computing the numbers $h^{p,q}_d$, $h^{p,q}_{\overline{\partial}}$, $h^{p,q}_{BC}$.

4 - Dolbeault harmonic $(1,1)$-forms

In this section, given a compact almost Hermitian 4-manifold (M,J,ω), we study the space of Dolbeault harmonic $(1,1)$-forms $\mathcal{H}^{1,1}_{\overline{\partial}}$ and its dimension $h^{1,1}_{\overline{\partial}}$ under different assumptions on the almost Hermitian metric ω. Thanks to Lemma 3.3 and to the existence of Gauduchon metrics in the conformal class of any given almost Hermitian metric, we can assume that the metric is Gauduchon if we want to compute $h^{1,1}_{\overline{\partial}}$. We begin by proving the following characterization of $\mathcal{H}^{1,1}_{\overline{\partial}}$.

Theorem 4.1 ([**18**, Theorem 3.6], cf. [**10**, Eq. (2)]). *Let (M,J,ω) be a compact almost Hermitian 4-manifold. If ω is Gauduchon, then*

$$\mathcal{H}^{1,1}_{\overline{\partial}} = \{f\omega + \gamma \in A^{1,1} \,|\, f \in \mathbb{C},\ *\gamma = -\gamma,\ fd\omega = id^c\gamma\}.$$

Proof. The inclusion $\supseteq$ is a simple computation. By Lemma 3.1, we can write

$$\psi = f\omega + \gamma,$$

where $f \in \mathcal{C}^\infty(M,\mathbb{C})$ and $*\gamma = -\gamma$. Then, $\psi \in \mathcal{H}^{1,1}_{\overline{\partial}}$ iff $\overline{\partial}\psi = \partial * \psi = 0$, iff

$$\overline{\partial}(f\omega) + \overline{\partial}\gamma = 0, \quad \partial(f\omega) - \partial\gamma = 0.$$

Therefore, if $\psi \in \mathcal{H}^{1,1}_{\overline{\partial}}$, then

$$\partial\overline{\partial}(f\omega) = -\partial\overline{\partial}\gamma = \overline{\partial}\partial\gamma = \overline{\partial}\partial(f\omega) = -\partial\overline{\partial}(f\omega),$$

that is, $\partial\overline{\partial}(f\omega) = 0$. By Lemma 3.2, we get $f \in \mathbb{C}$, which implies

$$f\overline{\partial}\omega + \overline{\partial}\gamma = 0, \quad f\partial\omega - \partial\gamma = 0,$$

that is $fd\omega = id^c\gamma$. This proves the other inclusion $\subseteq$. □

Recall that, given a compact complex surface (M,J), it admits a Kähler metric if and only if $h^{1,1}_{\overline{\partial}} = b^- + 1$. The following result generalizes to the non integrable case the property that if (M,J,ω) is Kähler, then $h^{1,1}_{\overline{\partial}} = b^- + 1$.

Corollary 4.2 ([**12**, Proposition 6.1], cf. [**18**, Proposition 3.3]). *Let (M,J,ω) be a compact almost Hermitian 4-manifold. If ω is globally conformally almost Kähler, then $h^{1,1}_{\overline{\partial}} = b^- + 1$. In particular, if ω is almost Kähler, then*

$$\mathcal{H}^{1,1}_{\overline{\partial}} = \mathbb{C} < \omega > \oplus \mathcal{H}^-_{\mathbb{C}}.$$

Proof. Assume that ω is almost Kähler. Then ω is in particular Gauduchon and by Theorem 4.1 we have the characterization

$$\mathcal{H}^{1,1}_{\overline{\partial}} = \{f\omega + \gamma \in A^{1,1} \mid f \in \mathbb{C},\ *\gamma = -\gamma,\ d^c\gamma = 0\}.$$

Note that $0 = d^c\gamma = i(\overline{\partial} - \partial)\gamma$ is equivalent to $0 = d\gamma = (\partial + \overline{\partial})\gamma$. Moreover, $d\gamma = 0$ and $*\gamma = -\gamma$ is equivalent to $d\gamma = d^*\gamma = 0$ and $*\gamma = -\gamma$, which again is equivalent to $\Delta_d\gamma = 0$ and $*\gamma = -\gamma$. This proves

$$\mathcal{H}^{1,1}_{\overline{\partial}} = \mathbb{C} < \omega > \oplus \mathcal{H}^-_{\mathbb{C}}$$

and $h^{1,1}_{\overline{\partial}} = b^- + 1$.
If ω is globally conformally almost Kähler, then $h^{1,1}_{\overline{\partial}} = b^- + 1$ by Lemma 3.3 and by the previous case where ω is almost Kähler. □

Thanks to Corollary 4.2, on a compact almost Hermitian 4-manifold (M, J, ω), if ω is globally conformally almost Kähler, then $h^{1,1}_{\overline{\partial}} = b^- + 1$. Otherwise, Theorem 4.1 yields a characterization of $\mathcal{H}^{1,1}_{\overline{\partial}}$ when the metric is Gauduchon. If we further assume that the metric is strictly locally conformally almost Kähler, then it turns out that $h^{1,1}_{\overline{\partial}} = b^-$.

Theorem 4.3 ([**18**, Theorem 3.6]). *Let (M, J, ω) be a compact almost Hermitian 4-manifold. If ω is strictly locally conformally almost Kähler, then $h^{1,1}_{\overline{\partial}} = b^-$. If ω is also Gauduchon, then*

$$\mathcal{H}^{1,1}_{\overline{\partial}} = \mathcal{H}^-_{\mathbb{C}}.$$

Proof. Since ω is strictly locally conformally almost Kähler, write $d\omega = \theta \wedge \omega$, where θ is d-closed but non d-exact. Assume that ω is also Gauduchon. Let $\psi \in \mathcal{H}^{1,1}_{\overline{\partial}}$. By Theorem 4.1, we have $\psi = f\omega + \gamma$, where $f \in \mathbb{C}$, $*\gamma = -\gamma$, and

$$f\overline{\partial}\omega + \overline{\partial}\gamma = 0, \quad f\partial\omega - \partial\gamma = 0.$$

Decompose θ as $\lambda + \overline{\lambda}$, where $\lambda \in A^{1,0}$. It follows that

$$f\overline{\lambda} \wedge \omega = -\overline{\partial}\gamma, \quad f\lambda \wedge \omega = \partial\gamma,$$

and, since $\lambda, \overline{\lambda}$ are primitive, applying the Hodge $*$ operator we find

$$f\overline{\lambda} = -i * \overline{\partial}\gamma, \quad f\lambda = -i * \partial\gamma.$$

This means

$$f\theta = -i * d\gamma = i * d * \gamma = -id^*\gamma.$$

Since d-closed and d^*-exact forms are L^2-orthogonal, and $f\theta$ is both d-closed and d^*-exact, we conclude that $f\theta = 0$. Since θ is non d-exact, it follows that $\theta \neq 0$ and so $f = 0$. Finally, we derive that $\psi = \gamma$ is harmonic, proving $\mathcal{H}^{1,1}_{\overline{\partial}} = \mathcal{H}^-_{\mathbb{C}}$.

If ω is strictly locally conformally almost Kähler but it is not Gauduchon, then there exists a conformal Gauduchon metric, which is still strictly locally conformally almost Kähler, such that $h^{1,1}_{\overline{\partial}} = b^-$. We conclude by Lemma 3.3. □

Thanks to Corollary 4.2 and Theorem 4.3, on a compact almost Hermitian 4-manifold (M, J, ω), we know that if ω is globally conformally almost Kähler, then $h^{1,1}_{\overline{\partial}} = b^- + 1$, and if ω is strictly locally conformally almost Kähler, then $h^{1,1}_{\overline{\partial}} = b^-$. Moreover, as noted in [**18**, Proposition 3.8], by (1) it is easy to derive

$$\mathcal{H}^{1,1}_{\overline{\partial}} \supseteq \mathcal{H}^-_{\mathbb{C}}.$$

It immediately follows that $h^{1,1}_{\overline{\partial}} \geq b^-$. The following Theorem tells us that $h^{1,1}_{\overline{\partial}} = b^-$ or $h^{1,1}_{\overline{\partial}} = b^- + 1$ are the two only possible options on a given compact almost Hermitian 4-manifold.

Theorem 4.4 ([**10**, Theorem 3.1]). *Let (M, J, ω) be a compact almost Hermitian 4-manifold. Then either $h^{1,1}_{\overline{\partial}} = b^-$ or $h^{1,1}_{\overline{\partial}} = b^- + 1$.*

Proof. By Lemma 3.3 we can assume, without loss of generality, that ω is Gauduchon. We know that

$$\mathcal{H}^-_{\mathbb{C}} \subseteq \mathcal{H}^{1,1}_{\overline{\partial}}.$$

When the inclusion is an equality, then $h^{1,1}_{\overline{\partial}} = b^-$. Suppose instead that there exists an element $f_0\omega + \gamma_0 \in \mathcal{H}^{1,1}_{\overline{\partial}}$ such that

$$f_0 \in \mathbb{C} \setminus \{0\}, \quad *\gamma_0 = -\gamma_0, \quad f_0 d\omega = id^c\gamma_0.$$

Recall that

$$\mathcal{H}^{1,1}_{\overline{\partial}} = \{f\omega + \gamma \in A^{1,1} \mid f \in \mathbb{C},\ *\gamma = -\gamma,\ f d\omega = id^c\gamma\}.$$

by Theorem 4.1. We claim that

$$\mathcal{H}^{1,1}_{\overline{\partial}} = \{f(f_0\omega + \gamma_0) + \gamma \in A^{1,1} \mid f \in \mathbb{C},\ *\gamma = -\gamma,\ d\gamma = 0\},$$

which yields $h^{1,1}_{\overline{\partial}} = b^- + 1$. The inclusion $\supseteq$ is immediate. Indeed, for any $f \in \mathbb{C}$ and $\gamma \in A^{1,1}$ such that $*\gamma = -\gamma$ and $d\gamma = 0$, consider

$$f(f_0\omega + \gamma_0) + \gamma = f f_0\omega + (f\gamma_0 + \gamma)$$

and note that

$$f f_0 d\omega = f i d^c \gamma_0 = i d^c (f\gamma_0 + \gamma),$$

since $d^c\gamma = 0$ iff $d\gamma = 0$, and $*(f\gamma_0 + \gamma) = -(f\gamma_0 + \gamma)$. This implies

$$f f_0 \omega + (f\gamma_0 + \gamma) \in \mathcal{H}^{1,1}_{\bar\partial}.$$

To prove the converse inclusion $\subseteq$, let $f_1\omega + \gamma_1 \in \mathcal{H}^{1,1}_{\bar\partial}$, i.e., $f_1 \in \mathbb{C}$, $*\gamma_1 = -\gamma_1$ and $f_1 d\omega = i d^c \gamma_1$. We compute

$$f_1\omega + \gamma_1 = \frac{f_1}{f_0}(f_0\omega + \gamma_0) + \gamma_1 - \frac{f_1}{f_0}\gamma_0 = f(f_0\omega + \gamma_0) + \gamma,$$

where we set $f = \frac{f_1}{f_0}$ and $\gamma = \gamma_1 - \frac{f_1}{f_0}\gamma_0$. Note that

$$f \in \mathbb{C}, \quad *\gamma = -\gamma, \quad i d^c \gamma = f_1 \omega - \frac{f_1}{f_0} f_0 d\omega = 0,$$

and $d^c\gamma = 0$ iff $d\gamma = 0$, proving the claim. □

Note that in the integrable case, i.e., on compact complex surfaces, it is always true that $h^{1,1}_{\bar\partial}$ is b^- or $b^- + 1$. Theorem 4.4 is a generalization, to the non integrable case, of the just mentioned fact. From Theorems 4.1 and 4.4, we derive as a corollary the following equivalent condition of $h^{1,1}_{\bar\partial} = b^- + 1$, when the almost Hermitian metric is Gauduchon.

Corollary 4.5 ([**10**, Corollary 3.2]). *Let (M, J, ω) be a compact almost Hermitian 4-manifold. Assume that ω is Gauduchon. Then, $h^{1,1}_{\bar\partial} = b^- + 1$ if and only if there exists an anti self dual $(1,1)$-form γ satisfying*

$$i d^c \gamma = d\omega.$$

In this case

$$\mathcal{H}^{1,1}_{\bar\partial} = \mathbb{C} < \omega + \gamma > \oplus \mathcal{H}^-_{\mathbb{C}}.$$

Otherwise, $h^{1,1}_{\bar\partial} = b^-$ and

$$\mathcal{H}^{1,1}_{\bar\partial} = \mathcal{H}^-_{\mathbb{C}}.$$

Proof. Assume that such a form γ exists. Therefore $\omega + \gamma \in \mathcal{H}^{1,1}_{\bar\partial}$ by Theorem 4.1. By the proof of Theorem 4.4 we derive that $h^{1,1}_{\bar\partial} = b^- + 1$.

Conversely, if $h^{1,1}_{\bar\partial} = b^- + 1$, then there must be some form in $\mathcal{H}^{1,1}_{\bar\partial}$ other than those contained in $\mathcal{H}^-_{\mathbb{C}}$. By Theorem 4.1, it means that there exists a form

$f_0\omega + \gamma_0 \in A^{1,1}$ with $f_0 \in \mathbb{C} \smallsetminus \{0\}$, $*\gamma_0 = -\gamma_0$ and $f_0 d\omega = id^c\gamma_0$. Thus $\gamma = \frac{1}{f_0}\gamma_0$ gives us the solution of $id^c\gamma = d\omega$.

Finally, if such γ exists, by the proof of Theorem 4.4 we derive

$$\mathcal{H}^{1,1}_{\overline{\partial}} = \mathbb{C} < \omega + \gamma > \oplus \mathcal{H}^-_{\mathbb{C}}.$$

Otherwise, $\mathcal{H}^{1,1}_{\overline{\partial}} = \mathcal{H}^-_{\mathbb{C}}$ and $h^{1,1}_{\overline{\partial}} = b^-$. □

The following result asserts that the number $h^{1,1}_d$, on a given compact almost Hermitian 4-manifold (M, J, ω), depends only on the fact that the metric is either globally conformally almost Kähler or not.

Theorem 4.6 ([**16**, Corollary 4.5], cf. [**10**, Theorem 3.4], [**12**, Proposition 6.1]). *Let (M, J, ω) be a compact almost Hermitian 4-manifold. If ω is globally conformally almost Kähler, then $h^{1,1}_d = b^- + 1$. In particular, if ω is almost Kähler, then*

$$\mathcal{H}^{1,1}_d = \mathbb{C} < \omega > \oplus \mathcal{H}^-_{\mathbb{C}}.$$

Otherwise, if ω is not globally conformally almost Kähler, then $h^{1,1}_d = b^-$. In particular, if ω is Gauduchon but non almost Kähler, then

$$\mathcal{H}^{1,1}_d = \mathcal{H}^-_{\mathbb{C}}.$$

Proof. Assume that ω is almost Kähler. Note that the inclusion $\supseteq$ of the characterization of $\mathcal{H}^{1,1}_d$ is trivial. Let $\psi \in A^{1,1}$. By Lemma 3.1, we can write

$$\psi = f\omega + \gamma,$$

where $f \in \mathcal{C}^\infty(M, \mathbb{C})$ and $*\gamma = -\gamma$. It follows that $\psi \in \mathcal{H}^{1,1}_d$ iff $d\psi = d * \psi = 0$, iff

$$df \wedge \omega + d\gamma = 0, \quad df \wedge \omega - d\gamma = 0.$$

Thus, if $\psi \in \mathcal{H}^{1,1}_d$, then $df \wedge \omega = 0$, which implies that f is a complex constant by the Lefschetz isomorphism, and $d\gamma = 0$, which implies that $\gamma \in \mathcal{H}^-_{\mathbb{C}}$. This proves the other inclusion $\supseteq$, and $h^{1,1}_d = b^- + 1$.

If ω is globally conformally almost Kähler, then $h^{1,1}_d = b^- + 1$ by Lemma 3.3.

Otherwise, assume that ω is Gauduchon but non almost Kähler. Let $\psi \in A^{1,1}$. By Lemma 3.1, we can write

$$\psi = f\omega + \gamma,$$

where $f \in \mathcal{C}^\infty(M, \mathbb{C})$ and $*\gamma = -\gamma$. Then, $\psi \in \mathcal{H}^{1,1}_d$ iff $d\psi = d * \psi = 0$, iff

$$d(f\omega) + d\gamma = 0, \quad d(f\omega) - d\gamma = 0,$$

iff $d(f\omega) = d\gamma = 0$. Therefore, if $\psi \in \mathcal{H}_d^{1,1}$, then $d(f\omega) = 0$, which implies $\partial\overline{\partial}(f\omega) = 0$ and so $f \in \mathbb{C}$ by Lemma 3.2. It follows that

$$f d\omega = 0,$$

and since $d\omega \neq 0$, then $f = 0$. This implies $\mathcal{H}_d^{1,1} = \mathcal{H}_{\mathbb{C}}^{-}$ and $h_d^{1,1} = b^{-}$.

If ω is not globally conformally almost Kähler, then there exists a conformal Gauduchon metric, which is still not globally conformally almost Kähler, and in particular not almost Kähler, such that $h_{\overline{\partial}}^{1,1} = b^{-}$. We conclude by Lemma 3.3. ☐

5 - Bott-Chern harmonic (1,1)-forms

In this section, given a compact almost Hermitian 4-manifold (M, J, ω), we study the space of Bott-Chern harmonic $(1,1)$-forms $\mathcal{H}_{BC}^{1,1}$ and its dimension $h_{BC}^{1,1}$.

Assuming that the metric is Gauduchon, we find the following characterization of the space $\mathcal{H}_{BC}^{1,1}$.

Theorem 5.1 ([**16**, Theorem 4.3]). *Let (M, J, ω) be a compact almost Hermitian 4-manifold. If ω is Gauduchon, then*

$$\mathcal{H}_{BC}^{1,1} = \{f\omega + \gamma \in A^{1,1} \mid f \in \mathbb{C},\ *\gamma = -\gamma,\ d(f\omega + \gamma) = 0\}.$$

Proof. Let $\psi \in A^{1,1}$. By Lemma 3.1, we can write

$$\psi = f\omega + \gamma,$$

where $f \in \mathcal{C}^{\infty}(M, \mathbb{C})$ and $*\gamma = -\gamma$. Then, $\psi \in \mathcal{H}_{BC}^{1,1}$ iff $\partial\psi = \overline{\partial}\psi = \partial\overline{\partial} * \psi = 0$, iff

$$\partial(f\omega) + \partial\gamma = 0, \quad \overline{\partial}(f\omega) + \overline{\partial}\gamma = 0, \quad \partial\overline{\partial}(f\omega) - \partial\overline{\partial}\gamma = 0. \tag{3}$$

Thus, if $\psi \in \mathcal{H}_{BC}^{1,1}$, then

$$\partial\overline{\partial}(f\omega) = \partial\overline{\partial}\gamma = -\partial\overline{\partial}(f\omega),$$

that is $\partial\overline{\partial}(f\omega) = 0$, which implies $f \in \mathbb{C}$ by Lemma 3.2. It follows that $\psi \in \mathcal{H}_{BC}^{1,1}$ iff

$$d(f\omega + \gamma) = 0,$$

since the third condition of (3) is implied by the first two, given that $f \in \mathbb{C}$ and ω is Gauduchon. This proves the theorem. ☐

As a consequence of Theorem 5.1, it is possible to deduce that either $h^{1,1}_{BC} = b^-$ or $h^{1,1}_{BC} = b^- + 1$ holds, see [**16**, Theorem 4.3]. Moreover, if the metric is globally conformally almost Kähler, then $h^{1,1}_{BC} = b^- + 1$, see [**16**, Corollary 4.4]. However, these results can be actually improved by the following argument.

Theorem 5.2 ([**10**, Theorem 4.3]). *Let (M, J, ω) be a compact almost Hermitian 4-manifold. Then $h^{1,1}_{BC} = b^- + 1$.*

Proof. By Lemma 3.3, we can assume, without loss of generality, that ω is Gauduchon. The Hodge decomposition for ω is

$$\omega = d\alpha + h + d^*\beta,$$

for some $\alpha \in A^1$, $h \in \mathcal{H}^2_d$, $\beta \in A^3$. Define

$$\gamma_0 = d * \beta + d^*\beta,$$

which satisfies

$$*\gamma_0 = *d * \beta + *d^*\beta = -d^*\beta - * * d * \beta = -\gamma_0.$$

Note that

$$d\omega = dd^*\beta = d\gamma_0,$$

which means that, for the value $f_0 = -1$, we have $f_0\omega + \gamma_0 \in \mathcal{H}^{1,1}_{BC}$ by Theorem 5.1.

We have just proved that there exists an element $f_0\omega + \gamma_0 \in \mathcal{H}^{1,1}_{BC}$ such that

$$f_0 \in \mathbb{C} \smallsetminus \{0\}, \quad *\gamma_0 = -\gamma_0, \quad d(f_0\omega + \gamma_0) = 0.$$

Recall that

$$\mathcal{H}^{1,1}_{BC} = \{f\omega + \gamma \in A^{1,1} \mid f \in \mathbb{C}, *\gamma = -\gamma, d(f\omega + \gamma) = 0\}.$$

We claim that

$$\mathcal{H}^{1,1}_{BC} = \{f(f_0\omega + \gamma_0) + \gamma \in A^{1,1} \mid f \in \mathbb{C}, \ *\gamma = -\gamma, \ d\gamma = 0\},$$

which yields $h^{1,1}_{BC} = b^- + 1$.

The inclusion $\supseteq$ is immediate. Indeed, for any $f \in \mathbb{C}$ and $\gamma \in A^{1,1}$ such that $*\gamma = -\gamma$ and $d\gamma = 0$, consider

$$f(f_0\omega + \gamma_0) + \gamma = ff_0\omega + (f\gamma_0 + \gamma)$$

and note that

$$d(ff_0\omega + (f\gamma_0 + \gamma)) = fd(f_0\omega + \gamma_0) + d\gamma = 0,$$

and $*(f\gamma_0 + \gamma) = -(f\gamma_0 + \gamma)$. This implies

$$ff_0\omega + (f\gamma_0 + \gamma) \in \mathcal{H}^{1,1}_{BC}.$$

To prove the converse inclusion $\subseteq$, let $f_1\omega + \gamma_1 \in \mathcal{H}^{1,1}_{BC}$, i.e., $f_1 \in \mathbb{C}$, $*\gamma_1 = -\gamma_1$ and $d(f_1\omega + \gamma_1) = 0$. We compute

$$f_1\omega + \gamma_1 = \frac{f_1}{f_0}(f_0\omega + \gamma_0) + \gamma_1 - \frac{f_1}{f_0}\gamma_0 = f(f_0\omega + \gamma_0) + \gamma,$$

where we set $f = \frac{f_1}{f_0}$ and $\gamma = \gamma_1 - \frac{f_1}{f_0}\gamma_0$. Note that

$$f \in \mathbb{C}, \quad *\gamma = -\gamma, \quad d\gamma = -f_1 d\omega + \frac{f_1}{f_0} f_0 d\omega = 0,$$

proving the claim. □

Note that in the integrable case, i.e., on compact complex surfaces, it is always true that $h^{1,1}_{BC} = b^- + 1$. Theorem 5.2 is a generalization, to the non integrable case, of the just mentioned fact.

6 - Examples

This section is devoted to the study of the number $h^{1,1}_{\overline{\partial}}$ on explicit examples of almost Hermitian 4-manifolds. In particular, we focus on the following two compact solvmanifolds: the Secondary Kodaira surface and the Inoue surface of type S_M.

6.1 - *Secondary Kodaira surface*

Let $M = \Gamma\backslash G$ be a secondary Kodaira surface. Here G is a solvable Lie group and Γ is a cocompact lattice. We refer to [**8**, pp. 756, 760] for the construction of M and for the structure equations of the global coframe $\{e^1, e^2, e^3, e^4\}$, i.e.,

$$de^1 = e^{24}, \quad de^2 = -e^{14}, \quad de^3 = e^{12}, \quad de^4 = 0.$$

The coframe $\{e^1, e^2, e^3, e^4\}$ is induced from a left invariant one on G.

We endow M with the almost complex structure J given by

$$\varphi^1 = e^1 + ie^3, \quad \varphi^2 = e^2 + ie^4$$

being a global coframe of the vector bundle of $(1,0)$ forms $T^{1,0}M$. The associated structure equations are

$$d\varphi^1 = \frac{i}{4}\left(\varphi^{12} + \varphi^{1\overline{2}} - \varphi^{2\overline{1}} + 2\varphi^{2\overline{2}} + \varphi^{\overline{12}}\right),$$
$$d\varphi^2 = \frac{i}{4}\left(\varphi^{12} - \varphi^{1\overline{2}} - \varphi^{2\overline{1}} - \varphi^{\overline{12}}\right),$$

therefore the almost complex structure J is non integrable.

We endow (M,J) with the diagonal almost Hermitian metric

$$\omega = i(\varphi^{1\overline{1}} + \varphi^{2\overline{2}}) = 2(e^{13} + e^{24}),$$

and consider the volume form $\frac{\omega^2}{2}$. The metric ω is not locally conformally almost Kähler, since

$$d\omega = -2e^{234} = \theta \wedge \omega,$$

where $\theta = e^3$ is not d-closed. The de Rham cohomology of M is

$$H_{dR}(M) = \mathbb{R} < 1 > \oplus \mathbb{R} < e^4 > \oplus \mathbb{R} < e^{123} > \oplus \mathbb{R} < e^{1234} > .$$

Note that $b^2 = 0$, therefore in particular $b^- = 0$.

Let us compute $h^{1,1}_{\overline{\partial}}$. Let $\eta \in A^{1,1}$ be of the form

$$\eta = A\varphi^{1\overline{1}} + B\varphi^{1\overline{2}} + C\varphi^{2\overline{1}} + D\varphi^{2\overline{2}},$$

with $A,B,C,D \in \mathbb{C}$. Assume that η is Dolbeault harmonic, i.e., $\overline{\partial}\eta = \partial * \eta = 0$. Note that

$$*\eta = D\varphi^{1\overline{1}} - B\varphi^{1\overline{2}} - C\varphi^{2\overline{1}} + A\varphi^{2\overline{2}}.$$

From the structure equations we derive

$$4i\overline{\partial}\varphi^{1\overline{1}} = 2\varphi^{2\overline{12}}, \qquad 4i\partial\varphi^{1\overline{1}} = 2\varphi^{12\overline{2}},$$
$$4i\overline{\partial}\varphi^{1\overline{2}} = -\varphi^{1\overline{12}} + \varphi^{2\overline{12}}, \qquad 4i\partial\varphi^{1\overline{2}} = -\varphi^{12\overline{1}} - \varphi^{12\overline{2}},$$
$$4i\overline{\partial}\varphi^{2\overline{1}} = -\varphi^{1\overline{12}} - \varphi^{2\overline{12}}, \qquad 4i\partial\varphi^{2\overline{1}} = -\varphi^{12\overline{1}} + \varphi^{12\overline{2}},$$
$$4i\overline{\partial}\varphi^{2\overline{2}} = 0, \qquad 4i\partial\varphi^{2\overline{2}} = 0.$$

Therefore

$$4i\overline{\partial}\eta = (-B-C)\varphi^{1\overline{12}} + (2A+B-C)\varphi^{2\overline{12}},$$
$$4i\partial * \eta = (B+C)\varphi^{1\overline{12}} + (2D+B-C)\varphi^{2\overline{12}},$$

and so

$$\begin{cases} B = -C, \\ A = C, \\ D = C. \end{cases}$$

Thus $h^{1,1}_{\overline{\partial}} \geq 1$, and $h^{1,1}_{\overline{\partial}} = 1 = b^- + 1$ by Theorem 4.4. The space of Dolbeault harmonic $(1,1)$-forms is

$$\mathcal{H}^{1,1}_{\overline{\partial}} = \mathbb{C} < \varphi^{1\overline{1}} - \varphi^{1\overline{2}} + \varphi^{2\overline{1}} + \varphi^{2\overline{2}} > .$$

6.2 - *Inoue surface S_M*

Let $M = \Gamma\backslash G$ be a Inoue surface of type S_M. Here G is a solvable Lie group and Γ is a cocompact lattice. We refer to [**8**, pp. 755, 760] for its construction and for the structure equations of the global coframe $\{e^1, e^2, e^3, e^4\}$, i.e., for any $\alpha, \beta \in \mathbb{R}$, $\alpha \neq 0$,

$$de^1 = \alpha e^{14} + \beta e^{24}, \quad de^2 = -\beta e^{14} + \alpha e^{24}, \quad de^3 = -2\alpha e^{34}, \quad de^4 = 0.$$

The coframe $\{e^1, e^2, e^3, e^4\}$ is induced from a left invariant one on G.

We endow M with the almost complex structure J given by

$$\varphi^1 = e^1 + ie^3, \quad \varphi^2 = e^2 + ie^4$$

being a global coframe of the vector bundle of $(1,0)$ forms $T^{1,0}M$. The associated structure equations are

$$\begin{aligned} d\varphi^1 &= \alpha \frac{i}{4}\left(\varphi^{12} - \varphi^{1\overline{2}} + 3\varphi^{2\overline{1}} + 3\varphi^{\overline{12}}\right) + \beta \frac{i}{2}\varphi^{2\overline{2}}, \\ d\varphi^2 &= \beta \frac{i}{4}\left(\varphi^{12} - \varphi^{1\overline{2}} - \varphi^{2\overline{1}} - \varphi^{\overline{12}}\right) + \alpha \frac{i}{2}\varphi^{2\overline{2}}, \end{aligned}$$

therefore the almost complex structure J is non integrable.

We endow (M, J) with the diagonal almost Hermitian metric

$$\omega = i(\varphi^{1\overline{1}} + \varphi^{2\overline{2}}) = 2(e^{13} + e^{24}),$$

and consider the volume form $\frac{\omega^2}{2}$. The de Rham cohomology of M is

$$H_{dR}(M) = \mathbb{R} < 1 > \oplus \mathbb{R} < e^4 > \oplus \mathbb{R} < e^{123} > \oplus \mathbb{R} < e^{1234} > .$$

Note that $b^2 = 0$, therefore in particular $b^- = 0$. The exterior derivative of ω is

$$d\omega = 2\alpha e^{134} - 2\beta e^{234} = \theta \wedge \omega,$$

where $\theta = \beta e^3 + \alpha e^4$. Note that e^3 is not d-closed, and e^4 is closed but non d-exact, therefore ω is strictly locally conformally almost Kähler for $\beta = 0$, and thus $h^{1,1}_{\overline{\partial}} = b^- = 0$ by Theorem 4.3, while it is not locally conformally almost Kähler for $\beta \neq 0$.

Let $\eta \in A^{1,1}$ be of the form

$$\eta = A\varphi^{1\overline{1}} + B\varphi^{1\overline{2}} + C\varphi^{2\overline{1}} + D\varphi^{2\overline{2}},$$

with $A, B, C, D \in \mathbb{C}$. Assume that η is Dolbeault harmonic, i.e., $\overline{\partial}\eta = \partial * \eta = 0$. Note that

$$*\eta = D\varphi^{1\overline{1}} - B\varphi^{1\overline{2}} - C\varphi^{2\overline{1}} + A\varphi^{2\overline{2}}.$$

From the structure equations we derive

$$\begin{aligned}
4i\overline{\partial}\varphi^{1\overline{1}} &= -2\alpha\varphi^{1\overline{1}\overline{2}} + 2\beta\varphi^{2\overline{1}\overline{2}}, &\qquad 4i\partial\varphi^{1\overline{1}} &= -2\alpha\varphi^{12\overline{1}} + 2\beta\varphi^{12\overline{2}},\\
4i\overline{\partial}\varphi^{1\overline{2}} &= -\beta\varphi^{1\overline{1}\overline{2}} - 3\alpha\varphi^{2\overline{1}\overline{2}}, & 4i\partial\varphi^{1\overline{2}} &= -\beta\varphi^{12\overline{1}} + \alpha\varphi^{12\overline{2}},\\
4i\overline{\partial}\varphi^{2\overline{1}} &= -\beta\varphi^{1\overline{1}\overline{2}} + \alpha\varphi^{2\overline{1}\overline{2}}, & 4i\partial\varphi^{2\overline{1}} &= -\beta\varphi^{12\overline{1}} - 3\alpha\varphi^{12\overline{2}},\\
4i\overline{\partial}\varphi^{2\overline{2}} &= 0, & 4i\partial\varphi^{2\overline{2}} &= 0.
\end{aligned}$$

Therefore

$$\begin{aligned}
4i\overline{\partial}\eta &= (-2\alpha A - \beta B - \beta C)\varphi^{1\overline{1}\overline{2}} + (2\beta A - 3\alpha B + \alpha C)\varphi^{2\overline{1}\overline{2}},\\
4i\partial * \eta &= (-2\alpha D + \beta B + \beta C)\varphi^{1\overline{1}\overline{2}} + (2\beta D - \alpha B + 3\alpha C)\varphi^{2\overline{1}\overline{2}}.
\end{aligned}$$

Since the determinant of

$$\begin{pmatrix} -2\alpha & -\beta & -\beta & 0\\ 0 & \beta & \beta & -2\alpha\\ 2\beta & -3\alpha & \alpha & 0\\ 0 & -\alpha & 3\alpha & 2\beta \end{pmatrix}$$

is $-32\alpha^2(\alpha^2 + \beta^2)$, which is surely different from 0, we get $A = B = C = D = 0$. Recall that if $\beta = 0$, then $h^{1,1}_{\overline{\partial}} = b^- = 0$ by Theorem 4.3, while for $\beta \neq 0$ this computation is not sufficient to compute $h^{1,1}_{\overline{\partial}}$.

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