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The space of tamed almost complex structures on symplectic 4-manifolds via symplectic spheres

Abstract. In this note, we study a fine decomposition of the space of tamed almost complex structures for symplectic 4 manifolds via symplectic spheres. We also show that every tamed almost complex structure on a rational surface other than $\mathbb{C}P^2$ is fibred.

Keywords. Symplectic 4-manifolds, almost complex structure.

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1 - Introduction

Now let (M, ω) be a symplectic 4-manifold, and let $\mathcal{S}_\omega \subset H_2(M, \mathbb{Z})$ denote the set of homology classes of embedded ω-symplectic spheres. Let $S_\omega^{<n} \subset \mathcal{S}_\omega$ respectively) be the collection of classes whose square is less than $n \in \mathbb{Z}$. Define the subsets $S_\omega^{\leq n}$, $S_\omega^n = S_\omega^{=n}$, $S_\omega^{>n}$, $S_\omega^{\geq n}$ similarly.

Let (M, ω) be a closed symplectic manifold, we denote by $\mathcal{J}_w$ the space of almost complex structures tamed by ω. $\mathcal{J}_w$ is an infinite dimensional Fréchet space, and it is contractible. One can decompose $\mathcal{J}_w$ into pieces that look like submanifolds. Gromov [**14**] first observed that this space, on some 4-manifolds such as $S^2 \times S^2$, has a decomposition via smooth symplectic spheres. He also related this decomposition to the topology of the symplectomorphism group $Symp(M, \omega)$. This approach is further probed in detail by Abreu and McDuff on Hirzebruch surfaces, cf. [**1**] and [**3**], and later on to rational or ruled 4-manifolds [**3**, **4**, **5**, **7**, **8**, **9**, **10**, **17**, **19**, **20**, **21**] etc. The stability in [**6**] highlights the role of $\mathcal{S}_\omega$. This is one of the motivations for this note.

The space $\mathcal{J}_w$ also appears in many other pieces of literature, for example, in [**12**] and [**2**], the strata in $\mathcal{J}_w$ correspond to extremal Kähler metrics. It also provides rich information in different contexts on toric manifolds, cf. [**15**,**16**,**30**],

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etc, where the strata of $\mathcal{J}_w$ guide the conjugacy classes of Hamiltonian torus actions and Hamiltonian circle actions.

Definition 1.1. *For $A \in \mathcal{S}_\omega$, let*

$$U_A := \{J \in \mathcal{J}_\omega | A \textit{ has a connected embedded } J\textit{-holomorphic representative of genus } 0\}, \textit{ and } \quad U_A^{eff} := \{J \in \mathcal{J}_\omega | A \textit{ has a } J\textit{-holomorphic representative}\}.$$

$$Cod_{\mathbb{R}}(A) = \min\{0, -2A \cdot A - 2\}. \tag{1}$$

We will often use the complex codimension of a curve or a set, denoted by $Cod_{\mathbb{C}}(A)$ or simply $Cod(A)$. We have the adjunction equality $K_\omega \cdot A + A \cdot A = -2$, where K_ω is the symplectic canonical class. This gives us equivalent expressions of the codimension if $A^2 \leq -2$:

$$Cod_{\mathbb{C}}(A) = K \cdot A + 1. \tag{2}$$

U_A^{eff} is the Brill Noether locus of A when J is projective. If $J \in U_A^{eff}$ then $A = \sum r_i[C_i]$ where r_i is a positive integer and C_i an irreducible $J-$holomorphic curve.

Note that A is in the $J-$curve cone if we allow $r_i \in \mathbb{Q}$.

Definition 1.2. *For a subset $\mathcal{C}$ of $\mathcal{S}_\omega^{\leq -2}$ such that any pair of classes intersect positively.*

$$U_{\mathcal{C}} := \{J \in \mathcal{J}_\omega | A \textit{ has an embedded } J\textit{-holomorphic representative if } A \in \mathcal{C}\},$$
$$U_{\mathcal{C}}^{eff} := \{J \in \mathcal{J}_\omega | A \in H_2 \textit{ has a } J\textit{-holomorphic representative if } A \in \mathcal{C}\}.$$

$$Cod_{\mathbb{R}}(\mathcal{C}) = \sum_{A_i \in \mathcal{C}} Cod_{\mathbb{R}}(A_i) = \sum_{A_i \in \mathcal{C}} 2(-A_i \cdot A_i - 1).$$

If we set $X_i := \amalg_{Cod(\mathcal{C}) \geq i} U_{\mathcal{C}}$ we have a filtration in $\mathcal{J}_\omega$ according to the (complex) codimension of prime submanifolds in Definition 1.1:

$$\emptyset = X_{n+1} \subset X_n \subset X_{n-1} \ldots \subset X_1 \subset X_0 = \mathcal{J}_\omega. \tag{3}$$

Set $X_i^{eff} = \cup_{Cod(\mathcal{C}) \geq i} U_{\mathcal{C}}^{eff}$. Although X_i^{eff} is not necessarily the closure of X_i, the condition that $X_i = X_i^{eff}$ for all i is analogous to the ∞ dimensional version of the stratification as in Definition 1.1.1 of [**31**], which means that the taking closure process respects codimension.

For $A \in \mathcal{S}_\omega^{<-1}$, we also mark the following subsets of $\mathcal{J}_\omega$

$$\mathcal{J}_A := \{J \in \mathcal{J}_\omega | A \in \mathcal{S}_\omega^{-1} \text{ has a } J\text{-hol embedded representative if and only if } A \in \mathcal{C}\},$$

$\mathcal{J}_{\mathcal{C}} := \{J \in \mathcal{J}_\omega | A \in H_2$ has a J-hol embedded representative if and only if $A \in \mathcal{C}\}$.

For M is a rational 4-manifold with $\chi(M) \leq 8$, any $A \in \mathcal{S}_\omega^{-2}$ and $\mathcal{C} = \{C_i\}$ such that $\mathcal{C}$ is the collection of homology classes of the Gromov limit of A, we have $Cod(\mathcal{J}_{\mathcal{C}}) > Cod(\mathcal{J}_A)$. This fact is very useful in determining the rank of $\pi_1(Symp(M,\omega))$.

In this note, we compare U_A^{eff} with U_A. In particular, we ask the following question.

Question 1.3. *Let (M,ω) be a symplectic 4-manifold and $A \in \mathcal{S}_\omega$. When do we have that $U_A^{eff} - U_A$ is covered by $U_{\mathcal{C}}$ with $Cod(\mathcal{C}) > Cod(A)$?*

This question is actually subtle to answer due to the possible presence of multiple covers of curves with negative self-intersection. For $M = \mathbb{C}P^2 \# 10\overline{\mathbb{C}P^2}$ and any symplectic form ω on M with $\omega \cdot K_\omega < 0$, consider the class $A = -2K_\omega$. Note that U_A is non-empty by [**13**] and U_A has codimension 3 since $A \cdot A = -4$. On the other hand, for any such ω, $U_A^{eff} - U_A$ contains a nonempty set Z of tamed almost complex structures for which $-K_\omega$ is represented by a smooth $J-$holomorphic torus by [**13**]. Note that Z has codimension 1 and $Z \cap U_{\mathcal{C}} = \emptyset$ for any $\mathcal{C}$ since $(-K_\omega) \cdot (-K_\omega) = -1$.

A simple observation is that we can use the diffeomorphism group action to probe this question. Namely, if ϕ is a diffeomorphism of M, then the question for (M,ω,A) has the same answer for $(M, \phi^*\omega, \phi_* A)$. By [**34**], [**22**], [**16**, and its Math Review], any symplectic form on a rational surface is diffeomorphic to a reduced symplectic form.

Definition 1.4. *Suppose $M = \mathbb{C}P^2 \# k\overline{\mathbb{C}P^2}$ and H, E_i is a standard basis. A class $\nu H - \sum c_i E_i$ is called* **reduced** *(with respect to the basis) if*

$$c_1 \geq c_2 \geq \cdots \geq c_k > 0 \quad \textit{and} \quad \nu \geq c_1 + c_2 + c_3.$$

Since diffeomorphic symplectic forms have homeomorphic symplectomorphism groups, one can also focus on reduced symplectic forms for applications of U_A to the topology of $Symp(M,\omega)$.

In this note, we present 3 positive results in different flavors. They are stated for a general symplectic form, and by the above discussion, we are going to present their proof for reduced symplectic forms in Section 3. To state the first result we introduce

Definition 1.5. *A is said to be $J-$nef if it pairs non-negatively with any irreducible $J-$holomorphic curve.*

A is said to be $\omega-$nef if it pairs non-negatively with any irreducible $J-$holomorphic curve for any $J \in \mathcal{J}_\omega$.

If $J \in U_A^{eff}$ and A is $J-$nef, then by [**26**, Theorem 1.5], there is a smooth $J-$holomorphic representative. Moreover, by Zhang [**33**, Theorem 4.4], the moduli space of $J-$holomorphic curves in the class A is homeomorphic to a projective space if A is primitive.

Lemma 1.6. *For $A \in \mathcal{S}_\omega$, $U_A = U_A^{eff}$ if only if A is $\omega-$nef.*

Proof. Clearly, $U_A = U_A^{eff}$ only if A is $\omega-$nef. Since $A \in \mathcal{S}_\omega$, the converse is also true by [**33**, Theorem 2.3], which states that a J-nef spherical class has a non-empty irreducible moduli space and hence has a smooth representative. □

It is observed in [**33**, Prop. 3.2] that, for an irrational ruled surface, the unique class in $\mathcal{S}_\omega^0$ is $\omega-$nef. On the other hand, it is also shown there that every class represented by a smooth embedded sphere is not $\omega-$nef for some symplectic form ω. We will show that such a class is $\omega-$nef for a large family of symplectic forms as long as the self-intersection is non-negative.

Theorem 1.7. *Let $M = \mathbb{C}P^2 \# k\overline{\mathbb{C}P^2}$ with $k \geq 1$ and H, E_i is a standard basis. Let ω be a reduced symplectic form. Then following classes in $\mathcal{S}_\omega$ are $\omega-$nef:*

- $H - E_1$;
- $A_l = lH - (l-1)E_1, l \geq 1$ *when* $\omega(A_{-l}) < 0$;
- $B_l = lH - (l-1)E_1 - E_2, l \geq 2$ *when* $\omega(B_{-l+1}) < 0$;
- $2H$ *when* $\omega(A_{-1}) < 0$.

Consequently, for any symplectic form τ (not necessarily reduced) and $A \in \mathcal{S}_\tau$, if ϕ is a diffeomorphism such that $\phi^*(\omega)$ is reduced (such ϕ always exists) and $\phi_*(A)$ is on the list above with respect to $\phi^*(\tau)$, then

$$U_A = U_A^{eff} = \mathcal{J}_\omega.$$

In particular, any tamed J on a rational surface with $\chi \geq 4$, i.e. other than $\mathbb{C}P^2$, is fibred.

This theorem essentially follows from Proposition 3.4. Theorem 1.7 also motivates a characterization of minimal $\omega-$area classes in $\mathcal{S}_\omega^0$ in Proposition 3.5.

Theorem 1.8. *Let (M, ω) be a symplectic rational surface and $A \in \mathcal{S}_\omega$. Suppose $F \in S_\omega^0$ has minimal area in S_ω^0 and $A \cdot F = 0$ or 1. If $A \in \mathcal{S}^{<0}$, then U_A^{eff} is the union of U_A and a collection of $U_\mathcal{C}$ with $Cod(\mathcal{C}) > Cod(A)$.*

Theorem 1.8 follows from Proposition 3.5 and Proposition 3.7.

By the example of $-2K_\omega$ on $(M,\omega) = (\mathbb{C}P^2\#10\overline{\mathbb{C}P^2},\omega)$, Question 1.3 can have a positive answers for all classes only if $\chi(M) \leq 12$. The next result settles Question 1.3 up to $\chi = 11$.

Theorem 1.9. *Let (M,ω) be a symplectic rational surface with $\chi(M) \leq 11$ and $A \in \mathcal{S}_\omega^{<0}$. Then $U_A^{eff} - U_A$ is covered by $U_{\mathcal{C}}$ with $Cod(\mathcal{C}) > Cod(A)$.*

Theorem 1.9 follows from Proposition 3.11.

2 - General properties of U_A, U_A^{eff}

2.1 - *Pseudo-holomorphic curves*

We review some general facts on J-holomorphic rational curves and symplectic spheres in symplectic 4-manifolds. The presentation is similar to [**1**] and [**7**].

Let (X,ω) be a symplectic 4-manifold and $J \in \mathcal{J}_\omega$. A parametrized J-holomorphic curve in X is a J-holomorphic map $u : (\Sigma, j) \to (X, J)$, where (Σ, j) is a smooth, connected Riemann surface. We will always assume that u is simple, i.e. it is non-constant and not a multiple covering. In this case, we say that $C = u(\Sigma)$ is an (unparameterized) J-holomorphic curve and denote by $[C]$ the homology class. Notice that the pairing $\omega([C])$ is positive.

Theorem 2.1 (Positivity of Intersection, [**14**,**28**,**29**]). *For an almost complex 4-manifold (X,J), two distinct irreducible $J-$holomorphic curves C, C' have only finitely many intersection points. Each such point p contributes $k_p \geq 1$ to the homological intersection number $[C]\cdot[C']$, and $k_p = 1$ if and only if C and C' meet transversally at p.*

Theorem 2.2 (Adjunction Inequality, [**27**]). *Let (X,J) be an almost complex 4-manifold with first Chern class $c_1(X,J)$ and $u : (\Sigma, j) \to X$ an irreducible $J-$holomorphic curve. Then the virtual genus of the image $C = u(\Sigma)$, defined as $g_v(C) = ([C]\cdot[C] - c_1(X,J)([C]))/2 + 1$, is a integer no less than $g(\Sigma)$. Moreover, $g_v(C) = g(\Sigma)$ if and only if u is an embedding.*

Theorem 2.3 ([**28**, Theor. 4.1.1 for immersion] [**23**, Prop. 2.1 for embedding]). *For a symplectic form ω and an $\omega-$tamed almost complex structure J, an irreducible $J-$holomorphic curve can be smoothed out to a connected, embedded symplectic surface.*

We are primarily interested in rational curves.

2.2 - *Symplectic spheres*

2.2.1 - Properties of U_A

We next list various facts about representing a class in $\mathcal{S}_\omega$ by irreducible or stable J-holomorphic rational curves.

For classes with self-intersection at least -1 we have the following well-known fact from the Gromov-Witten theory.

Proposition 2.4. *Let (X, ω) be a symplectic 4-manifold and A a class in $\mathcal{S}_\omega$.*

- *By the adjunction inequality, any irreducible J-holomorphic curve in the class A is embedded.*
- *For $A \in \mathcal{S}_\omega^{\geq -1}$, A has non-trivial Gromov-Witten invariant, by* [**24**, Prop. 3.2], *U_A is path connected, open, dense in $\mathcal{J}_\omega$, and $U_A^{eff} = \mathcal{J}_\omega$.*

For classes with self-intersections at most -2 we have the following fact in [**7**, Appendix B.1]:

Proposition 2.5. *Let (X, ω) be a symplectic 4-manifold. Suppose $U_\mathcal{C} \subset \mathcal{J}_\omega$ is a subset characterized by the existence of a configuration of embedded J-holomorphic rational curves $C_1 \cup C_2 \cup \cdots \cup C_N$ of negative self-intersection with $\{[C_1], [C_2], \cdots, [C_N]\} = \mathcal{C}$. Then $U_\mathcal{C}$ is a co-oriented Fréchet submanifold of $\mathcal{J}_\omega$ of (real) codimension $2N - 2c_1([C_1] + \cdots + [C_N])$.*

2.2.2 - A few Conditions

To ensure nice properties of the decomposition into prime subsets we introduce the following conditions for symplectic surfaces in (X, ω).

Definition 2.6. *Let (X, ω) be a symplectic $4-$manifold.*

Condition 1 : *Any embedded symplectic surface S with $S \cdot S < 0$ is a symplectic sphere.*

Condition 2 : *Any embedded symplectic surface S with $S \cdot S \leq 0$ is a symplectic sphere.*

Condition 3 : *Any symplectic surface S with $S \cdot S \geq 0$ has $K_\omega \cdot S < 0$.*

Condition 4 : *Any symplectic surface S with $S \cdot S \geq 0$ has $K_\omega \cdot S \leq 0$.*

Now we show that on small rational 4 manifolds and K3 surfaces, such Conditions hold:

Lemma 2.7. *Condition* 1 *in Definition* 2.6 *holds for a symplectic rational surface* (X, ω) *with* $\chi(X) \leq 12$ *or a K3 surface.*

Condition 2 *and Condition* 3 *holds hold for a symplectic rational surface* (X, ω) *with* $\chi(X) \leq 11$.

Condition 4 *holds for a symplectic rational surface* (X, ω) *with* $\chi(X) \leq 12$.

Proof. For rational surfaces, Conditions 1 and 2 follow from [**32**, Prop. 4.2]. Then conditions 3 and 4 follow directly from adjunction equality since the surfaces are embedded.

For Condition 1 on $K3$, we just need the adjunction inequality $2g-2 \leq K_\omega \cdot A + A \cdot A$. Note $g \geq 0$, $K_\omega \cdot A = 0$, and $A \cdot A < 0$. Hence the only possibility is $g = 0$ and $A \cdot A = -2$. This means any negative square irreducible curves are embedded (-2) spheres. Hence Condition 1 holds. □

The Conditions in 2.6 are useful in investigating the relation between U_A and U_A^{eff} toward Question 1.3, particularly when $Cod(\mathcal{C})$ is small.

We now construct a cover of $U_A^{eff} \setminus U_A$ by $U_{\mathcal{C}}$ with codimension as large as possible. First note if $J \in U_A^{eff} \setminus U_A$, we have a non-trivial decomposition of A:

$$A = \sum_\alpha r_\alpha C_\alpha + \sum r_\beta C_\beta + \sum r_\gamma C_\gamma, \tag{4}$$

where $C_\alpha, C_\beta, C_\gamma$ are irreducible J-holomrophic curves with $C_\alpha^2 \leq -2, C_\beta^2 = -1$, and $C_\gamma \cdot C_\gamma \geq 0$. Note that we will not distinguish the curve representative with their homology class if it is clear form context.

We take $\mathcal{C}$ to be the collection of embedded components among C_α. Apparently, $J \in U_{\mathcal{C}}$. Note that our choice of $\mathcal{C}$ is maximal in the sense that any subset $\mathcal{C}' \subset \mathcal{C}$ has the property $J \in U_{\mathcal{C}'}$. We call such $\mathcal{C}$ admissible.

Lemma 2.8. *Suppose Conditions* 1 *and* 3 *are satisfied. If* $A^2 = -2$ *and* $\mathcal{C}$ *an admissible set constructed as above, then* $Cod(\mathcal{C}) > Cod(A) = 1$.

Suppose Conditions 1 *and* 4 *are satisfied. If* $A^2 = -3$ *and* $\mathcal{C}$ *an admissible set constructed as above, then* $Cod(\mathcal{C}) \geq Cod(A)$. *The equality holds only if* $\mathcal{C}$ *is a single element subset of* $\mathcal{S}_\omega^{-3}$, *with multiplicity at least* 2 *in equation* (4).

Proof. By Theorem 2.3, the curves C_α and C_β are represented by embedded sympletic surfaces. Hence Condition 1 tells us that C_α and C_β are embedded sphere classes, i.e. $C_\alpha \in S_\omega^{<-1}$ and $C_\beta \in S_\omega^{-1}$. Note $\mathcal{C}$ being admissible means that **all** $C_\alpha \in \mathcal{C}$.

Pairing with K on both sides of equation (4),

$$K_\omega \cdot A = \sum_\alpha r_\alpha K_\omega \cdot C_\alpha + \sum r_\beta K_\omega \cdot C_\beta + \sum r_\gamma K_\omega \cdot C_\gamma \leq \sum_\alpha r_\alpha K_\omega \cdot C_\alpha.$$

Since $C_\alpha \in S_\omega^{\leq -2}$ and $C_\beta \in S_\omega^{-1}$, $K_\omega \cdot C_\beta = -1$, $K_\omega \cdot C_\alpha = -C_\alpha^2 - 2$. By Condition 3, $K_\omega \cdot C_\gamma < 0$.

When $A^2 = -2$, $K_\omega \cdot A = 0$. If there is at least one C_α with $C_\alpha^2 \leq -3$ then we are done. Otherwise, each $C_\alpha^2 = -2$ and there are no C_β or C_γ. If there is only one C_α, then $r_\alpha = 1$. So there are at least two such C_α.

When $A^2 = -3$, $K_\omega \cdot A = 1$. Then there is at least one C_α with $C_\alpha^2 \leq -3$ And equality holds only if there is only one C_α term, and $C_\alpha^2 = -3$, $r_\alpha \geq 2$. □

Consequently, we have the following (lower-level) stratification result for small rational surfaces, generalizing [**19**, Theorem 3.8]:

Proposition 2.9. *If* (M, ω) *satisfies Conditions* 1 *and* 3, *then* $X_1^{eff} = X_1$ *and* $X_2^{eff} = X_2$.

Proof. Then case $X_1^{eff} = X_1$ is covered by Lemma 2.8 the first statement. Both X_1 and X_1^{eff} are the collections of $J \in \mathcal{J}_\omega$ such that there is at least one embedded -2 curve. Hence they are the same.

For the second statement, notice that the case $U_{\mathcal{C}}$ where $\mathcal{C} = \{C\}$ with $C \in S_\omega^{\leq -3}$ is covered by the second statement of 2.8.

We just need to deal with the case that $\mathcal{C}$ contains at least two classes in $\mathcal{S}_\omega^{-2}$. In any $U_{\mathcal{C}}^{eff} \setminus U_{\mathcal{C}}$, there is a curve in class $A \in \mathcal{S}_\omega^{-2}$ that has a homology decomposition $A = c'A' + \sum c_i A_i$ with $\{A_i\} \subset \mathcal{S}_\omega^{-1} \coprod \mathcal{S}_\omega^0, c' \geq 0, c_i \geq 1$. Here A' is another class in $\mathcal{C} \cap \mathcal{S}_\omega^{-2}$. We claim that the set $\{A_i\}$ is not empty. This is true because $A \neq A'$ and $c'A'$ is not in $\mathcal{S}_\omega^{-2}$ for any $c' \neq 1$. Now pair the cusp curve decompositions with K_ω. We again get a contradiction since $K_\omega \cdot A = 0$ while $K_\omega \cdot A' = 0$ and $K_\omega \cdot A_i < 0$.

Hence Both X_2 and X_2^{eff} are the collections of $J \in \mathcal{J}_\omega$ such that the codimension is at least 4. □

3 - Rational surfaces

Notice that a rational 4-manifold is diffeomorphic to either $S^2 \times S^2$, $\mathbb{C}P^2$ or a blowup of it. For the case of $S^2 \times S^2$, the stratification is relatively simple and is fully understood in [**1**].

For $M = \mathbb{C}P^2 \# k\overline{\mathbb{C}P^2}, k \in \mathbb{Z}^{\geq 0}$, let $\{H, E_1, \cdots, E_k\}$ be a standard basis of $H_2(\mathbb{C}P^2 \# n\overline{\mathbb{C}P^2}; \mathbb{Z})$ where H is the line class and E_i's are the exceptional classes.

3.1 - *Reduced symplectic forms*

Recall that a class $\nu H - \sum c_i E_i$ is called **reduced** (with respect to the basis) if

$$c_1 \geq c_2 \geq \cdots \geq c_k > 0 \quad \text{and} \quad \nu \geq c_1 + c_2 + c_3.$$

Reduced cohomology classes are defined as the Poincaré dual of reduced homology classes. A symplectic form ω on X is called reduced if $[\omega]$ is reduced. A reduced symplectic class is the class of a reduced symplectic form.

3.1.1 - Irreducible pseudo-holomorphic curves for a reduced form

We have the following results from Chen [**11**] on connected embedded symplectic surfaces for a reduced symplectic form. Note that any irreducible $J-$holomorphic curve can be smoothed out to such a symplectic surface. So these results also apply to irreducible $J-$holomorphic curves.

Lemma 3.1 (Lemma 3.2 of Chen [**11**], see also Lemma 4.1 in [**32**]). *Let $X = \mathbb{C}P^2 \# k\overline{\mathbb{C}P^2}$ with a reduced symplectic form. If C is a connected, embedded symplectic surface with $C \cdot H < 0$, then $C = -nH + (n+1)E_1 - \sum_{k_j > 1} E_{k_j}$. In particular, $C^2 \leq -3$ and C is a symplectic sphere.*

On the other hand, we can also characterize the curve classes with a nonnegative coefficient on H.

Lemma 3.2. *For a reduced form ω, let $A = aH - \sum b_i E_i$ be represented by a connected, embedded symplectic surface with $a \geq 0$.*

1. *If $a > 0$, then $a \geq A \cdot (-E_i) = b_i \geq 0$ for each i.*
2. *If $a = A \cdot (-E_i) = b_i$ for some i, then $a = 1$ or 0.*
3. *If $a = 1$ then $b_i = 0$ or 1.*
4. *If $a = 0$, then $A = E_i - \sum_{q_j > i} E_{q_j}$ for some i.*

Proof. All the statements, except $q_j > i$ in (4), follow from the adjunction inequality with $K_\omega = -3H + \sum E_i$ of the reduced form ω.
The condition $q_j > i$ in (4) comes from $\omega(A) > 0$ and $c_i \geq c_{i+1}$. □

Using the smoothing result in Theorem 2.3, we are able to translate the above classification of symplectic surfaces to irreducible pseodoholomorphic curves.

Proposition 3.3. *Suppose J is tamed. Then any irreducible $J-$holomorphic curve satisfies the conditions in Lemmas* 3.1 *and* 3.2.

Proof. Any irreducible $J-$holomorphic curve can be smoothed out to be an embedded, connected $\omega-$symplectic surface. $\square$

3.1.2 - Theorem 1.7 for $\omega-$nef classes in $\mathcal{S}_\omega$

Now let us focus on the ω-nef classes and give a sufficient condition. Those classes will be useful in the discussion of Fiber/section type classes in 3.1.3.

Proposition 3.4. *Let ω be a reduced form on a rational surface $M = \mathbb{C}P^2 \# k\overline{\mathbb{C}P^2}$ with $k \geq 1$. The following classes in $\mathcal{S}_\omega$ are $\omega-$nef:*

- $H - E_1$;
- $A_l = lH - (l-1)E_1, l \geq 1$ *when* $\omega(A_{-l}) < 0$;
- $B_l = lH - (l-1)E_1 - E_2, l \geq 2$ *when* $\omega(B_{-l+1}) < 0$;
- $2H$ *when* $\omega(A_{-1}) < 0$.

Consequently, in these cases, $U_A = U_A^{eff} = \mathcal{J}_\omega$.

Proof. By Proposition 3.3, all irreducible curves for an ω-tame J are classified by Lemma 3.1 and Lemma 3.2. It is straightforward to check that all the possible classes pair non-negatively with $H - E_1$.

Now we establish the $\omega-$nef property of A_l when $\omega(A_{-l}) < 0$. A_l pairs positively on positive H coefficient curve since $b_1 \leq a-1$ for such a curve class. A_l is positive on zero H coefficient curve since $b_1 \geq 0$.

$$A_l \cdot A_{-p} = l(-p) + (l-1)(p+1) = l - p - 1 \geq 0$$

for $p \leq l-1$. If $\omega(A_{-l}) < 0$, then any negative H coefficient curve is of the form $A_{-p} - \sum E_u$ with $1 \leq p \leq l-1$.

The case of B_l is similar. B_l pairs positively on positive H coefficient curve since $b_1 \leq a-1$ for such a curve class. B_l is positive on zero H coefficient curve since $b_1 \geq 0$. And $B_l \cdot B_{-p} = l(-p) + (l-1)(p+1) - 1 = l - p - 2 \geq 0$ for $p \leq l-2$.

As for the case of $2H$ it is the same as $A_1 = H$.

The last statement follows from Lemma 1.6. $\square$

Except for the last statement, Theorem 1.7 is now a consequence of Proposition 3.4 and the facts that (i) any symplectic form on a rational surface is diffeomorphic to a reduced symplectic form and (ii) the equality $U_A = U_A^{eff} = \mathcal{J}_\omega$ is preserved under a diffeomorphism ϕ.

To establish the last statement of Theorem 1.7 that any tamed J on a rational surface with $\chi \geq 4$, i.e. other than $\mathbb{C}P^2$, is fibred, by [**25**], it suffices to show that there is always a $J-$nef fiber class for any tamed J. For $S^2 \times S^2$, the factor with a smaller area provides such a fiber class. For $\mathbb{C}P^2 \# k\overline{\mathbb{C}P^2}$ with $k \geq 1$, the desired fiber class if $H - E_1$ if J is tamed by a reduced symplectic form, or $\phi_*(H - E_1)$ if J is tamed by a symplectic form ω such that $\phi^*\omega$ is reduced.

We next turn to Theorem 1.8.

3.1.3 - Theorem 1.8 for fiber/section type classes

Let us start with the class $H - E_1$. The next proposition is one of the reasons that it is special from our perspective. We will further define fiber/section type classes according to the pairing with $H - E_1$.

Proposition 3.5. *Let ω be a reduced form on a rational surface $X = \mathbb{C}P^2 \# k\overline{\mathbb{C}P^2}$ with $k \geq 1$. $H - E_1$ has minimal $\omega-$area in $\mathcal{S}^0_\omega$. Furthermore, if $A \in \mathcal{S}^0_\omega$ and $\omega(A) = \omega(H - E_1)$, then there is symplectomorphism taking A to $H - E_1$.*

Proof. Suppose $A = aH - \sum_{i=1}^k b_i E_i$ is in $\mathcal{S}^0_\omega$. By the adjunction formula, $K_\omega \cdot A = -2$ and hence

$$3a = 2 + \sum_{i=1}^n b_i.$$

By Theorem 3.2, $b_1 + 1 \leq a$ and $b_2 + 1 \leq a$. In particular, $b_1 + b_2 \leq 2a - 2$.

The key observation is that we can express A as

$$A = U_1 + \cdots + U_{a-1} + V, \tag{5}$$

where each U_a is of the form $H - E_p - E_q - E_r$ with no repeated E_1 or E_2, and V is of the form $H - E_i$. Notice that $\omega(V) \geq \omega(H - E_1)$ by the reduced condition.

Now we prove that we are able to find a symplectomorphism sending A to $H - E_1$ if A has minimal area. Notice that for $a = 1$, A can only be $H - E_i$; for $a = 2$, A can only be $2H - \sum_{j=1,2,3,4} E_{i_j}$.

Let's assume $a \geq 2$ now, and show that when $A = aH - \sum b_i E_i$, at least three b_i's are positive. By adjunction, $3a = 2 + \sum b_i$. Assume by contradiction that only two b_i's are positive, say $3a = 2 + x + y$. Because A has non-trivial Gromov-Witten, it pairs non-negative with exceptional classes. Hence we have $x + y \leq a$, and hence $x + y + 2 \leq a + 2 < 3a$, contradiction.

Now if A is reduced, by [**18**] or [**34**, Theorem 1.1], there is a unique reduced representative in S^0_ω and $A = H - E_1$.

Now we assume A is not reduced. Note that we can assume $a \geq 2$, because any $H - E_i$ is symplectomorphic to $H - E_1$ if they have the same area.

Now we take the largest three b_i's, denote them by $b_{i_1}, b_{i_2}, b_{i_3}$. By the non-reduced assumpsion, $b_{i_1} + b_{i_2} + b_{i_3} > a$. Now in the decomposition (5), we can write U_{a-1} as $H - E_{i_1} - E_{i_2} - E_{i_3}$ and $V = H - E_j$. And we can still distribute other E_k's into U_i's with $i < a - 1$ and obtain a decomposition of type (5). Hence $H - E_{i_1} - E_{i_2} - E_{i_3}$ is Lagrangian.

Now by [**24**, Theorem 4.14], we are able to reflect along U_{a-1} to obtain W_{a-1}. Note that W_{a-1} is still in S^0_ω, and it has the same area as A.

By induction, we are able to repeat the above process if A is not reduced, and obtain W_1 which has $a = 1$, i.e. $H - E_1$. Notice W_1 has the same area as A. Then we are able to find a symplectomorphism reflecting along Lagrangian $E_1 - E_j$, sending W_1 to $H - E_1$. □

When the pairing of a class A with $H - E_1$ is 0 or 1, we have an affirmative answer to Question 1.3. Let us make the following definition:

Definition 3.6. *We call A an $(H-E_1)$-fiber type class if $A\cdot(H-E_1) = 0$, and A an $(H - E_1)$-section type class if $A \cdot (H - E_1) = 1$.*

Note that the next proposition is a special version of Theorem 1.8, after pulling an arbitrary symplectic form to a reduced form.

Proposition 3.7. *Let (M, ω) be a symplectic rational surface and $A \in \mathcal{S}_\omega$. Suppose ω is a reduced symplectic form and $A \cdot (H - E_1) = 0, 1$. If $A \in \mathcal{S}^{<0}$, then U^{eff}_A is the union of U_A and a collection of $U_{\mathcal{C}}$ with $Cod(\mathcal{C}) > Cod(A)$.*

For the proof, we further separate it into fiber type classes (Proposition 3.9) and section type classes (Proposition 3.10). By Theorem 3.4 we have

Lemma 3.8. *Let $A = aH - \sum b_iE_i$ be an $(H - E_1)-$fiber class represented by a connected, embedded symplectic surface. Then $A \in \mathcal{S}_\omega$ and A is of the form:*

- $(H - E_1) - \sum E_{i_j}, i_j > 1$
- $E_l - \sum E_{i_k}, i_k > l,\ l \geq 2.$

If $A = \sum_{i\geq 1} r_iC_i$, then each C_i is also an $(H - E_1)-$fiber type class. Moreover, there is a principle class that has the same leading term as A. If we call this class C_1 then $r_1 = 1$.

Consequently, U^{eff}_A is a union of U_A and admissible $U_{\mathcal{C}}$.

Proof. For the classification statement, note that $a = b_1$ since $A \cdot (H - E_1) = 0$. Thus $a \geq 0$ by Lemma 3.1. Further, $a = 0$ or 1 by Lemma 3.2 (2).

The second statement follows from the fact that $H - E_1$ is J-nef and $0 = (H - E_1) \cdot A = (H - E_1) \cdot (r_1 C_1 + \sum_{i \geq 2} r_i C_i) \geq 0$. Hence all C_i's are $(H - E_1)$-fibered. For the principal component, we discuss two cases:

- If $A = (H - E_1) - \sum E_{i_j}, i_j > 1$: First note that there are no C_i with negative H coefficient, because all curves in Lemma 3.1 are of $H - E_1$-section type. It immediately follows that $C_i \cdot H = 0, 1$, and there is exactly one such C_i pairing H being 1. We will call this one C_1.

- If $A = E_l - \sum E_{i_k}, i_k > l$, $l \geq 2$: By clasification in Lemma 3.2, all C_i has to be $E_i - \sum E_{i_l}, i_l > i, i \geq l$. Notice that there has to be exactly one class with $E_l - \sum E_{i_l}$, because there is a leading E_l in A. The multiplicity of such class has to be 1 since there is no class with $-E_l$.

For the last statement, it suffices to prove for any admissible $\mathcal{C}$ we take to cover U_A^{eff}, $U_{\mathcal{C}} \subset U_A^{eff}$. Take a $J \in U_A^{eff}$ with a decomposition $A = \sum r_i C_i = \sum r_\alpha C_\alpha + \sum r_\gamma C_\gamma$ with $C_\alpha^2 < -1$ and $C_\gamma^2 \geq -1$.

If A is $H - E_1$, we are done because $U_A = U_A^{eff} = \mathcal{J}_\omega$ by Proposition 3.4. Assume A is not $H - E_1$, then we have $A^2 \leq -1$. Note that there must be at least one C_α class. The reason is the following: assuming C_i's all have square at least -1, then $C_i \cdot K < 0$ for each C_i and this is a contradiction because $-1 \leq A \cdot K = K \cdot (\sum r_i C_i) \leq -2$ because there are at least two terms on the right hand side. By the above disscusion, $C_\alpha \in S_\omega^{<-1}$ and $C_\gamma \in S_\omega^{\geq -1}$. Hence C_γ all have non-trivial Gromov-Witten invariant by Lemma 2.4, and one can find J-representatives in each class C_γ for any J. Now we take $\mathcal{C} := \{C_\alpha\}$ as the admissible, and we have $U_{\mathcal{C}} \subset U_A^{eff}$.

Running the above argument for every $J \in U_A^{eff}$, we conclude that U_A^{eff} is a union of U_A and admissible $U_{\mathcal{C}}$. □

Now we present the (complex) codimension estimate for $(H - E_1)$-fiber type classes:

Proposition 3.9. *Let $M = \mathbb{C}P^2 \# k\overline{\mathbb{C}P^2}$ with a reduced symplectic form. Let A be an $(H - E_1)-$fiber type class and $A \neq H - E_1$. If $U_{\mathcal{C}} \subset U_A^{eff}$ and $\mathcal{C} \neq \{A\}$ then $Cod(\mathcal{C}) > Cod(A)$.*

Proof. Take a set $\mathcal{C} = \{C_\alpha\}$ such that $U_{\mathcal{C}} \subset U_A^{eff}$, then we have $A = \sum r_\alpha C_\alpha + \sum r_\gamma C_\gamma$. Then we know each C_i is also an $(H - E_1)-$fiber type class. By the preceding lemma, $A = H - E_1 - \sum_{u \in U} E_u$ or $A = E_p - \sum_{u \in U} E_u$ with

$p \geq 2$. Moreover, the preceding lemma also tells us that there is a principal class C_1 with $r_1 = 1$ (see the two bullets in the proof).

Assume first $A = E_p - \sum_{u \in U} E_u$ for some $p \geq 2$ and a collection of integers $u > p$. Then the principal class

$$C_1 = E_p - \sum_{v \in V} E_v$$

for a collection of integers $v > p$.

Then

$$\sum_{i \geq 2} r_i C_i = E_{m_1} + \cdots + E_{m_M} - E_{n_1} - \cdots - E_{n_N}, \tag{6}$$

where $\{E_{m_i}\} = \{E_v\} - \{E_v\} \cap \{E_u\}, \{E_{n_i}\} = \{E_u\} - \{E_u\} \cap \{E_v\}$, and $0 \leq M < N$. Note that $N - M = |V| - |U|$.

Since $Cod(A) = -A^2 - 1 = 2 + |U|$ and $Cod(C_1) = 2 + |V|$, we have

$$Cod(C_1) - Cod(A) = |V| - |U|.$$

So the assumption $Cod(C_1) \leq Cod(A)$ means that

$$N - M = |V| - |U| \geq 0.$$

Each E_{m_i} term could appear alone so could contribute 0 to $\sum_{i \geq 2} Cod(C_i)$. On the other hand, each $-E_{n_j}$ appears in a class of the form $E_q - E_{n_j} - \sum_s E_s$, so it at least contributes to $\sum_{i \geq 2} Cod(C_i)$ by 1. Since $C_i \in \mathcal{S}_\omega$ for all i we have $Cod(\mathcal{C}) = \sum Cod(C_i)$. Now $Cod(\mathcal{C}) - Cod(A)$ is given by

$$Cod(C_1) - Cod(A) + \sum_{i \geq 2} Cod(C_i) \geq -(N - M) + N = M$$

Note that equality occurs only if $0 = M$. This implies that $\sum_{i \geq 2} r_i C_i = -E_{n_1} - \cdots - E_{n_N}$ and hence the zero class due to positivity of area. Of course, this just means that $A = C_1$.

The case $A = H - E_1 - \sum E_u$ is similar. If we write $C_1 = H - E_1 - \sum E_v$ we still get the expression of $\sum_{i \geq 2} r_i C_i$ as in (6). $\square$

For section type classes, we have a similar but slightly more complicated proof:

Proposition 3.10. *Let $M = \mathbb{C}P^2 \# k\overline{\mathbb{C}P^2}$ with a reduced symplectic form. Assume $A \in \mathcal{S}_\omega$ is an $(H - E_1)-$section class, i.e. $A = -lH + (l+1)E_1 - \sum E_u$ for some integer l. If $A = \sum_{i \geq 1} r_i C_i$, then there is exactly one $(H - E_1)-$section* *type class C_1 with $r_1 = 1$, and all other C_i's are $(H - E_1)-$fiber type class. Furthermore, we also have $C_i \in \mathcal{S}_\omega$.*

Consequently, if $A^2 \leq -1$, U_A^{eff} is a union of U_A and admissible $U_{\mathcal{C}}$ with $Cod(\mathcal{C}) > Cod(A)$.

Proof. The proof of the first statement is similar to the fiber type class situation. Write

$$C_1 = (-l - d)H + (l + 1 + d)E_1 - \sum k_v E_v$$

with $d \geq 0$ and $k_v \neq 0$. By Theorem 2.2, $k_v = 1$ and $C_1 \in \mathcal{S}_\omega$. Again C_1 cannot be multiply covered by Lemma 3.2 statement 1). Other C_i's belongs to $\mathcal{S}_\omega$ follows fom Lemma 3.8. This completes the first statement.

For the statement U_A^{eff} is a union of U_A and admissible $U_{\mathcal{C}}$, it is the same as the proof of the last statement in Lemma 3.8: We are able to find a decomposition $A = \sum r_\alpha C_\alpha + \sum r_\gamma C_\gamma$. We take $\mathcal{C} = \{C_\alpha\} \neq$ and C_γ all have J-representative. Hence $U_C \subset U_A^{eff}$.

Now we finish the codimension comparison: Denote the cardinality of $\{E_u\}$ by U and the cardinality of $\{E_v\}$ by V. The statement is to say when

$$\sum_{i\geq 2} r_i C_i = dH - dE_1 + E_{m_1} + \cdots + E_{m_M} - E_{n_1} - \cdots - E_{n_N},$$

where $\{E_{m_i}\} = \{E_v\} - \{E_v\} \cap \{E_u\}, \{E_{n_i}\} = \{E_u\} - \{E_u\} \cap \{E_v\}$, and $0 \leq M < N$. Note that $N - M = U - V$. Since $Cod(A) = -A^2 - 1 = 2l - U$ and $Cod(C_1) = 2(l + d) - V$, we have

$$Cod(C_1) - Cod(A) = 2d - (U - V).$$

So the assumption $Cod(C_1) \leq Cod(A)$ means that

$$N - M = U - V \geq 2d.$$

Each $C_i, i \geq 2$, is in classes $H - E_1 - \sum k_l E_l, k_l \in \{0, 1\}$ or $E_p - \sum E_q$.

The strategy is still to inspect each $-E_{n_j}, n_j \geq 2$. Such a term at least appears in some component of $E_p - \sum E_q$ or $H - E_1 - \sum k_l E_l, k_l \in \{0, 1\}$. This means each $-E_{n_j}$ at least contributes to $\sum_{i\geq 2} Cod(C_i)$ by 1 except that it appears in the -1 curve class $H - E_1 - E_{n_j}$. Let Z be the number of such n_j. Then

$$0 \leq Z \leq d$$

and

$$\sum_{i\geq 2} Cod(C_i) \geq N - Z.$$

Since $C_i \in \mathcal{S}_\omega$ for all i we have $Cod(\mathcal{C}) = \sum Cod(C_i)$. Now $Cod(\mathcal{C}) - Cod(A)$ is given by

$$Cod(C_1) - Cod(A) + \sum_{i\geq 2} Cod(C_i) \geq 2d - (N-M) + (N-Z) = 2d + M - Z \geq d + M.$$

Note that equality occurs only if $0 = d = M = d - Z$. This implies that $\sum_{i\geq 2} r_i C_i = -E_{n_1} - \cdots - E_{n_N}$ and hence the zero class due to positivity of area. Of course, this just means that $A = C_1$, which is not allowed by our assumption. □

Proposition 3.7 follows from Lemma 3.8, Proposition 3.9 and Proposition 3.10. Theorem 1.8 follows from Proposition 3.5 and Proposition 3.7.

Notice that the conclusion of Theorem 1.8 does not hold for an arbitrary class on an arbitrary rational surface. Let us recall the example in the introduction, where $(M,\omega) = (\mathbb{C}P^2 \# 10\overline{\mathbb{C}P^2}, \omega)$ and $A = -2K_\omega$. Here U_A has codimension 3 but $U_A^{eff} - U_A$ contains a set of $K-$effective almost complex structures which has codimension 1 and not in any $U_{\mathcal{C}}$.

We next prove Theorem 1.9 for a symplectic rational surface with $\chi \leq 11$.

3.1.4 - When $\chi \leq 11$

Proposition 3.11. *For $M = \mathbb{C}P^2 \# k\overline{\mathbb{C}P^2}, k \leq 8$ with a reduced symplectic form and $A \in S_\omega^{<0}$, $Cod(\mathcal{C}) > Cod(A)$ if $\mathcal{C}$ is an admissible set as defined in Lemma 2.8.*

Proof. When $A \cdot H \leq 0$, A is a section or fiber type class by Propositions 3.10, 3.9. If $A^2 = -2$, we can apply Lemma 2.8.

So we only need to consider the case $A \cdot H > 0$ and $A^2 \leq -3$. From [**32**, Prop. 4.6], the only such class is $A = 2H - \sum_{i=2}^{8} E_i$, which is a 2−section class $A^2 = -3$. By Lemma 2.8, we have $2 = Cod(A) \leq Cod(\mathcal{C})$ and $=$ holds only if $\mathcal{C}$ is a single element subset of $\mathcal{S}_\omega^{-3}$. We will call this element C_0, and it has multiplicity at least 2.

Assume that C_0 is a section type class. Then $r_0 = 2$ and other curves are fiber type with square at least -1. There are only 3 cases for C_0: $H - E_2 - E_3 - E_4 - E_5$, $E_1 - E_2 - E_3$, $-H + 2E_1$. We then deal with each case as below:

- If $C_0 = H - E_2 - E_3 - E_4 - E_5$: the fiber terms have no H or E_1 terms, so they are of the form $E_i, i \geq 2$. Also, there are no curves contain $-E_6, -E_7, -E_8$ terms.

- If $C_0 = E_1 - E_2 - E_3 - E_4$: there are at most two fiber terms containing $-E_i$, they are of the form $H - E_1 - E_i$.
- If $C_0 = -H + 2E_1$: there are at most four fiber terms containing $-E_i$, of the form $H - E_1 - E_i$.

It is straightforward to check that those cases all have $Cod(\mathcal{C}) > Cod(A)$.

Assume that C_0 is a fiber type class. Then $C_0 = E_2 - E_3 - E_4$ and it contains two $-E_i$ terms. We will consider the distribution of $-E_i$ terms among C_i's. The possible (unique) section class can be H, $H - E_5$, $H - E_5 - E_6$, $2H - E_1 - E_5 - E_6 - E_7 - E_8$. The fiber terms can be $E_i, i \geq 2$, $H - E_1$, $H - E_1 - E_i$. Notice that there are no negative H terms. If there is a unique $2H$ term, then it contains at most four $-E_i$ terms so at least one $-E_i$ term is not accounted. If there are two H terms, then they contain together at most four $-E_i$ terms so at least one $-E_i$ term is not accounted for. Hence this case is not possible, and we finished our proof. $\square$

Note that this Proposition generalizes a result in [**19**, Theorem 3.8], where A is only allowed to be a -2 sphere, and the manifold is chosen to be $\mathbb{C}P^2\#k\overline{\mathbb{C}P^2}$,$k \leq 5$. Theorem 1.9 follows from Proposition 3.11.

3.2 - *The sets X_i*

For the sets X_i, we can prove the following result when the Euler number is very small:

Proposition 3.12. *For $\mathbb{C}P^2\#k\overline{\mathbb{C}P^2}$, $k \leq 4$, $X_i^{eff} = X_i$ for each i.*

The proof involves a detailed case-by-case analysis which we omit here. It might be possible to extend this to $k = 5, 6$ using the classification of configurations in [**32**, Prop. 4.6].

Notice that in Proposition 3.12, it is necessary to assume the number of blowup points (or equivalently the Euler number) is small. We have the following example for 9 times blowup of $\mathbb{C}P^2$. Note $\mathbb{C}P^2\#9\overline{\mathbb{C}P^2}$ is diffeomorphic to $S^2 \times S^2\#8\overline{\mathbb{C}P^2}$, with basis $B, F, E_1 \cdots E_8$, where B, F are the base classes in $S^2 \times S^2$.

On $S^2 \times S^2\#8\overline{\mathbb{C}P^2}$, let $\mathcal{C} = \{B - F, B + F - E_1 - E_2 - E_3 - E_4,$

$$B + F - E_1 - E_2 - E_5 - E_6, B + F - E_1 - E_2 - E_7 - E_8\},$$

and

$$\mathcal{C}' = \{B - F - E_1 - E_2\}.$$

Meanwhile,

$$cod(\mathcal{C}) = 4 \times (2-1) = 4 > cod(\mathcal{C}') = 3.$$

Notice that we have

$$B+F-E_1-E_2-E_3-E_4 = B-F-E_1-E_2+(F-E_3)+(F-E_4),$$
$$B+F-E_1-E_2-E_5-E_6 = B-F-E_1-E_2+(F-E_5)+(F-E_6),$$
$$B+F-E_1-E_2-E_7-E_8 = B-F-E_1-E_2+(F-E_7)+(F-E_8),$$
$$B-F = (B-F-E_1-E_2)+E_1+E_2.$$

It is not hard to show that $U_{\mathcal{C}}^{eff} \cap U_{\mathcal{C}'}$ contains an open subset of $U_{\mathcal{C}'}$. This implies that $X_8^{eff} \neq X_8$.

Also, note that for any $k \geq 8$ blowup of $S^2 \times S^2$, this counterexample is always there, as long as one endows the manifold with a symplectic form obtained by blowing up points on a pencil of curves in class $B+F$ in $S^2 \times S^2$ such that those classes have positive symplectic area.

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