

NOBUHIRO HONDA and AYATO MINAGAWA

**On the cuspidal locus in the dual varieties
of Segre quartic surfaces**

Abstract. Motivated by a kind of Penrose correspondence, we investigate the space of hyperplane sections of Segre quartic surfaces which have an ordinary cusp. We show that the space of such hyperplane sections is empty for two kinds of Segre surfaces, and it is a connected surface for all other kinds of Segre surfaces. We also show that when it is non-empty, the closure of the space is either birational to the surface itself or birational to a double covering of the surface, whose branch divisor consists of some specific lines on the surface.

Keywords. Segre surfaces, cuspidal curves, projectively dual varieties, minitwistor spaces.

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1 - Introduction

The so-called Penrose correspondence gives a relationship between conformal differential geometry and complex geometry, and typically it means that the parameter space of smooth rational curves on a complex manifold (i.e. a twistor space) is equipped with a special geometric structure, if the normal bundles of the rational curves are of a particular form. The most famous one would be the case where the rational curve is in a complex 3-manifold with the normal bundle being isomorphic to $\mathcal{O}(1)^{\oplus 2}$, and under this situation the space of rational curves is a 4-dimensional manifold, equipped with a self-dual conformal structure [1, 8]. Another interesting instance is established by Hitchin [4], in which case the rational curve is in a complex surface with normal bundle $\mathcal{O}(2)$. The space of rational curves is a 3-dimensional manifold equipped with an Einstein-Weyl structure, which may be regarded as a conformal version of Einstein metrics. The complex surface in which the rational curves move is called the *minitwistor space*, and the rational curves are called *minitwistor lines*.

In the article [6], we showed that the same result about the presence of Einstein-Weyl structure still holds even when the rational curves on a complex surface have ordinary nodes, as long as the complete family of such nodal rational curves on the surface is 3-dimensional. These nodal rational curves are still called minitwistor lines, and the number of nodes is called the *genus* of a minitwistor space. In [7], we showed that, under a natural minimality condition, compact minitwistor spaces with genus one are exactly a special kind of algebraic surfaces, classically called *Segre surfaces*. In modern language, these surfaces are nothing but the anti-canonical models of weak del-Pezzo surfaces

of degree four, and all of them are complete intersections of two quadrics in $\mathbb{C}\mathbb{P}_4$. The nodal rational curves in a Segre surface are sections by hyperplanes which are tangent to the surface at one smooth point. This means that the space of minitwistor lines, namely the Einstein-Weyl space associated to the minitwistor space, is a Zariski-open subset of the projectively dual variety of the Segre surface. Here, an interesting point is that the Zariski-open subset can never be the entire dual variety, because nodal curves can always be deformed in the surface, into a reducible curve or a curve with an ordinary cusp. This means that, the Einstein-Weyl space is always non-compact and admits a natural compactification as a projective variety. As in [7], we call 2-dimensional components of the added locus in the compactification as *divisors at infinity* of the Einstein-Weyl space. Note that degeneration into a cuspidal curve can happen only when the genus of a minitwistor space is positive.

In the article [7], for any Segre surface, we studied divisors at infinity which parameterize reducible hyperplane sections of the surface. In particular we determined all components of such divisors, and showed that they are isomorphic to either a projective plane or a smooth quadratic surface. Also, we showed that the dual variety of a Segre surface intersects itself along these components in such a way that the dual variety has ordinary nodes along generic points of these components.

A purpose of the present article is to investigate the space of hyperplane sections of any Segre surface, which have an ordinary cusp at a smooth point of the surface. The closure of this space constitutes a subvariety in the dual variety of the Segre surface, and we call it the *cuspidal locus* in the dual variety of the surface. The details on the cuspidal locus will be discussed in Section 2.1 for general surfaces embedded in projective spaces. In Section 2.2, by using deformation theory of curves with singularities, we show that *the cuspidal locus in the dual variety of any Segre surface is reduced and 2-dimensional* if it is non-empty (Theorem 2.2), and that *the dual variety has ordinary cusps along the cuspidal locus* (Theorem 2.3). Thus, the cuspidal locus always constitutes a divisor at infinity of the Einstein-Weyl space.

In Section 3.1, we prove some basic properties on Segre surfaces that will be needed in the rest of this article. In Section 3.2, we determine the structure of certain divisor in the incidence variety of the Segre surface, which is closely related to the cuspidal locus. In Section 3.3, we identify hyperplanes which cut out cuspidal curves from Segre surfaces, by using the incidence variety. Consequently, we find that the cuspidal locus can have three kinds of structures, depending on the number of such pencils on the Segre surfaces. Subsequently in Section 3.4, we express the number of pencils of double conics in terms of Segre symbols for Segre surfaces. As a result, it turns out that *the cuspidal*

locus for two kinds of Segre surfaces are empty (Corollary 3.19). Also we see that among sixteen kinds of Segre surfaces, *the cuspidal locus for seven kinds of Segre surfaces are naturally birational to the surfaces themselves* (Corollary 3.20).

For the remaining seven kinds of Segre surfaces, the cuspidal locus turns out to be birational to a double covering over the surfaces whose branch divisor consist of some lines on the surfaces. In Section 4.1, we show that every line on a Segre surface which does not pass any singularities of the surface is a simple branch divisor of the double covering (Proposition 4.3). In Sections 4.2 and 4.3, we obtain similar results for any other lines on Segre surfaces.

In Section 5.1, by using the results in Section 4.1, we show that *the cuspidal locus in the dual variety of any Segre surface is an irreducible surface if it is not empty* (Corollary 5.2). This is the main result in this article. Next we determine which lines on Segre surfaces are really branch divisors of the above structure of double covering (Proposition 5.3). By using these results, we construct a surface which is birational to the cuspidal locus in the dual variety of a smooth Segre surface, and find that the cuspidal locus is a surface of general type (Proposition 5.4). In Section 5.2, we investigate singularities of the cuspidal locus in the dual varieties of Segre surfaces, which arise from the above double covering structure, and show that *the cuspidal locus always has ordinary cusps along some conic in the dual variety* (Proposition 5.9).

In the Appendix, we investigate structure of the cuspidal locus for some Segre surfaces, by using explicit equations of the surfaces. As a result, we can precisely identify some divisor in the incidence variety which is studied in Section 3.2. Also, we identify all lines on Segre surfaces which are not branch divisor of the above double covering. This completes identification of the branch divisor of the above double covering which was postponed in Section 5.1.

In general, the singular locus of the dual variety of a projective variety is known to be highly singular, but they are far from being well-understood. See [11, Section 10] for information about this topic. Because Segre surfaces are varieties of low degree and are of small codimension, they seem to be a nice class of varieties to investigate the singular locus of the dual variety in detail. Indeed, the fact that Segre surfaces are of degree four is essential in our investigation in many places. Combined with the results in [7], the present results would give a fairly complete understanding of the singular locus of the dual varieties of Segre surfaces.

2 - Cuspidal locus

2.1 - Some generalities on the cuspidal locus in the dual variety

First we briefly recall basic facts on projectively dual varieties. For more details, see Tevelev's book [11] for example. Let $X \subset \mathbb{C}\mathbb{P}_n$ be a non-degenerate irreducible projective variety, and write X_{reg} for the locus of smooth points of X . We say that a hyperplane $H \subset \mathbb{C}\mathbb{P}_n$ is tangent to X if H includes the tangent space $T_p X$ at some point $p \in X_{\text{reg}}$. This condition implies that the hyperplane section $H|_X$ has a singularity at the point p . The dual variety of X , which is denoted by X^* , is defined as the closure of the set of hyperplanes in $\mathbb{C}\mathbb{P}_n$ which are tangent to X , where the closure is taken in the dual projective space $\mathbb{C}\mathbb{P}_n^*$. The dual variety X^* is a subvariety in $\mathbb{C}\mathbb{P}_n^*$ and it is a hypersurface if the variety X is not ruled, namely when there is some point of X which is not passed by a line in X [11, Theorem 1.18]. If X is not ruled, for a generic hyperplane $H \in X^*$, the section $H|_X$ has exactly one ordinary node as its all singularity [12, Section 2.1.1].

It is also useful to introduce the incidence variety $I(X) \subset \mathbb{C}\mathbb{P}_n \times \mathbb{C}\mathbb{P}_n^*$. This is by definition the closure of the locus in $\mathbb{C}\mathbb{P}_n \times \mathbb{C}\mathbb{P}_n^*$ formed by a pair (p, H) of a point $p \in X_{\text{reg}}$ and a hyperplane H containing $T_p X$. This is a subvariety in $\mathbb{C}\mathbb{P}_n \times \mathbb{C}\mathbb{P}_n^*$. Then the dual variety is nothing but the image of $I(X)$ under the projection to $\mathbb{C}\mathbb{P}_n^*$, and there is a diagram

$$(2.1) \quad \begin{array}{ccc} & I(X) & \\ \pi_1 \swarrow & & \searrow \pi_2 \\ X & & X^* \end{array}$$

where π_1 and π_2 are restrictions to $I(X)$ of the projections from $\mathbb{C}\mathbb{P}_n \times \mathbb{C}\mathbb{P}_n^*$ to the two factors respectively. Over the smooth locus X_{reg} , the projection π_1 is a projective space bundle. The other projection π_2 is birational as long as $\dim X^* = n - 1$.

Next, in order to define the cuspidal locus for a non-degenerate irreducible projective variety $X \subset \mathbb{C}\mathbb{P}_n$, we consider the following locally closed subset of $I(X)$:

$$\{(p, H) \in I(X) \mid p \in X_{\text{reg}}, \\ \text{and the singularity of } H|_X \text{ at } p \text{ is not an ordinary node}\}.$$

Taking the closure in $I(X)$ of this subset, we obtain a subvariety in $I(X)$. In [11, §10.2.4], the image of this subvariety under the projection π_2 to X^* is

denoted by X_{cusp}^* when X is smooth (in that case one does not need to take the closure), and for the moment we use this notation. From the definition, for any hyperplane $H \in X_{\text{cusp}}^*$, the section $H|_X$ has a non-nodal singularity at a smooth point of X , or perhaps a singularity at a singular point of X . The variety X_{cusp}^* is not necessarily irreducible. Moreover, even when $\dim X = 2$, a generic point of an irreducible component of X_{cusp}^* can correspond to a hyperplane section whose singularity is not an ordinary cusp. So in this article, when $\dim X = 2$, we call the *cuspidal locus* in the dual variety X^* the union of all components of the subvariety X_{cusp}^* whose generic point corresponds to a hyperplane section which has an ordinary cusp at a smooth point of X . So under this definition, the cuspidal locus really parameterizes hyperplane sections which have an ordinary cusp.

Next, we give a description of the cuspidal locus in terms of local coordinates when X is a surface in \mathbb{CP}_4 . It will be used to investigate the structure of the cuspidal locus of Segre surfaces. We remark that it is not difficult to generalize the following description to the case where no constraint is supposed on dimension and codimension of the subvariety X .

Let $S \subset \mathbb{CP}_4$ be a non-degenerate irreducible projective surface, and pick any $p \in S_{\text{reg}}$. By choosing suitable non-homogeneous coordinates (x, y, z, w) , we may suppose that, in a neighborhood of the point p , the pair (x, y) works as holomorphic coordinates on S in a neighborhood of p , and that there exist two holomorphic functions

$$F = F(x, y) \quad \text{and} \quad G = G(x, y),$$

defined in a neighborhood of p , such that, around p , the surface S is defined by the equations

$$z = F(x, y), \quad w = G(x, y).$$

Under these choices, for a point $q = (a, b) \in S_{\text{reg}}$ in the neighborhood, any hyperplane which contains the tangent plane $T_q S$ is of the form

$$(2.2) \quad \lambda \{z - F(q) - F_x(q)(x - a) - F_y(q)(y - b)\} \\ + \mu \{w - G(q) - G_x(q)(x - a) - G_y(q)(y - b)\} = 0$$

for some $(\lambda, \mu) \in \mathbb{C}^2 \setminus \{(0, 0)\}$. The pair (λ, μ) can be regarded as homogeneous coordinates on the fiber of the projection $\pi_1 : I(S) \rightarrow S$, which is a projective line \mathbb{CP}_1 . Substituting the equations $z = F(x, y)$ and $w = G(x, y)$ into (2.2), we obtain a defining function of the intersection of the hyperplane (2.2) with S . This intersection has a non-nodal singularity at the point q iff the Hessian (i.e. the determinant of the Hessian matrix) of the defining function vanishes at

q . It is immediate to see from (2.2) that this Hessian is equal to the Hessian

$$\mathbf{H} := \text{Hess}(\lambda F + \mu G).$$

Thus, to each $(\lambda, \mu) \in \mathbb{C}^2 \setminus \{(0, 0)\}$, the section of S by the hyperplane (2.2) has a non-nodal singularity at $q \in S_{\text{reg}}$ if and only if $\mathbf{H}(q) = 0$.

Since the Hessian matrix is of size 2×2 , as a homogeneous equation in (λ, μ) , the function \mathbf{H} is quadratic in λ and μ . More explicitly, we easily have

$$(2.3) \quad \mathbf{H} = \text{Hess}(F)\lambda^2 + (F_{xx}G_{yy} + G_{xx}F_{yy} - 2F_{xy}G_{xy})\lambda\mu + \text{Hess}(G)\mu^2.$$

The condition $\mathbf{H} = 0$ is independent of a choice of the coordinates (x, y, z, w) . Therefore, the equation $\mathbf{H} = 0$ defines a locally closed subset in the incidence variety $I(S)$. Throughout this paper, we denote the closure of it by the bold letter \mathbf{D} . This is a divisor in $I(S)$. If the equations

$$(2.4) \quad \text{Hess}(F) = \text{Hess}(G) = F_{xx}G_{yy} + G_{xx}F_{yy} - 2F_{xy}G_{xy} = 0$$

hold at some point $q \in S_{\text{reg}}$, then whole the fiber $\pi_1^{-1}(q) = \mathbb{CP}_1$ is included in the divisor \mathbf{D} . So, if there exists a curve on S along which (2.4) holds, then the fibers over the curve constitute a component of the divisor \mathbf{D} . This component can have a multiplicity greater than one in general. Throughout this article, we denote \mathbf{D}_1 for the sum of all these components. These components are mapped to curves by the projection $\pi_1 : I(S) \rightarrow S$. Of course, \mathbf{D}_1 might be the zero divisor. We write $\mathbf{D}_S := \mathbf{D} - \mathbf{D}_1$, so that

$$\mathbf{D} = \mathbf{D}_1 + \mathbf{D}_S$$

holds. We always have $\mathbf{D}_S \neq 0$, and since the equation (2.3) is quadratic in λ and μ , the restriction of the projection $\pi_1 : I(S) \rightarrow S$ to \mathbf{D}_S is of degree two. But this restriction is still not necessarily a finite morphism, because there might exist an isolated point $q \in S_{\text{reg}}$ such that the degeneracy condition (2.4) holds, and also because \mathbf{D}_S might include a curve in a fiber over some singular point of S . Thus, what we can say in general is that the projection $\pi_1|_{\mathbf{D}_S} : \mathbf{D}_S \rightarrow S$ is a generically finite double covering over S . The divisor \mathbf{D}_S can be reducible, and in that case, each of the two components is birational to S by the projection π_1 . Regardless of whether \mathbf{D}_S is irreducible or not, we think \mathbf{D}_S as a double covering over S . If $\mathbf{D}_1 = 0$, then the branch divisor of $\mathbf{D}_S \rightarrow S$ is the zero divisor of the discriminant of the quadratic polynomial (2.3). If $\mathbf{D}_1 \neq 0$, in order to obtain the equation of the branch divisor of $\mathbf{D}_S \rightarrow S$, we have to divide the quadratic polynomial (2.3) by a defining equation of \mathbf{D}_1 , before taking the discriminant.

Mapping the divisor $\mathbf{D} = \mathbf{D}_1 + \mathbf{D}_S$ in $I(S)$ to the dual variety S^* by the projection $\pi_2 : I(S) \rightarrow S^*$, we obtain a subvariety in S^* . This is nothing but S_{cusp}^* in the notation of [11] we have mentioned above. This subvariety can be reducible in general, and the cuspidal locus in S^* under our current definition is some component of this subvariety when it is non-empty. A component of the image $\pi_2(\mathbf{D})$ is not necessarily a component of the cuspidal locus in S^* because a generic point of the component can correspond to a non-cuspidal hyperplane section.

We end this subsection by showing that a straight-line on a surface S in $\mathbb{C}\mathbb{P}_4$ is always a candidate of a branch divisor of the generically finite double covering $\mathbf{D}_S \rightarrow S$:

Proposition 2.1. *If a non-degenerate irreducible surface $S \subset \mathbb{C}\mathbb{P}_4$ contains a straight-line l which is not contained in the singular locus of S , then the discriminant of the quadratic polynomial (2.3) vanishes along l .*

Proof. The discriminant of the polynomial \mathbf{H} in (2.3) can be easily calculated, and it can be written as

$$(2.5) \quad (F_{xx}G_{yy} - G_{xx}F_{yy})^2 + 4(F_{xx}G_{xy} - F_{xy}G_{xx})(F_{yy}G_{xy} - F_{xy}G_{yy}).$$

If l is a line as in the proposition and $p \in l \cap S_{\text{reg}}$, then we can always take the non-homogeneous linear coordinates (x, y, z, w) in such a way that $T_p S = \{z = w = 0\}$ and $l = \{y = z = w = 0\}$ hold. These imply $F|_l = G|_l = 0$, which mean

$$\frac{\partial^k F}{\partial x^k}(x, 0, 0, 0) = \frac{\partial^k G}{\partial x^k}(x, 0, 0, 0) = 0 \quad \text{for any } k \geq 0.$$

From these, the discriminant (2.5) vanishes along the line l . □

From the above relationship between the branch divisor of the generically finite double covering $\mathbf{D}_S \rightarrow S$ and the discriminant of the quadratic polynomial (2.3), a line $l \subset S$ is a branch divisor of $\mathbf{D}_S \rightarrow S$, provided that some of the three coefficient functions of (2.3) do not vanish identically on the line l . If all the three functions vanish along the line l , and $m (> 0)$ is the minimal vanishing order of these functions along l , then $\pi_1^{-1}(l)$ is included in the divisor \mathbf{D}_1 with multiplicity precisely m . Then the line l is really a branch divisor of $\mathbf{D}_S \rightarrow S$ if and only if the discriminant of the quadratic polynomial \mathbf{H}/y^m vanishes along l in the coordinates of the previous proof. In Sections 4 and 6, we will use these to determine whether lines on Segre surfaces are really branch divisors of the covering $\mathbf{D}_S \rightarrow S$.

2.2 - Cuspidal locus in the dual variety of a Segre surface

A non-degenerate irreducible 2-dimensional subvariety $S \subset \mathbb{CP}_4$ is called a *Segre quartic surface* or simply a *Segre surface* if it is of degree four in \mathbb{CP}_4 , and is not a cone over a quartic curve in \mathbb{CP}_3 nor a projection of a quartic surface in \mathbb{CP}_5 . All of them are complete intersections of two quadrics. The normal forms of a pair of quadratic equations on \mathbb{CP}_4 are known, and by using them, Segre surfaces can be classified into 16 types. Any Segre surface has at most a finite number of singularities, and all of them are rational double points. Any Segre surface has a finite number of straight-lines on it. For more details on Segre surfaces, see [5, Chapter XIII, Section 10] and [3, Section 8.6]. For a brief account, see [7].

In this subsection, we prove two basic properties on the cuspidal locus in the dual varieties of Segre surfaces. The first one concerns smoothness and dimension:

Theorem 2.2. *Let $S \subset \mathbb{CP}_4$ be any Segre surface, and H a hyperplane not passing any singularity of S . If H belongs to the cuspidal locus in S^* and $H|_S$ indeed has a single ordinary cusp as its all singularity, then $H|_S$ is a rational curve and the cuspidal locus in S^* is smooth and 2-dimensional at the point $H \in S^*$.*

Note that the existence of a hyperplane section as in the theorem is not obvious at all. Indeed, we will see in Section 3.4 that some Segre surfaces do not have such a hyperplane H .

The second property is about singularity the dual variety has along the cuspidal locus. By Theorem 2.2, the cuspidal locus itself is smooth at points formed by hyperplane sections $H|_S$ as in the theorem. (We note that this does not mean that S^* is smooth along this locus.) The next result is about singularities of S^* along this smooth locus of the cuspidal locus.

Theorem 2.3. *The dual variety S^* of any Segre surface S has ordinary cusps along smooth locus in the cuspidal locus formed by hyperplanes appearing in Theorem 2.2.*

This is similar to the property that the dual variety S^* of a Segre surface S has ordinary nodes along the locus whose points correspond to hyperplane sections of S which have two ordinary nodes [7, Proposition 3.15].

Proof of Theorem 2.2. Let H be as in the theorem and put $C = H|_S$. By Lefschetz theorem, the curve C is connected. Let $p \in C$ be the ordinary cusp of C . By assumption, the curve C is smooth except p , and hence C

is irreducible. Since the arithmetic genus of C is easily seen to be one, the presence of a single cusp means that C is a rational curve.

We denote Ω_S and Ω_C for the sheaves of Kähler differentials on S and C respectively, and \mathcal{I}_C for the ideal sheaf of C in \mathcal{O}_S . Associated to the embedding $C \subset S$, there is a standard exact sequence

$$(2.6) \quad 0 \longrightarrow \mathcal{I}_C/\mathcal{I}_C^2 \longrightarrow \Omega_S|_C \longrightarrow \Omega_C \longrightarrow 0.$$

By applying $\mathcal{H}om_{\mathcal{O}_C}(\cdot, \mathcal{O}_C)$ to this sequence, noting that $\mathcal{E}xt^1(\Omega_S|_C, \mathcal{O}_C) = 0$ since the surface S is supposed to be smooth at points on C and it implies that $\Omega_S|_C$ is locally free, we obtain an exact sequence

$$0 \longrightarrow \Theta_C \longrightarrow \Theta_S|_C \longrightarrow N_C \longrightarrow T_C^1 \longrightarrow 0,$$

where Θ is the tangent sheaf $\mathcal{H}om_{\mathcal{O}}(\Omega, \mathcal{O})$, N_C means the normal sheaf $[C]|_C$ of C in S , and $T_C^1 = \mathcal{E}xt^1(\Omega_C, \mathcal{O}_C)$. Then the equi-singular normal sheaf N'_C of C in S is defined as the kernel sheaf of the surjective homomorphism $N_C \longrightarrow T_C^1$ in this exact sequence. In particular, we have a short exact sequence

$$(2.7) \quad 0 \longrightarrow N'_C \longrightarrow N_C \longrightarrow T_C^1 \longrightarrow 0.$$

In the following, for simplicity, we write N and N' for N_C and N'_C respectively. By [13] (see also [9, Proposition 1.1.9] and [10, Section 1]), equi-singular displacements of C in S are governed by the cohomology groups of the sheaf N' . More precisely, the Zariski tangent space at the point C of the space of such displacements is identified with $H^0(N')$, and an obstruction for smoothness of the last space at the point C is in $H^1(N')$. In particular, if $H^1(N') = 0$, the space of such displacements is smooth at the point C , with the dimension being equal to $h^0(N')$.

Let \mathcal{J} be the ideal sheaf on S which is locally generated by the derivatives $\partial f/\partial x$ and $\partial f/\partial y$, where (x, y) are local coordinates on S and $f = f(x, y)$ is a local defining function of the cuspidal curve C . Note that we are assuming $C \cap \text{Sing } S = \emptyset$. Further, we put $\mathcal{J}_C := \mathcal{J} \otimes_{\mathcal{O}_C} \mathcal{O}_C$. These are called the Jacobian ideal sheaves of the singular curve C . As in [10, p. 111] and [9, §4.7.1], the equi-singular normal sheaf satisfies

$$(2.8) \quad N' \simeq N \otimes_{\mathcal{O}_C} \mathcal{J}_C.$$

The way for calculating cohomology groups $H^q(N')$ using (2.8) as well as the normalization of C is briefly described in [9, §4.7.1] for plane curves with ordinary nodes or cusps, but we write it here with some detail since in Section 3.3 we will need to calculate $H^q(N')$ for a curve which has an A_3 -singularity (tacnode).

To be explicit, let $\mu : S' \rightarrow S$ be the blowup of S at the cusp p of C . Then the restriction of μ to the strict transform \tilde{C} of C gives the normalization $\nu : \tilde{C} \rightarrow C$. As seen above, $\tilde{C} \simeq \mathbb{C}P_1$. Take coordinates (x, y) in a neighborhood of the cusp p such that C is locally defined by the equation $y^2 - x^3 = 0$. Putting $y = ux$ for a new coordinate u on the exceptional curve of μ , the map μ is locally given by $(x, u) \mapsto (x, ux)$, while \tilde{C} is defined by $x = u^2$. We also have $\mathcal{I}_p = (x^2, y)$ for the germ at p of the Jacobian ideal sheaf, and therefore $(\mu^* \mathcal{I})_{\tilde{p}} = (x^2, ux)$ for the germ at the point $\tilde{p} := \nu^{-1}(p)$. Restricting $\mu^* \mathcal{I}$ to \tilde{C} means a substitution of $x = u^2$, so we obtain

$$(\nu^* \mathcal{I}_C)_{\tilde{p}} \simeq \left((\mu^* \mathcal{I}) \otimes_{\mathcal{O}_{S'}} \mathcal{O}_{\tilde{C}} \right)_{\tilde{p}} \simeq (u^4, u^3) = (u^3).$$

This means

$$(2.9) \quad \nu^* \mathcal{I}_C \simeq \mathcal{O}_{\tilde{C}}(-3\tilde{p}).$$

Therefore, from (2.8) and $C^2 = 4$, we obtain

$$(2.10) \quad \begin{aligned} \nu^* N' &\simeq \nu^* N \otimes_{\mathcal{O}_{\tilde{C}}} \nu^* \mathcal{I}_C \\ &\simeq \nu^* N \otimes_{\mathcal{O}_{\tilde{C}}} \mathcal{O}_{\tilde{C}}(-3\tilde{p}) \end{aligned}$$

$$(2.11) \quad \simeq \mathcal{O}_{\tilde{C}}(1).$$

Taking the direct image of the isomorphism (2.10), from projection formula (which is available since the sheaf N is invertible), we obtain $\nu_* \nu^* N' \simeq N \otimes \nu_* \nu^* \mathcal{I}_C$. On the other hand, since the cusp has a single branch at p , we have $\nu_* \nu^* \mathcal{I}_C \simeq \mathcal{I}_C$. From these, we obtain $\nu_* \nu^* N' \simeq N \otimes \mathcal{I}_C \simeq N'$. By Leray spectral sequence, since we have $R^q \nu_* (\nu^* N') = 0$ for any $q > 0$ as $\nu^* N'$ is invertible, it holds $H^q(\tilde{C}, \nu^* N') \simeq H^q(C, \nu_* \nu^* N')$ for any $q \geq 0$. Thus we obtain $H^q(\tilde{C}, \nu^* N') \simeq H^q(C, N')$ for any $q \geq 0$. Hence from (2.11), we obtain $H^q(C, N') \simeq H^q(\tilde{C}, \mathcal{O}(1))$ for any $q \geq 0$. This implies $H^1(C, N') = 0$ and $h^0(C, N') = 2$. These mean that equi-singular displacements of the cuspidal curve C in S are parameterized by a smooth complex surface, and this is what we need to show. \square

For the proof Theorem 2.3, we recall basic results about deformations of A_n -singularity of a curve. As is well-known, the versal family of A_n -singularity is smooth, n -dimensional, and if we express the singularity by the equation $y^2 = x^{n+1}$ in \mathbb{C}^2 , the versal family is concretely given by

$$(2.12) \quad y^2 = x^{n+1} + s_1 x^{n-1} + s_2 x^{n-2} + \dots + s_n, \quad s_1, \dots, s_n \in \mathbb{C}.$$

If C denotes the singular curve $y^2 = x^{n+1}$ for the moment, the tangent space of the parameter space \mathbb{C}^n of this family at the origin $(s_1, \dots, s_n) = (0, \dots, 0)$

is naturally identified with the space $H^0(T_C^1)$. If we take a Galois cover of the parameter space \mathbb{C}^n by introducing new parameters t_1, \dots, t_{n+1} by putting

$$(2.13) \quad s_1 = \sum_{1 \leq i < j \leq n+1} t_i t_j, \quad s_2 = \sum_{1 \leq i < j < k \leq n+1} t_i t_j t_k, \quad \dots, \quad s_n = t_1 t_2 \cdots t_{n+1}$$

and imposing the condition $t_1 + \dots + t_{n+1} = 0$, then as a base change of the family (2.12), we obtain another n -dimensional family

$$(2.14) \quad y^2 = (x - t_1)(x - t_2) \cdots (x - t_{n+1}), \quad (t_1, \dots, t_{n+1}) \in \mathbb{C}^{n+1}, \quad \sum_{1 \leq i \leq n+1} t_i = 0.$$

If k is an integer satisfying $0 < k < n$, the locus of points whose fibers have an A_k -singularity as their all singularity is concretely given by the conditions that exactly k among t_1, \dots, t_{n+1} are equal and that the remaining parameters are not equal. In particular, when $n = 2$, the fiber over the point (t_1, t_2, t_3) with $t_1 + t_2 + t_3 = 0$ of the family (2.14) has an A_1 -singularity iff (t_1, t_2, t_3) is of the form $(t, t, -2t)$, $(t, -2t, t)$ or $(-2t, t, t)$ for some $t \neq 0$. Hence, from (2.13), the fiber over the point (s_1, s_2) of the versal family (2.12) has an A_1 -singularity iff $(s_1, s_2) = (-3t^2, -2t^3)$ for some $t \neq 0$. We denote the locus formed by these points in \mathbb{C}^2 by \mathcal{A}_1 . Then \mathcal{A}_1 is locally closed, smooth, 1-dimensional, and its closure in \mathbb{C}^2 has an A_2 -singularity at the origin. With these preliminaries, we provide:

Proof of Theorem 2.3. We keep notations in the proof of Theorem 2.2. We first show $H^1(N) = 0$ and $H^0(N) \simeq \mathbb{C}^4$. Let C be any hyperplane section of the Segre surface S which has a single ordinary cusp at a smooth point of S as its all singularity. By using that the surface $S \subset \mathbb{CP}_4$ is a complete intersection of two quadrics, we readily obtain that C belongs to the anti-canonical class on S . This means that there is an exact sequence

$$(2.15) \quad 0 \longrightarrow \mathcal{O}_S \longrightarrow K_S^{-1} \longrightarrow N \longrightarrow 0.$$

This time we mean by $\mu : \tilde{S} \rightarrow S$ the minimal resolution of all singularities of S . Since all the singularities of S are rational double points ([3, Theorem 8.1.11]), we have $K_{\tilde{S}} \simeq \mu^* K_S$. Further, $K_{\tilde{S}}^{-1}$ is nef and big ([3, p.355]). Furthermore, by Leray spectral sequence, for any invertible sheaf \mathcal{L} on S , we have $H^q(\tilde{S}, \mu^* \mathcal{L}) \simeq H^q(S, \mathcal{L})$ for any $q \geq 0$. From these and Kodaira-Ramanujan vanishing theorem, it readily follows that $H^q(K_S^{-1}) = H^q(\mathcal{O}_S) = 0$ for any $q > 0$. Hence from the cohomology exact sequence of (2.15), we obtain $H^1(N) = 0$. Also, as showed in the proof of [7, Lemma 3.8], the restriction homomorphism

$$(2.16) \quad H^0(\mathbb{CP}_4, \mathcal{O}(1)) \longrightarrow H^0(K_S^{-1})$$

is isomorphic, and hence $h^0(K_S^{-1}) = 5$. So again from the cohomology exact sequence of (2.15), we obtain $h^0(N) = 4$.

The fact $H^1(N) = 0$ means that any first order displacement of C in the surface S (without any constraint for displacements this time) is unobstructed, and the tangent space at the point C of the parameter space of the versal family of displacements of C in S is identified with $H^0(N) \simeq \mathbb{C}^4$.

From the short exact sequence (2.7), since $H^1(N') = 0$ as we have shown in the proof of Theorem 2.2, we obtain an exact sequence

$$(2.17) \quad 0 \longrightarrow H^0(N') \longrightarrow H^0(N) \longrightarrow H^0(T_C^1) \longrightarrow 0.$$

Let $H \subset \mathbb{C}\mathbb{P}_4$ be the hyperplane which satisfies $H|_S = C$, and B a neighborhood of the point H in the dual space $\mathbb{C}\mathbb{P}_4^*$. To each point of B , we can naturally associate a hyperplane section of S , so B can be regarded as a parameter space of displacements of C in S . As above, we have a natural isomorphism $T_H B \simeq H^0(N)$. By versality, after shrinking B if necessary, there is an induced holomorphic map $f : B \rightarrow \mathbb{C}^2$ from B to the parameter space of the versal family of an A_2 -singularity, while the differential $df : T_H B \rightarrow T_0 \mathbb{C}^2$ is identified with the map $H^0(N) \rightarrow H^0(T_C^1)$ in (2.17). Writing $\mathcal{A}_1 \subset \mathbb{C}^2$ for the locus formed by points whose fibers have A_1 -singularity as presented right before the present proof, because a generic point of the dual variety S^* corresponds to a hyperplane section which has an A_1 -singularity, we have

$$S^* \cap B = f^{-1}(\mathcal{A}_1 \cup \{H\}).$$

Since the map $H^0(N) \rightarrow H^0(T_C^1)$ in (2.17) is surjective, again shrinking the neighborhood B if necessary, there are coordinates (z_1, z_2, z_3, z_4) on B such that f takes the form $(z_1, z_2, z_3, z_4) \mapsto (z_1, z_2)$. As seen right before the present proof, the locus $\mathcal{A}_1 \cup \{H\}$ is 1-dimensional and has an ordinary cusp at the point H as its only singularity. These imply that the dual variety S^* has ordinary cusps along the smooth surface $f^{-1}(0, 0) = \{z_1 = z_2 = 0\}$ in B , formed by hyperplane sections which have ordinary cusp as its all singularity. \square

3 - The structure of the cuspidal locus

3.1 - Some general properties of Segre surfaces

In this subsection, we prove a few basic properties of Segre surfaces that will be used throughout the rest of this article. We begin with an easy but useful property about curves with low degree on Segre surfaces.

Proposition 3.1. *Let $S \subset \mathbb{C}\mathbb{P}_4$ be a Segre surface, and $\mu : \tilde{S} \rightarrow S$ the minimal resolution of all singularities of S . The self-intersection number in \tilde{S} of the strict transform of a line, an irreducible conic, or a rational normal curve of degree three, lying on S , is (-1) , 0 and 1 respectively.*

Of course, when the curve (a line, a conic, or a rational normal curve of degree three) does not pass any singularity of S , this implies that the self-intersection number of the curve in S is (-1) , 0 and 1 respectively.

Remark 3.2. From Proposition 3.1, we obtain that an irreducible conic on any Segre surface S is always a member of a pencil of conics on S , and the pencil can have a base point only at a singularity of S .

Proof of Proposition 3.1. We write C for the curve in S and \tilde{C} for the strict transform of C into \tilde{S} . As remarked in the proof of Theorem 2.3. if H is the hyperplane class on $\mathbb{C}\mathbb{P}_4$, we have $H|_S = K_S^{-1}$ and $K_{\tilde{S}} \simeq \mu^* K_S$. Noting that the restriction $\mu|_{\tilde{C}}$ gives an isomorphism $\tilde{C} \simeq C$ as C is smooth, we have

$$\begin{aligned} K_{\tilde{C}} &\simeq K_{\tilde{S}}|_{\tilde{C}} \otimes N_{\tilde{C}/\tilde{S}} \\ &\simeq (\mu^* K_S)|_{\tilde{C}} \otimes N_{\tilde{C}/\tilde{S}} \\ &\simeq \mu^*(K_S|_C) \otimes N_{\tilde{C}/\tilde{S}} \\ &\simeq (\mu|_{\tilde{C}})^*(-H|_C) \otimes N_{\tilde{C}/\tilde{S}}. \end{aligned}$$

Hence, since \tilde{C} is a smooth rational curve in the present situation, comparing the degrees of both sides, we obtain

$$-2 = -H.C + \tilde{C}^2.$$

From this we readily obtain the assertions of the proposition. \square

We also have

Proposition 3.3. *Any Segre surface S does not have a reduced cubic curve which is contained in a 2-plane in $\mathbb{C}\mathbb{P}_4$.*

Proof. Let $C \subset S$ be such a cubic curve, and P the 2-plane containing C . Then for any hyperplane H containing P , we have $H|_S = C + l$ for a line l . Since we have an isomorphism (2.16), the line l moves as H moves. This contradicts finiteness of lines on S . Hence, such a curve C does not exist. \square

In the following, by a tangent space of a Segre surface at a smooth point, we always mean a closed linear subspace in the projective space $\mathbb{C}\mathbb{P}_4$. So it is a projective 2-plane, not just a vector space. In general, if l is a line lying on a

subvariety, then clearly the tangent space at any smooth point belonging to l contains l . The next lemma says that for any Segre surface S , lines on S are all the curves on S which are contained in a tangent plane at some smooth point of S .

Lemma 3.4. *Let $S \subset \mathbb{CP}_4$ be a Segre surface, p a smooth point of S , and T_pS the tangent plane in the above sense. If the intersection $S \cap T_pS$ contains a curve, then any of its irreducible component is a line through the point p .*

For the proof of this lemma, it is convenient to introduce the following notations. They will be used throughout the rest of this article.

Definition 3.5. For a smooth point p of a Segre surface $S \subset \mathbb{CP}_4$, we denote $(T_pS)^*$ for the pencil of hyperplanes in \mathbb{CP}_4 which contain the tangent plane T_pS . Also, we write $(T_pS)^*|_S$ for the pencil on S whose members are of the form $H|_S$ for some $H \in (T_pS)^*$.

Proof of Lemma 3.4. Since $\deg S = 4$, the intersection $S \cap T_pS$ can contain a curve whose degree is at most four. If $S \cap T_pS$ contains an irreducible quartic curve C , then by the same reason, any hyperplane $H \in (T_pS)^*$ satisfies $S \cap H = C$. In particular, the intersection $S \cap H$ is independent of a choice of such a hyperplane H . This contradicts the basic isomorphism (2.16). Hence, $S \cap T_pS$ does not contain an irreducible quartic curve. Next, if $S \cap T_pS$ contains an irreducible cubic curve C , for any hyperplane H containing T_pS , we have $S \cap H = C + l$ for some line l . Again from the isomorphism (2.16), if we move H in the pencil $(T_pS)^*$, the line l also really moves. This contradicts finiteness of the number of lines on any Segre surfaces. Hence $S \cap T_pS$ does not contain an irreducible cubic curve.

Next, suppose that $S \cap T_pS$ contains an irreducible conic C . Then for any $H \in (T_pS)^*$, $H|_S$ is of the form $C + D$ for some conic D which is possibly reducible or non-reduced. Again from the isomorphism (2.16), this conic D has to move as H moves in $(T_pS)^*$. Hence, again from the finiteness of lines, the conic D has to be irreducible for a generic $H \in (T_pS)^*$. But then D would also be smooth, and therefore D has to pass the point p since the curve $C + D$ is singular at p as $H \in (T_pS)^*$. This means that there is a pencil of conics on S which has a base point belonging to S_{reg} . This contradicts Remark 3.2. Thus, the intersection $S \cap T_pS$ does not contain an irreducible conic.

Therefore, any curve in $S \cap T_pS$ is a line. Let l be such a line and suppose that $p \notin l$. If $H \in (T_pS)^*$, then $H|_S = l + D$ for some cubic D which is possibly reducible or non-reduced. Since $p \notin l$, the curve D has to have at least a double point at p . We show that the pencil

$$(T_pS)^*|_S - l = \{H|_S - l \mid H \in (T_pS)^*\}$$

formed by the residual cubic curves has no fixed component. Suppose not and let F be the fixed component. Then F is a curve on the 2-plane T_pS , so it has to be a line as above. Moreover, F goes through p , since otherwise the movable part of the pencil $(T_pS)^*|_S - l - F$ consists of conics whose general members are irreducible, and this contradicts that any member of the pencil $(T_pS)^*|_S$ is singular at p . Thus, under the assumption on the presence of the fixed component F , a generic member of $(T_pS)^*|_S$ can be written as $l + F + C$, where F is a line through p and C is an irreducible conic through p . This again contradicts Remark 3.2, and we obtain that the residual pencil $(T_pS)^*|_S - l$ has no fixed component. From the finiteness of lines on S , this means that a generic member D of this pencil is an irreducible cubic curve. Because we are supposing $p \notin l$, the cubic curve D itself is singular at p . Since D is an irreducible cubic curve, this implies that D lies on some 2-plane P since otherwise D would be a rational normal curve, which is smooth. This contradicts Proposition 3.3. Therefore, the point p belongs to l , as desired. \square

By using this lemma, we next show that there is no point $p \in S_{\text{reg}}$ such that any member of the pencil $(T_pS)^*|_S$ has a triple point at p .

Lemma 3.6. *Let $S \subset \mathbb{C}\mathbb{P}_4$ be a Segre surface and p a smooth point of S . Then the multiplicity in p of a generic member of the pencil $(T_pS)^*|_S$ is precisely two.*

Proof. First we show that if all members of the pencil $(T_pS)^*|_S$ have a triple point at p , then there has to exist a line on S through p . Let $\mu : S' \rightarrow S$ be a blowup at p , and E the exceptional curve. If the hyperplane section $H|_S$ would have a triple point at p for any $H \in (T_pS)^*$, then the linear system $|\mu^*H - 3E|$ on S' is also a pencil. Then since $(\mu^*H - 3E)^2 = 4 - 9 = -5 < 0$, the pencil $|\mu^*H - 3E|$ would have a fixed component. If E would be a fixed component, then the system $|\mu^*H - 4E|$ is a pencil, and again from the self-intersection number, this has to have a fixed component. This process cannot continue infinitely many times. So there exists some $m > 2$ such that the system $|\mu^*H - mE|$ is a pencil and has a fixed component other than E . If we take the image of this component to S by μ , we obtain a curve on S , and it has to be contained in the 2-plane T_pS since it has to be a base curve of the pencil $(T_pS)^*|_S$. By Lemma 3.4, the last curve is a line through p .

Hence, if any member of the pencil $(T_pS)^*|_S$ has a triple point at p , a generic member of the pencil is of the form $l + D$ for some line l through p and a cubic curve D which may be reducible or non-reduced. The curve D has to be singular at p since $l + D$ is assumed to have a triple point at p . If a generic member of the residual pencil $(T_pS)^*|_S - l$ of cubic curves would be reducible, by finiteness of lines on S , the pencil has to have another line l' as a fixed component. If

$p \notin l'$, then any member of the pencil $(T_p S)^*|_S - l - l'$ of conics has to have a double point at p , and this cannot happen by finiteness of lines on S . Hence $p \in l'$. But still the pencil $(T_p S)^*|_S - l - l'$ of conics has p as a base point, which again contradicts Remark 3.2. Hence a generic member of the pencil $(T_p S)^*|_S - l$ is irreducible. So there exists an irreducible cubic curve D on S which is singular at p . The curve D has to lie on a 2-plane, and this contradicts Proposition 3.3. Therefore, a generic member of the pencil $(T_p S)^*|_S$ does not have a triple point at p . \square

3.2 - Structure of the divisor D_1

As before, let $S \subset \mathbb{C}\mathbb{P}_4$ be a Segre surface. In this subsection, by using the results in the previous subsection, we describe the structure of the divisor D_1 in the incidence variety $I(S)$. (See Section 2.1 for the definition of the divisor D_1 .)

Let p be a smooth point of S , $\mu : S' \rightarrow S$ a blowup at p , and E the exceptional curve. Then there is the following natural isomorphism between pencils:

$$(3.1) \quad (T_p S)^*|_S \simeq |\mu^* H - 2E|.$$

If the point p is on a line $l \subset S$, then the strict transform l' of l into S' is obviously a fixed component of the pencil $|\mu^* H - 2E|$. The following proposition can be thought of as a kind of its converse.

Lemma 3.7. *Let $S \subset \mathbb{C}\mathbb{P}_4$ be a Segre surface and p a smooth point of S . If the pencil $|\mu^* H - 2E|$ on the blowup has a base point, then there exists a line lying on S which passes the point p .*

Proof. It is enough to show that the pencil $|\mu^* H - 2E|$ has a fixed component which is different from E , since the image of such a component to S by μ is a fixed component of the pencil $(T_p S)^*|_S$, and therefore it is contained in the tangent plane $T_p S$, which has to be a line through p by Lemma 3.4.

First, by Lemma 3.6, the exceptional curve E of μ cannot be a base curve of the pencil $|\mu^* H - 2E|$ because if so, any member of the pencil $(T_p S)^*|_S$ would have a triple point at p . So assume that the pencil $|\mu^* H - 2E|$ has no fixed component and derive a contradiction. Since we are supposing $\text{Bs } |\mu^* H - 2E| \neq \emptyset$, this assumption means that the pencil $|\mu^* H - 2E|$ has an isolated fixed point. Suppose that some (and hence a generic) member of the pencil $|\mu^* H - 2E|$ is irreducible. Since the arithmetic genus of the line bundle $\mu^* H - 2E$ is readily seen to be zero, this implies that the irreducible member of $|\mu^* H - 2E|$ is

smooth and rational. We also obtain that the self-intersection number of the rational curve is zero. These mean that $\text{Bs } |\mu^*H - 2E| = \emptyset$. Hence any member of the pencil $|\mu^*H - 2E|$ is reducible. Therefore, any member of the pencil $(T_pS)^*|_S$ is also reducible. Hence, from the finiteness of lines on S , the only possibility for the form of a generic member of the pencil $(T_pS)^*|_S$ is $C_1 + C_2$, where C_1 and C_2 are irreducible conics. Since C_1 and C_2 are then smooth, both of them must pass the point p . Therefore at least one of the conics C_1 and C_2 moves on S while passing the smooth point p of S . But this again contradicts Remark 3.2. \square

From this, it is not difficult to obtain the following

Lemma 3.8. *Let $S \subset \mathbb{C}\mathbb{P}_4$ be a Segre surface and p a smooth point of S . If there exists no line on S through p , then the pencil $|\mu^*H - 2E|$ has exactly two members whose restrictions to E are double point.*

Proof. It is elementary to prove that, under a suitable choice of homogeneous coordinates (u, v) on $\mathbb{C}\mathbb{P}_1$, any 1-dimensional subsystem of the linear system $|\mathcal{O}(2)|$ on $\mathbb{C}\mathbb{P}_1$ is generated by either u^2 and v^2 , or u^2 and uv . These are distinguished by presence of a base point of the subsystem. By Lemma 3.7, under the present assumption on the point p , the pencil $|\mu^*H - 2E|$ is base point free. So it does not have a base point also on E . This means that the restriction homomorphism

$$(3.2) \quad r_E : H^0(\mu^*H - 2E) \longrightarrow H^0(\mathcal{O}_E(2))$$

has a 2-dimensional image, and that the image is generated by u^2 and v^2 for suitable homogeneous coordinates (u, v) on E . Obviously, the two members of the pencil $(T_pS)^*|_S$ which correspond to these generators are the members which satisfy the property in the lemma. \square

From this lemma, we obtain the following result about the defining function \mathbf{H} of the divisor \mathbf{D} in $I(S)$. (See Section 2.1 for the definition of \mathbf{H} and \mathbf{D} .)

Proposition 3.9. *Let $S \subset \mathbb{C}\mathbb{P}_4$ be a Segre surface and $p \in S_{\text{reg}}$ a point which does not belong to any line on S . Then we have $\mathbf{H}(p) \neq 0$ in the sense that the quadratic polynomial $\mathbf{H}(p)$ in λ, μ is not identically zero. Further, the quadratic equation $\mathbf{H}(p) = 0$ does not have a double root.*

Proof. As in the previous proof, under the assumption on the point p , the restriction map r_E as in (3.2) has a two-dimensional image. So not all elements of $\text{Image}(r_E)$ can have a double root on E . Because zeroes of $\mathbf{H}(p)$ correspond to hyperplanes whose sections have non-nodal singularity at p , this means that a generic member of the pencil $(T_pS)^*|_S$ has an ordinary node at p . This implies

$\mathbf{H}(p) \neq 0$. Since each solution of the equation $\mathbf{H}(p) = 0$ about $(\lambda, \mu) \in \mathbb{CP}_1$ corresponds to a hyperplane $H \in (T_p S)^*$ such that the singularity of $H|_S$ at p is non-nodal, the equation $\mathbf{H}(p) = 0$ has a double root iff such a hyperplane is unique. On the other hand, via the blowup μ at p , such a hyperplane gives an element of the pencil $|\text{Image}(r_E)|$ on E which is a double point. By Lemma 3.8, when p is not on a line on S , the last pencil has exactly two such members. These mean that the equation $\mathbf{H}(p) = 0$ does not have a double root. \square

As an immediate consequence, we can identify any component of the divisor \mathbf{D}_1 in $I(S)$ as follows.

Corollary 3.10. *For any Segre surface $S \subset \mathbb{CP}_4$, any irreducible component of the divisor \mathbf{D}_1 is of the form $\pi_1^{-1}(l)$, where l is a line lying on S .*

Remark 3.11. This corollary does not assert that $\pi_1^{-1}(l)$ is a component of the divisor \mathbf{D}_1 for any line l on S . Indeed, we will see that this is not correct in general.

From the corollary, we obtain the following information about the cuspidal locus in the dual varieties of Segre surfaces.

Proposition 3.12. *Let $S \subset \mathbb{CP}_4$ be any Segre surface. Then the cuspidal locus in the dual variety S^* is contained in the image $\pi_2(\mathbf{D}_S)$, where $\pi_2 : I(S) \rightarrow S^*$ is the projection introduced in Section 2.1.*

Proof. From our definition of the cuspidal locus given in Section 2.1, any component of the cuspidal locus has a point H such that $H|_S$ has an ordinary cusp at some point $p \in S_{\text{reg}}$. Then the point $(p, H) \in I(S)$ belongs to either \mathbf{D}_1 or \mathbf{D}_S . If $(p, H) \in \mathbf{D}_1$, then from Corollary 3.10, the smooth point p belongs to a line l on S , and the hyperplane H includes l . This means that the section $H|_S$ contains l as a component. Therefore, $H|_S$ does not have an ordinary cusp at p , which contradicts our choice of H . This implies $(p, H) \in \mathbf{D}_S$. So the hyperplane H belongs to the image $\pi_2(\mathbf{D}_S)$. \square

3.3 - Hyperplane sections from the divisor \mathbf{D}_S

From Proposition 3.12, for any Segre surface S , the investigation of the cuspidal locus in the dual variety S^* is reduced to that of the divisor \mathbf{D}_S in $I(S)$ and its image to S^* by the projection $\pi_2 : I(S) \rightarrow S^*$. In this subsection, we first investigate the structure of hyperplane sections of a Segre surface which are obtained from generic points of the divisor \mathbf{D}_S . Next, by using it, we show that there are three possibilities for the concrete forms of hyperplane sections

which are obtained from generic points of \mathbf{D}_S , and then derive some conclusions about irreducibility of the cuspidal locus in the dual varieties of Segre surfaces.

Take any point $p \in S_{\text{reg}}$ which does not belong to any line on S . By Proposition 3.9, there exist precisely two hyperplanes belonging to the pencil $(T_p S)^*$ whose intersections with S have non-nodal singularity at p . We denote H_1 and H_2 for these hyperplanes. We then have $(p, H_1) \in \mathbf{D}_S$ and $(p, H_2) \in \mathbf{D}_S$.

Proposition 3.13. *Let C be any one of the two hyperplane sections $H_1|_S$ and $H_2|_S$. If C is irreducible, then C has an ordinary cusp at p as its all singularity. If C is reducible, then exactly one of the following holds:*

- C is an irreducible conic with multiplicity two,
- C consists of two irreducible conics which are not co-planar and which are tangent to each other at p with order two.

Proof. Since $p_a(C) = 1$ for the arithmetic genus of C , when C is irreducible, it is a rational curve which has an ordinary node or an ordinary cusp at p as its all singularity. But since the singularity is non-nodal as above, it has to be an ordinary cusp.

Next assume that C is reducible and has some line l as a component. Write $C = l + D$, where D is a cubic curve. Since l cannot go through p from the choice of p , the cubic D has to have a non-nodal singularity at p . From absence of lines through p , this means that the cubic D is irreducible and p is an ordinary cusp of D . Therefore, D is contained in a 2-plane. This contradicts Proposition 3.3. Hence, the curve C does not contain any line. So we have $C = C_1 + C_2$, where both C_1 and C_2 are irreducible conics. Since both are smooth, they pass the point p . If $C_1 = C_2$, then C is a double irreducible conic as in the first item in the proposition.

Suppose $C_1 \neq C_2$. They do not intersect transversely at the point p as C is non-nodal at p . Because of the isomorphism (2.16), they cannot lie on the same 2-plane. Since C_1 and C_2 are tangent at p , this means $C_1 \cap C_2 = \{p\}$. Because p is a smooth point of S and $C_1 + C_2$ is a hyperplane section of S , this implies that the smooth curves C_1 and C_2 do not pass any singularity of S . From Proposition 3.1, this means $C_1^2 = C_2^2 = 0$. We also have

$$4 = (C_1 + C_2)^2 = C_1^2 + C_2^2 + 2C_1C_2.$$

So we obtain $C_1C_2 = 2$. Thus, the curve $C = C_1 + C_2$ is tangent to each other at p with order two. \square

By using the double covering structure on some Segre surfaces over a quadric surface or a quadric cone in \mathbb{CP}_3 given in [7], it is possible to see that the

situation as in the two items of Proposition 3.13 really happen. For the situation in the first item, the conic with multiplicity two, which we simply call a *double conic*, satisfies the following property.

Proposition 3.14. *Irreducible double conics on a Segre surface always form a pencil on the surface. Further, any double conic passes a singular point of the surface.*

Proof. Write the double conic as $2C$. Let $\mu : \tilde{S} \rightarrow S$ be the minimal resolution of all singularities of S and \tilde{C} the strict transform of C into \tilde{S} . Then by Proposition 3.1, we have $\tilde{C}^2 = 0$. Hence we have $\dim |\tilde{C}| = 1$. Mapping twice of members of the pencil $|\tilde{C}|$ by μ , we obtain the required pencil of double conics on S .

For the latter assertion, writing $H|_S = 2C$ with a hyperplane H , we have

$$4 = (H|_S)^2 = (2C)^2 = 4C^2.$$

This means $C^2 = 1$. On the other hand, if the conic C does not pass any singular point of S , then we have $C^2 = 0$ by Proposition 3.1. Hence the conic C passes through a singular point of S . \square

Next we show the following proposition meaning that, among the three possibilities in Proposition 3.13 for the two hyperplane sections, the last possibility cannot occur if the point p is generic.

Proposition 3.15. *Let S be any Segre surface and C a hyperplane section of S which is as in the second item in Proposition 3.13. Then the parameter space of the versal family of equi-singular displacements of C in S is smooth and 1-dimensional.*

Proof. The idea of the proof is similar to Theorem 2.2. Write $C = C_1 + C_2$ as above. So C_1 and C_2 are irreducible conics which are tangent to each other at a point $p \in S_{\text{reg}}$ with order two, and which have no other intersection. As in the proof of Proposition 3.13, C_1 and C_2 do not pass any singularity of S . We may replace the cuspidal curve C in the short exact sequence (2.6) by the reducible curve $C_1 + C_2$, and we still have an exact sequence

$$(3.3) \quad 0 \longrightarrow \mathcal{I}_{C_1+C_2} / \mathcal{I}_{C_1+C_2}^2 \longrightarrow \Omega_S|_{C_1+C_2} \longrightarrow \Omega_{C_1+C_2} \longrightarrow 0.$$

Applying $\mathcal{H}om_{\mathcal{O}_{C_1+C_2}}(\cdot, \mathcal{O}_{C_1+C_2})$ to this sequence, the equi-singular normal sheaf $N'_{C_1+C_2}$ is again defined as the kernel sheaf of the surjective homomorphism $N_{C_1+C_2} \longrightarrow T_{C_1+C_2}^1$, and we have an exact sequence

$$(3.4) \quad 0 \longrightarrow N'_{C_1+C_2} \longrightarrow N_{C_1+C_2} \longrightarrow T_{C_1+C_2}^1 \longrightarrow 0.$$

Let $\nu : \tilde{C}_1 \sqcup \tilde{C}_2 \rightarrow C_1 \cup C_2$ be the normalization of the curve $C_1 + C_2$. \tilde{C}_1 and \tilde{C}_2 are isomorphic to C_1 and C_2 respectively, and the point $p = C_1 \cap C_2$ is an A_3 -singularity (i.e. tacnode) of $C_1 + C_2$. So in a neighborhood of p , the curve $C_1 + C_2$ can be identified with the curve $y^2 = x^4$ in \mathbb{C}^2 around the origin. As $C_1^2 = C_2^2 = 0$ from Proposition 3.1 and $C_1 \cdot C_2 = 2$, we have

$$(3.5) \quad N_{C_1+C_2}|_{C_i} \simeq [C_1 + C_2]|_{C_i} \simeq \mathcal{O}_{C_i}(2), \quad i = 1, 2.$$

Let \mathcal{J} be the Jacobian ideal sheaf of the curve $C_1 + C_2$, and write $\mathcal{J}_{C_1+C_2}$ for $\mathcal{J}|_{C_1+C_2}$. We readily have $\mathcal{J}_p = (x^3, y)$ for the germ at p . The desingularization of the tacnode may be obtained by blowing-up the surface S twice. Let μ be the composition of the two blowups. Then the restrictions of μ to $\tilde{C}_1 \sqcup \tilde{C}_2$ gives the normalization ν of the curve $C_1 + C_2$. By introducing the coordinates $u = y/x$ and $v = u/x$ on the exceptional curves of the first and the second blowup respectively, the composition μ is concretely written as $(x, v) \mapsto (x, x^2v)$, while the strict transforms \tilde{C}_1 and \tilde{C}_2 are defined by $v = 1$ and $v = -1$. Hence, if $\tilde{p}_1 \in \tilde{C}_1$ and $\tilde{p}_2 \in \tilde{C}_2$ denote the points on the normalization which correspond to the tacnode p , the germs of $\mu^* \mathcal{J}$ at \tilde{p}_1 and \tilde{p}_2 are generated by two functions x^3 and x^2v . Restricting $\mu^* \mathcal{J}$ to the strict transforms means a substitution of $v = \pm 1$, and therefore, over the two components \tilde{C}_1 and \tilde{C}_2 , we have

$$(3.6) \quad \begin{aligned} (\nu^* \mathcal{J}_{C_1+C_2})|_{\tilde{C}_i} &\simeq (\mu^* \mathcal{J})|_{\tilde{C}_i} \\ &\simeq \mathcal{O}_{\tilde{C}_i}(-2\tilde{p}_i). \end{aligned}$$

As in (2.8) for the case of an A_2 -singularity, we have $N' \simeq N \otimes_{\mathcal{O}_{C_1+C_2}} \mathcal{J}_{C_1+C_2}$. Hence, combined with (3.5), over each component \tilde{C}_i , $\nu^* N'$ is isomorphic to $\mathcal{O}_{\tilde{C}_i}(2) \otimes \mathcal{O}_{\tilde{C}_i}(-2) \simeq \mathcal{O}_{\tilde{C}_i}$. So we have

$$(3.7) \quad \nu^* N'_{C_1+C_2} \simeq \mathcal{O}_{\tilde{C}_1 \sqcup \tilde{C}_2}.$$

Therefore we obtain an exact sequence

$$(3.8) \quad 0 \rightarrow N'_{C_1+C_2} \rightarrow \nu_* \mathcal{O}_{\tilde{C}_1 \sqcup \tilde{C}_2} \rightarrow \mathbb{C}_p \rightarrow 0.$$

Since $R^q \nu_* \mathcal{O}_{\tilde{C}_1 \sqcup \tilde{C}_2} = 0$ for $q > 0$, the Leray spectral sequence means $H^i(\tilde{C}_1 \sqcup \tilde{C}_2, \mathcal{O}_{\tilde{C}_1 \sqcup \tilde{C}_2}) \simeq H^i(\nu_* \mathcal{O}_{\tilde{C}_1 \sqcup \tilde{C}_2})$ for any $i \geq 0$. This readily means $H^i(\nu_* \mathcal{O}_{\tilde{C}_1 \sqcup \tilde{C}_2}) = 0$ for any $i > 0$. From the cohomology exact sequence of (3.8), noting that the homomorphism $H^0(\nu_* \mathcal{O}_{\tilde{C}_1 \sqcup \tilde{C}_2}) \rightarrow H^0(\mathbb{C}_p)$ is evidently surjective, we obtain

$$H^0(N'_{C_1+C_2}) \simeq \mathbb{C} \quad \text{and} \quad H^1(N'_{C_1+C_2}) = 0.$$

The former implies that any first order equi-singular displacement of the curve $C_1 + C_2$ in S is unobstructed, and the latter means that the versal family of such displacements of $C_1 + C_2$ in S is 1-dimensional. \square

One might expect that the possibility of occurrence of the double conic as in the first item in Proposition 3.13 in case the point p is generic can be eliminated as well. However, as we shall see soon, this is not at all correct. Rather, occurrence of the double conic for a generic point p will play a key role for proving irreducibility of the cuspidal locus in the dual variety of an arbitrary Segre surface.

We still assume that $p \in S_{\text{reg}}$ is a point which does not belong to any line on S , and H_1 and H_2 are the two hyperplanes which cut out from S curves having non-nodal singularity at p . From Propositions 3.13 and 3.15, assuming genericity to the point p , there are the following three possibilities for the two hyperplane sections:

- (i) both $H_1|_S$ and $H_2|_S$ are double conics,
- (ii) exactly one of $H_1|_S$ and $H_2|_S$ has an ordinary cusp at p , and the other one is a double conic,
- (iii) both $H_1|_S$ and $H_2|_S$ have ordinary cusp at p .

When the situation (i) occurs, the image $\pi_2(\mathbf{D}_S)$ is not a component of the cuspidal locus because singularity of a double curve is not an ordinary cusp. Together with Proposition 3.12, this means that *the cuspidal locus in S^* is empty* in this case. For the case (ii), the divisor \mathbf{D}_S is necessarily reducible. It consists of two irreducible components since the projection $\mathbf{D}_S \rightarrow S$ is of degree two. Both components are birational to S by the projection. The image under the projection $\pi_2 : I(S) \rightarrow S^*$ of the irreducible component corresponding to the double conics is not a component of the cuspidal locus by the same reason to the case (i), while the image of the other component is exactly the cuspidal locus in S^* . It follows that *the cuspidal locus in S^* is irreducible* in this case. Finally, if the situation (iii) occurs, *whole of the image $\pi_2(\mathbf{D}_S)$ is exactly the cuspidal locus in S^** . But at this stage, we do not know whether it is irreducible or not, because we do not know whether the divisor \mathbf{D}_S is irreducible or not. This ambiguity will be resolved in Sections 4 and 5.

3.4 - Pencils of double conics on Segre surfaces

As one can expect from results in the previous section, irreducibility of the cuspidal locus in S^* is closely related to the existence of double conics in S . In

this subsection, we first provide a relation between the pencil of double conics and irreducibility of the cuspidal locus in the dual varieties of Segre surfaces. Next we determine the number of pencils of double conics for any Segre surfaces.

Proposition 3.16. *Pencils of double conics on a Segre surface S are in one-to-one correspondence with irreducible components of the divisor \mathbf{D}_S whose generic point (p, H) satisfies the property that $H|_S$ is a double conic.*

Proof. Let \mathcal{P} be a pencil of double conics on S . For a generic point $p \in S_{\text{reg}}$, there exists precisely one member of \mathcal{P} which passes p . Let H_p be the hyperplane which cuts out this member. Then we have $(p, H_p) \in \mathbf{D}_S$ by Corollary 3.10, and the assignment $p \mapsto (p, H_p)$ gives a birational map from S to an irreducible component of \mathbf{D}_S . If \mathcal{P}_1 and \mathcal{P}_2 are distinct pencils of double conics on S , then the components of \mathbf{D}_S obtained this way are distinct because \mathcal{P}_1 and \mathcal{P}_2 do not have a common member by Proposition 3.1. Conversely, any component of the divisor \mathbf{D}_S as in the proposition comes from a pencil of double conics by Proposition 3.14. Therefore, the correspondence is bijective. \square

In particular, any Segre surface S has at most two pencils of double conics, and if S has two such pencils, then the divisor \mathbf{D}_S is reducible. The number of pencils of double conics on each Segre surface can be determined from the Segre symbol of the surface by the following result.

Proposition 3.17. *Pencils of double conics on a Segre surface S are in one-to-one correspondence with the units*

$$(3.9) \quad (11), (12), (13), (14)$$

which are contained in the Segre symbol of S .

For example, if the Segre symbol of S is $[(12)11]$, then S has a single pencil of double conics since the symbol contains one unit (12) among (3.9). Similarly, if the Segre symbol of S is $[(11)(11)1]$, then S has two pencils of double conics since the symbol contains a unit (11) twice.

Proof. First we explain how we can obtain a pencil of double conics from any one of the units (3.9). The Segre symbol determines two symmetric matrices of size 5×5 which correspond to two defining quadratic polynomials for the Segre surface. An entry in the Segre symbol determines submatrices of the two full matrices. For the unit (12) for example, the two symmetric submatrices are given by

$$P_{(12)} = \begin{pmatrix} \alpha & 0 & 0 \\ 0 & 0 & \alpha \\ 0 & \alpha & 1 \end{pmatrix}, \quad Q_{(12)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$

where α is a complex number. If the Segre symbol of S contains the unit (12), then the two full symmetric matrices P and Q which define S have the matrices $P_{(12)}$ and $Q_{(12)}$ as submatrices respectively. Therefore, the matrix $P - \alpha Q$ contains $P_{(12)} - \alpha Q_{(12)}$ as a submatrix, and we have

$$P_{(12)} - \alpha Q_{(12)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

This means that a projection from $\mathbb{C}\mathbb{P}_4$ to $\mathbb{C}\mathbb{P}_2$ which drops two coordinates maps S to a conic in $\mathbb{C}\mathbb{P}_2$. The conic has to be irreducible since S is irreducible. By pulling back tangent lines of the conic by the last projection, we obtain a pencil of double conics on S . The same argument works when the Segre symbol contains any one of the units (11), (13) or (14). (For the case (14), the submatrices are equal to the full matrices.)

In order to provide the reverse direction, assume that a Segre surface S has a pencil $\mathcal{P} = \{2C_t \mid t \in \mathbb{C}\mathbb{P}_1\}$ of double conics. Then pulling back the conics to the minimal resolution \tilde{S} of S as in the proof of Proposition 3.14, we obtain a pencil $\{\tilde{C}_t \mid t \in \mathbb{C}\mathbb{P}_1\}$ of rational curves on \tilde{S} whose self-intersection numbers are zero. The linear system $\{\tilde{C}_t + \tilde{C}_s \mid t, s \in \mathbb{C}\mathbb{P}_1\}$ induces a rational map $\tilde{S} \rightarrow \mathbb{C}\mathbb{P}_2$ whose image is an irreducible conic, and this factors through S via the minimal resolution. Hence there is a rational map $S \rightarrow \mathbb{C}\mathbb{P}_2$ whose image is an irreducible conic. Since the curves $C_t + C_s$ belong to the class of hyperplane sections, the last projection $S \rightarrow \mathbb{C}\mathbb{P}_2$ is induced from a linear projection $\mathbb{C}\mathbb{P}_4 \rightarrow \mathbb{C}\mathbb{P}_2$ as a restriction of the domain. Let $f(X_0, \dots, X_4)$ and $g(X_0, \dots, X_4)$ be quadratic polynomials that define S , and P and Q the symmetric matrices that correspond to F and G respectively. Then from the presence of the projection to the conic, there has to exist constants a and b such that $\text{rank}(aF + bG) = 3$. Now, using the list of the symbols for all Segre surfaces (see Table 1), it is not difficult to verify that this occurs only when the symbol includes one of (11), (12), (13) and (14). \square

Remark 3.18. From the proof, it follows that the component of \mathbf{D}_S as in Proposition 3.16 is contracted to a conic in S^* by the projection $\pi_2 : I(S) \rightarrow S^*$.

Proposition 3.17 implies the following

Corollary 3.19. *Let $S \subset \mathbb{C}\mathbb{P}_4$ be a Segre surface. The cuspidal locus in S^* is empty if and only if the Segre symbol of S is either [(11)(11)1] or [(12)(11)].*

Proof. These two symbols actually contain two units among (3.9). Conversely, from the list of symbols for all Segre surfaces as in Table 1, all other symbols contain at most one unit among (3.9). \square

Corollary 3.20. *If the symbol of a Segre surface S is among the following list, then the cuspidal locus in S^* is an irreducible surface, and it is birational to S .*

$$(3.10) \quad [111(11)], [12(11)], [11(12)], [1(13)], [(11)3], [(12)2], [(14)].$$

Proof. From the list of symbols for Segre surfaces (Table 1), these seven symbols are the ones which have exactly one unit among (3.9). By Proposition 3.16, the divisor D_S for these Segre surfaces is reducible, and exactly one of the irreducible components is from the pencil of double conics. From the results in the previous subsection, the image of the other irreducible component of D_S by the projection $\pi_2 : I(S) \rightarrow S^*$ is exactly the cuspidal locus in S^* . The projection from that irreducible component of D_S to the cuspidal locus is birational, since the hyperplane section of S determined from a generic point of the component has a single cusp, which means that the projection is generically one-to-one. \square

Segre symbol	Sing (S)	# lines	D_S	culp. locus	x	y
[11111]	none	16 + 0 + 0	irred.	irred.	-	16 + 0 + 0
[1112]	A_1	8 + 4 + 0	irred.	irred.	-	8 + 2 · 4 + 0
[111(11)]	$2A_1$	0 + 8 + 0	red.	irred.	-	0 + 2 · 8 + 0
[12(11)]	$3A_1$	0 + 4 + 2	red.	irred.	2	0 + 2 · 4 + 4 · 2
[1(11)(11)]	$4A_1$	0 + 0 + 4	red.	empty	2	0 + 0 + 4 · 4
[113]	A_2	4 + 4 + 0	irred.	irred.	-	4 + 3 · 4 + 0
[122]	$2A_1$	4 + 4 + 1	irred.	irred.	2	4 + 2 · 4 + 4 · 1
[11(12)]	A_3	0 + 4 + 0	red.	irred.	-	0 + 4 · 4 + 0
[14]	A_3	2 + 3 + 0	irred.	irred.	-	2 + (4 + 4 + 6) + 0
[1(13)]	D_4	0 + 2 + 0	red.	irred.	-	0 + 8 · 2 + 0
[(11)3]	$2A_1 + A_2$	0 + 2 + 2	red.	irred.	3	0 + 2 · 2 + 6 · 2
[(12)2]	$A_1 + A_3$	0 + 2 + 1	red.	irred.	4	0 + 4 · 2 + 8 · 1
[[11](12)]	$2A_1 + A_3$	0 + 0 + 2	red.	empty	4	0 + 0 + 8 · 2
[(14)]	D_5	0 + 1 + 0	red.	irred.	-	0 + 16 · 1 + 0
[23]	$A_1 + A_2$	2 + 3 + 1	irred.	irred.	3	2 + (2 + 2 + 3) + 6 · 1
[5]	A_4	1 + 2 + 0	irred.	irred.	-	1 + (10 + 5) + 0

Table 1. See Remark 3.21 for details.

Remark 3.21. This is a remark for Table 1. For the number of lines, the first number is that of lines disjoint from singularities of S . The second one is the number of lines which pass exactly one singularity of S . The third one

is the number of lines which pass two singularities of S . The x -column is the multiplicity of the component $\pi_1^{-1}(l)$ in the divisor \mathbf{D}_1 , where l is a line passing two singularities of S . These are calculated in Section 6, and are independent of the choice of such a line l . Finally, the y -column shows multiplicities of lines, counted from a realization of each Segre surface as a degeneration from smooth ones. For instance, for the surface with symbol $[12(11)]$, $0 + 2 \cdot 4 + 4 \cdot 2$ means that the multiplicities of the 4 lines which pass one singularity is two, and those of the two lines which pass two singularities are 4. We are not presenting multiplicities of lines which do not pass any singularity, because they are all one. See the remark at the end of this article for more details and as to why we present these multiplicities. \square

From Corollaries 3.19 and 3.20, the structure of the cuspidal locus in the dual variety of a Segre surface is well-understood when the surface admits a pencil of double conics. When the surface does not admit a pencil of double conics, we already know that the cuspidal locus is exactly the image of the divisor \mathbf{D}_S under the projection $\pi_2 : I(S) \rightarrow S^*$. The projection from the divisor \mathbf{D}_S to the cuspidal locus is birational, by the same reason to the similar assertion in Corollary 3.20. On the other hand, \mathbf{D}_S has a structure of generically finite double covering over S by the projection $\pi_1 : \mathbf{D}_S \rightarrow S$. Thus, if S is a Segre surface which does not have a pencil of double conics, then the cuspidal locus in S^* is birational to a double covering over S . In the next section, we discuss the branch divisor of this covering for arbitrary Segre surface S .

4 - Lines on Segre surfaces as branch divisors

We begin with the following

Proposition 4.1. *For any Segre surface S , every component of the branch divisor of the generically finite double covering $\mathbf{D}_S \rightarrow S$ is a line lying on S .*

Proof. By Proposition 3.9, the function \mathbf{H} on $I(S)$ which defines the divisor \mathbf{D} does not vanish identically on the fiber over a point $p \in S_{\text{reg}}$ if p is not on a line on S , and moreover, the quadratic polynomial $\mathbf{H}(p)$ does not have a double root. Let (p, H_1) and (p, H_2) be the two points corresponding to two roots of the equation $\mathbf{H}(p) = 0$. Then we have $(p, H_1), (p, H_2) \in \mathbf{D}$ but these do not belong to \mathbf{D}_1 because $\mathbf{H}(p) \neq 0$ as above. Hence $(p, H_1), (p, H_2) \in \mathbf{D}_S$. Since $H_1 \neq H_2$, this means that p is not a branch point of the projection $\mathbf{D}_S \rightarrow S$. \square

We note that the proposition does not assert that a line on S is always a branch divisor. Indeed, as mentioned in Section 2.1, a sufficient condition for a line l on S to be a branch divisor is that not all three coefficient functions of \mathbf{H}

vanish identically on l . If this condition is not satisfied, then l is a branch divisor iff the discriminant of the quadratic function \mathbf{H}/y^m vanishes on l identically, where $y = 0$ is a local equation of the line l on S , and m is the minimal vanishing order of the three functions along l . In this sections, we determine whether a line on S is really a branch divisor of the generically finite double covering $D_S \rightarrow S$. Conclusions will vary depending on the number of singularities of S belonging to the line.

4.1 - The case $l \cap \text{Sing}(S) = \emptyset$

We begin with some property for lines on S which do not pass any singularity of S . The number of such lines on each Segre surface is displayed in Table 1. (See Remark 3.21.)

Lemma 4.2. Let $S \subset \mathbb{C}\mathbb{P}_4$ be a Segre surface and l a line lying on S . Suppose $l \cap \text{Sing}(S) = \emptyset$. Then there exists no hyperplane $H \subset \mathbb{C}\mathbb{P}_4$ such that the divisor $H|_S$ includes the double line $2l$ as a sub-divisor. Further, if a point p on l is not on another line on S , then there exists no hyperplane section which has a triple point at p .

Proof. For the first assertion, take any hyperplane H containing l , and write $H|_S$ in the form $l + D$ with D a cubic curve. It suffices to show that D does not include the line l as a component. The indeterminacy of the projection from the line l is eliminated by blowing up $\mathbb{C}\mathbb{P}_4$ along l , and we obtain a morphism from the blowup $\tilde{\mathbb{C}}\mathbb{P}_4$ to $\mathbb{C}\mathbb{P}_2$. As $l \cap \text{Sing}(S) = \emptyset$, the strict transform \tilde{S} of S into $\tilde{\mathbb{C}}\mathbb{P}_4$ is isomorphic to S . Let $\phi : \tilde{S} \rightarrow \mathbb{C}\mathbb{P}_2$ be the restriction of the morphism $\tilde{\mathbb{C}}\mathbb{P}_4 \rightarrow \mathbb{C}\mathbb{P}_2$ to \tilde{S} . Under the identification $\tilde{S} \simeq S$, ϕ is naturally identified with the map from S induced by a 2-dimensional subsystem of $|D|$, and this subsystem is base point free. Since l is a (-1) -curve from Proposition 3.1, we have

$$(4.1) \quad D^2 = (K_S^{-1} - l)^2 = 4 - 2 - 1 = 1 \quad \text{and} \quad D.l = (K_S^{-1} - l).l = 1 - (-1) = 2.$$

The former means that the morphism $\phi : \tilde{S} \rightarrow \mathbb{C}\mathbb{P}_2$ is birational. Hence the restriction $\phi|_l$ cannot be of degree-two, and hence, from the latter of (4.1), the image $\phi(l)$ is a conic. This means that no member of the above 2-dimensional subsystem of $|D|$ can contain l as a component.

For the second assertion, suppose that a point $p \in l$ is not on another line on S , and let $H|_S$ be a hyperplane section which has a triple point at p . Then the hyperplane H has to contain $T_p S$. Hence we may write $H|_S = l + D$ as

before, where D is a possibly reducible cubic curve which has at least a double point at p . If D is reducible or non-reduced, then D has to include a line l' as a component. We have $l' \neq l$ from what we have already proved in this proof, and also we have $p \notin l'$ as we are assuming that p is not on another line on S . This means that the curve $D - l'$ is a conic which has a double point at p . So $D - l'$ consists of two lines and both of them pass p . This again contradicts the assumption that p is not on another line on S . Therefore the cubic curve D has to be non-reduced, irreducible, and singular. This means that D is planer, and contradicts Proposition 3.3. Therefore no hyperplane section of S can have a triple point at p . \square

By using this lemma, we next show:

Proposition 4.3. *Let $S \subset \mathbb{C}\mathbb{P}_4$ be a Segre surface. If l is a line on S that does not intersect $\text{Sing}(S)$, then $\pi_1^{-1}(l) \not\subset \mathbf{D}_1$ holds, and l is a simple branch divisor of the generically finite double covering $\mathbf{D}_S \rightarrow S$.*

Proof. First we show that if l is a line as in the proposition, then for any point $p \in l$ which is not on another line on S , any member of the residual pencil

$$(T_p S)^*|_S - l = \{D - l \mid D \in (T_p S)^*|_S\}$$

is smooth at p , and any two different members of the same pencil intersect each other transversely at p . The former follows immediately from Lemma 4.2, because a member of the residual pencil which is singular at p gives a hyperplane section which has a triple point at p . For the latter, let $\mu : S' \rightarrow S$ be a blowup at p , and E the exceptional curve. Again by Lemma 4.2, the restriction homomorphism

$$(4.2) \quad r_E : H^0(\mu^*H - 2E) \longrightarrow H^0(\mathcal{O}_E(2))$$

is injective, and therefore, has a 2-dimensional image. Let l' be the strict transform of l into S' . Any element of the image of r_E vanishes at the point $E \cap l'$. So this point is a base point of the pencil on E formed by the image of r_E . These mean that members of the pencil $(T_p S)^*|_S - l$ are naturally in one-to-one correspondence with points on E . Therefore, any two different members of the pencil $(T_p S)^*|_S - l$ intersect each other transversely at p .

We still fix any point $p \in l$ which is not on another line on S . Since S is smooth at p and l is a line, we can choose non-homogeneous coordinates (x, y, z, w) on $\mathbb{C}\mathbb{P}_4$, centered at the point p , such that

$$(4.3) \quad T_p S = \{z = w = 0\} \quad \text{and} \quad l = \{y = z = w = 0\}$$

hold. The pair (x, y) can be used as coordinates on S around p . As in Section 2.1, there exists a pair of holomorphic functions $F(x, y)$ and $G(x, y)$ such that

$$(4.4) \quad S = \{(x, y, z, w) \mid z = F(x, y), w = G(x, y)\}$$

holds around p . Then $F(x, y) = 0$ and $G(x, y) = 0$ are local equations around p of generators of the pencil $(T_p S)^*|_S$. Let $\mathfrak{m} = (x, y)$ be the maximal ideal at p . Then we have $F, G \in \mathfrak{m}^2$, and also $F(x, 0) = 0$ and $G(x, 0) = 0$ hold since $l \subset S$. So we can write $F = yf$ and $G = yg$ for some $f, g \in \mathfrak{m}$. From the property on members of the residual pencil $(T_p S)^*|_S - l$ obtained in the first part of the present proof, we may suppose that the curve $\{f = 0\}$ intersects l transversely at p . This condition implies $f_x(0, 0) \neq 0$. We consider the map

$$(x, y) \longmapsto (\tilde{x}, \tilde{y}) := (f(x, y), y)$$

defined in a neighborhood of p in S . As $f_x(0, 0) \neq 0$, the Jacobian of this map does not vanish at the origin, and hence we can use (\tilde{x}, \tilde{y}) as holomorphic coordinates in a neighborhood of p . For simplicity of presentation, we redefine x to be the above \tilde{x} , so that we have $F = xy$, and we still write $G = yg$, $g \in \mathfrak{m}$. Then by replacing G by $G - cF$ for a constant c to eliminate the term xy from $G = yg$, we may suppose that the leading term of G in a Taylor expansion at p is of the form ay^2 for some constant a . We have $a \neq 0$ since otherwise the zero of G would have a triple point at p , which contradicts Lemma 4.2. Thus, an expansion of $g (= G/y)$ at p is of the form

$$(4.5) \quad ay + h(x, y) \quad \text{with } h \in \mathfrak{m}^2 \text{ and } a \neq 0.$$

Now recall from (2.3) that a defining equation of the divisor $D = D_1 + D_S$ on $I(S)$ is given by

$$(4.6) \quad \text{Hess}(F)\lambda^2 + (F_{xx}G_{yy} + G_{xx}F_{yy} - 2F_{xy}G_{xy})\lambda\mu + \text{Hess}(G)\mu^2 = 0.$$

Thanks to the above change of coordinates, we have $\text{Hess}(F) = -1 \neq 0$, and this means that $\pi_1^{-1}(l) \not\subset D_1$. We recall from (2.5) that the discriminant of (4.6) is given by

$$(4.7) \quad (F_{xx}G_{yy} - G_{xx}F_{yy})^2 + 4(F_{xx}G_{xy} - F_{xy}G_{xx})(F_{yy}G_{xy} - F_{xy}G_{yy}).$$

In the present situation, it is easy to see that this is written as

$$(4.8) \quad 4G_{xx}G_{yy} = 4yg_{xx}(2g_y + yg_{yy}).$$

This means that the line $l = \{y = 0\}$ is included in the zero divisor of the discriminant (4.7) with multiplicity at least one. If this multiplicity is strictly

greater than one, from (4.8), we obtain a divisibility $y \mid g_{xx}$ or $y \mid g_y$. The former happens iff the Taylor expansion of g at the origin is of the form

$$b_1 + b_2x + b_3y + y\tilde{h}(x, y),$$

with $\tilde{h} \in \mathfrak{m}$ and $b_1, b_2, b_3 \in \mathbb{C}$. Comparing this with (4.5), we obtain $b_1 = b_2 = 0$ and $b_3 = a$. This means $g = ay + y\tilde{h} = y(a + \tilde{h})$. Hence $y \mid g$. Therefore $y^2 \mid G$ holds, which implies that the hyperplane section $\{w = 0\}|_S$ contains the double line $2l$. This contradicts Lemma 4.2. Hence we have $y \nmid g_{xx}$. Furthermore, again from (4.5), we have $g_y(0, 0) \neq 0$. Hence we have $y \nmid g_y$. Thus we have shown that the discriminant (4.8) is not divisible by y^2 . Hence the line l is included in the zero divisor of the discriminant with multiplicity precisely one. This means that l is a simple branch divisor of $D_S \rightarrow S$. \square

We postpone an immediate implication from Proposition 4.3 until Section 5.

4.2 - The case where $l \cap \text{Sing}(S)$ is a single point

In this subsection, we consider lines on Segre surfaces to which only one singular point of the surface belongs. Again, the number of such lines on each Segre surface is displayed in Table 1. (See Remark 3.21.) Each of these lines are not Cartier divisors on the surfaces. We begin with the following lemma which is analogous to Lemma 4.2.

Lemma 4.4. *Let $S \subset \mathbb{CP}_4$ be a Segre surface and l a line on S . If $l \cap \text{Sing}(S)$ consists of one point, then there exists a unique hyperplane H such that the divisor $H|_S$ is of the form $2l + l' + l''$, where l' and l'' are possibly identical lines which are different from l . Further, if a point $p \in l \cap S_{\text{reg}}$ does not belong to any other line on S , then there exists no hyperplane section which has a triple point at p .*

Proof. Let $\mu : \tilde{S} \rightarrow S$ be the minimal resolution of the unique singularity of S lying on l , E the exceptional divisor, and \tilde{l} the strict transform of l into \tilde{S} . Of course, E is not necessarily irreducible. Let Z be the fundamental cycle in the exceptional divisor E , so that if \mathfrak{m} is the maximal ideal at the singularity, for a generic element $f \in \mathfrak{m}$, the function μ^*f vanishes on E with the same multiplicity as Z on each component of E . Then as in the same way to showing the property $\mu^*\mathfrak{m} \simeq \mathcal{O}(-Z)$ [2, III (3.8) Proposition], we have

$$\mu^*(K_S^{-1} \otimes \mathcal{O}_S(-l)) \simeq K_{\tilde{S}}^{-1} \otimes \mathcal{O}_{\tilde{S}}(-\tilde{l} - Z).$$

Let $\tilde{\mathbb{CP}}_4 \rightarrow \mathbb{CP}_4$ be a blowup along l , and $S' \subset \tilde{\mathbb{CP}}_4$ the strict transform of S . This time the projection $S' \rightarrow S$ is not an isomorphism, but the minimal

resolution μ factors through the morphism $S' \rightarrow S$. This means that the system $|K_{\tilde{S}}^{-1} - \tilde{l} - Z|$ does not have a base point. Using $\tilde{l}^2 = -1$ from Proposition 3.1 as well as the properties $Z^2 = -2$ and $Z \cdot \tilde{l} = 1$, we calculate

$$\begin{aligned} (K_{\tilde{S}}^{-1} - \tilde{l} - Z)^2 &= (K_S^{-1})^2 + \tilde{l}^2 + Z^2 - 2K_{\tilde{S}}^{-1} \cdot \tilde{l} + 2\tilde{l} \cdot Z - 2K_{\tilde{S}}^{-1} \cdot Z \\ &= 4 + (-1) + (-2) - 2 \cdot 1 + 2 \cdot 1 - 2 \cdot 0 = 1, \\ (K_{\tilde{S}}^{-1} - \tilde{l} - Z) \cdot \tilde{l} &= 1 - (-1) - 1 = 1. \end{aligned}$$

If $\phi : \tilde{S} \rightarrow \mathbb{C}P_2$ means the morphism induced by the system $|K_{\tilde{S}}^{-1} - \tilde{l} - Z|$, these imply that ϕ is birational and the image $\phi(\tilde{l})$ is a line. Thus, the divisor $\phi^{-1}(\phi(l))$ is the unique member of $|K_{\tilde{S}}^{-1} - \tilde{l} - Z|$ which has \tilde{l} as a component. The multiplicity of \tilde{l} in this member is precisely one since ϕ is birational. Adding $\tilde{l} + Z$ to $\phi^{-1}(\phi(l))$ and projecting it to S by μ , we obtain the unique hyperplane section of S which includes the line l with multiplicity precisely two. Write this hyperplane section as $2l + C$, where C is a conic. This conic has to be reducible since otherwise C will move in S from Remark 3.2, which contradicts the uniqueness just proved. Therefore C consists of two lines, and the unique hyperplane section is of the form $2l + l' + l''$ with $l' \neq l$ and $l'' \neq l$ as in the lemma.

For the latter assertion of the lemma, let $p \in l \cap S_{\text{reg}}$ be as in the lemma, and suppose that $H|_S$ is a hyperplane section which has a triple point at p . The hyperplane section $2l + l' + l''$ has a triple point, but the triple point is on a line different from l . So we have $H|_S \neq 2l + l' + l''$. Hence, from the above uniqueness, $H|_S$ is of the form $l + D$, where D is a cubic curve which does not contain l as a component. D has at least a double point at p . If D is reducible, it contains a line, which does not pass p . But then the residual conic $D - l$ cannot have a double point at p . Hence D is an irreducible singular cubic curve. Such a curve is planar. This contradicts Proposition 3.3. Therefore, no hyperplane section has a triple point at p . \square

By using this lemma, we show an analogous result to Proposition 4.3.

Proposition 4.5. *Let $S \subset \mathbb{C}P_4$ be a Segre surface and l a line on S . If $l \cap \text{Sing}(S)$ consists of one point, then $\pi_1^{-1}(l) \not\subset \mathbf{D}_1$ holds, and l is a branch divisor of the generically finite double covering $\mathbf{D}_S \rightarrow S$ with multiplicity at least two.*

Proof. The idea of the proof is similar to Proposition 4.3. First we show that if the line l is as in the proposition, then for any point $p \in l \cap S_{\text{reg}}$ which is not on another line on S , we can choose two generators of the residual pencil

$(T_p S)^*|_S - l$ in such a way that one of them has the line l as the unique component which passes p , while the other one has a component which intersects l transversely at p as the unique component which passes p .

First let $H_1|_S$ be the unique hyperplane section which includes the double line $2l$ as in the first half of Lemma 4.4. By the assumption that p does not belong to another line on S and the latter half of Lemma 4.4, there does not exist a hyperplane section of S which has a triple point at p . Hence the residual curve $H_1|_S - 2l$ does not pass p . Let H_2 be any element of the pencil $(T_p S)^*$ which is different from H_1 . Then since the multiplicity of $H_2|_S$ at p is precisely two by Lemma 4.4, the curve $H_2|_S - l$ has a unique component which passes p , and it is smooth at p . If this component were tangent to l at p , by subtracting a constant multiple of a defining equation of H_1 from that of H_2 , there would exist a hyperplane $H \in (T_p S)^*$ for which $H|_S$ has a triple point at p . This contradicts Lemma 4.4. Hence the component always intersects l transversely at p . Thus we have obtained generators of the pencil $(T_p S)^*|_S - l$ which satisfy the above properties.

As in the proof of Proposition 4.3, we can choose non-homogeneous coordinates (x, y, z, w) centered at p , such that the properties (4.3) hold. Let $F(x, y)$ and $G(x, y)$ be a pair of holomorphic functions such that (4.4) holds. We may suppose that the hyperplane $\{z = 0\}$ is exactly the one whose section contains the double line $2l$. Then we have the divisibility $y^2 | F$ and $y | G$. Further, if we write $G = yg(x, y)$, then g belongs to the maximal ideal \mathfrak{m} at p . Moreover, we have $g_x(0, 0) \neq 0$ from the above transversality for the component different from l . Hence in the same way to the proof of Proposition 4.3, we may use $(g(x, y), y)$ as coordinates on S around p instead of (x, y) . Again we redefine x as $g(x, y)$, so that $G = xy$. We remain to write $F = y^2 f$. Then we have $f \notin \mathfrak{m}$ since all components of $\{z = 0\} \cap S$ passing p are $2l$. From $G = xy$, we have $\text{Hess}(G) = -1 \neq 0$, and hence again from (4.6), we have $\pi_1^{-1}(l) \notin \mathbf{D}_1$. Moreover, this time from $F = y^2 f$, we obtain that the discriminant (4.7) is given by

$$(4.9) \quad 4F_{xx}F_{yy} = 4y^2 f_{xx}(2f + 4yf_y + y^2 f_{yy}).$$

Since this is divisible by y^2 , we obtain that the generically finite double covering $\mathbf{D}_S \rightarrow S$ has the line l as a branch divisor and the multiplicity of l as a branch divisor is at least two. \square

We remark that since the function f_{xx} in (4.9) may be divisible by y or its power, we cannot conclude that the multiplicity of the line l as a branch divisor is exactly two. Note that $2f + 4yf_y + y^2 f_{yy}$ is not divisible by y since f is not. Since quadric polynomials which define a Segre surface are concretely obtained, it might be possible to obtain the functions F and G in explicit forms

in the present circumstance, and by using them to determine the multiplicity of the line as a branch divisor, for each Segre surface and each line of the present kind. In Section 6, we will do that sort of calculations for lines on Segre surfaces which pass two singularities of the surfaces.

4.3 - The case where $l \cap \text{Sing}(S)$ consists of two points

Let l be a line on a Segre surface S which passes two singular points of S . In this subsection, we investigate the structure of hyperplane sections of S which belong to the pencil $(T_p S)^*|_S$ for some point $p \in l \cap S_{\text{reg}}$. Note that since S is quartic, no three singularities of S belong to the same line lying on S . Note also that the line joining two singularities of S does not necessarily lie on S . We begin with the following property which is characteristic to the present kind of lines.

Lemma 4.6. *Let $S \subset \mathbb{CP}_4$ be a Segre surface and l a line on S . If $l \cap \text{Sing}(S)$ consists of two points, then the tangent plane $T_p S$ is independent of the choice of a point $p \in l \cap S_{\text{reg}}$. In particular, for any such a point p and any hyperplane $H \in (T_p S)^*$, the section $H|_S$ includes the double line $2l$ as a sub-divisor.*

Proof. Let p_1 and p_2 be the two singularities of S on l , and $\mu : \tilde{S} \rightarrow S$ the minimal resolution of p_1 and p_2 . Let \tilde{l} be the strict transform of l into \tilde{S} , and Z_1 and Z_2 the fundamental cycles over p_1 and p_2 respectively. Then in the same way to the proof of Lemma 4.4, we have

$$\mu^*(K_S^{-1} \otimes \mathcal{O}_S(-l)) \simeq K_{\tilde{S}}^{-1} \otimes \mathcal{O}_{\tilde{S}}(-\tilde{l} - Z_1 - Z_2),$$

and the system $|K_{\tilde{S}}^{-1} - \tilde{l} - Z_1 - Z_2|$ does not have a base point. Similarly, we also have $(K_{\tilde{S}}^{-1} - \tilde{l} - Z_1 - Z_2)^2 = 1$, but this time we have

$$(K_{\tilde{S}}^{-1} - \tilde{l} - Z_1 - Z_2) \cdot \tilde{l} = 1 - (-1) - 1 - 1 = 0.$$

If $\phi : \tilde{S} \rightarrow \mathbb{CP}_2$ means the morphism induced by the 2-dimensional system $|K_{\tilde{S}}^{-1} - \tilde{l} - Z_1 - Z_2|$, these imply that ϕ is a birational morphism and it contracts the curve \tilde{l} to a point. Hence the system $|K_{\tilde{S}}^{-1} - 2\tilde{l} - Z_1 - Z_2|$ (note that \tilde{l} is further subtracted) is a pencil. Adding $2\tilde{l} + Z_1 + Z_2$ to the members of this pencil and projecting them to S by μ , we obtain a pencil of hyperplane sections which contains the double line $2l$ as fixed components. Clearly, this pencil is equal to the pencil $(T_p S)^*|_S$. Since this holds for any $p \in l \cap S_{\text{reg}}$, we obtain

that the pencil $(T_p S)^*|_S$ is independent of a choice of the point $p \in l \cap S_{\text{reg}}$. Hence the tangent plane $T_p S$ is independent of a choice of a point $p \in S_{\text{reg}}$. \square

It is possible to obtain more detailed information about the structure of the hyperplane sections by $H \in (T_p S)^*$, in the situation of Lemma 4.6 as follows. We notice that from Table 1, when two singularities of S belong to a line, the types of them are $\{A_1, A_1\}, \{A_1, A_2\}$ or $\{A_1, A_3\}$.

Lemma 4.7. *Let S and l be as in Lemma 4.6, and $p \in l \cap S_{\text{reg}}$ a point such that no other line on S passes p . Then there exists a unique hyperplane section of S whose multiplicity at p is at least three. Further, we have:*

- *If the singularities on l are $\{A_1, A_1\}$, then such a section is of the form $2l + C$, where C is an irreducible conic that intersects l transversely at p .*
- *If the singularities on l are $\{A_1, A_2\}$, then such a section is of the form $3l + l'$ where l' is a line different from l .*
- *If the singularities on l are $\{A_1, A_3\}$, then such a section is of the form $4l$.*

Proof. Because not every member of the pencil $(T_p S)^*|_S$ has a triple point at p by Lemma 3.6, there exists at most one member of the pencil which has a triple point at p . Let $\mu : S' \rightarrow S$ be blowup at p , E the exceptional curve, and l' the strict transform of l . By Lemma 4.6, any member of the pencil $(T_p S)^*|_S$ contains the double line $2l$. So the strict transform to S' of any member of the pencil $(T_p S)^*|_S$ has at least a double point at $E \cap l'$. This implies that the image of the restriction homomorphism

$$r_E : H^0(\mu^* H - 2E) \longrightarrow H^0(\mathcal{O}_E(2))$$

is at most 1-dimensional. Further the kernel of r_E is at most 1-dimensional from Lemma 3.6. Hence r_E has exactly 1-dimensional kernel. This means that the pencil $(T_p S)^*|_S$ has a unique member whose multiplicity at p is at least three.

Next in order to show the latter half of the proposition, we change the meaning of the above notations and let $\mu : \tilde{S} \rightarrow S, \tilde{l}, Z_1, Z_2$ have the same meaning as in the proof of Lemma 4.6. Further, for brevity, we put $L := K_{\tilde{S}}^{-1} - \tilde{l} - Z_1 - Z_2$, and let $\phi : \tilde{S} \rightarrow \mathbb{C}P_2$ again mean the morphism induced by the 2-dimensional linear system $|L|$. Recall that ϕ is birational and contracts \tilde{l} to a point.

First suppose that both singularities on l are A_1 -points as in the first item in the lemma. Then each of the two divisors Z_1 and Z_2 is just a single (-2) -curve respectively. We readily have

$$L \cdot Z_1 = L \cdot Z_2 = 1.$$

Therefore both $\phi(Z_1)$ and $\phi(Z_2)$ are lines. By the factorization theorem of birational morphism between smooth surfaces, the morphism ϕ is a composition of blowups. We have $\tilde{l}^2 = -1$ from Proposition 3.1. Further, since \tilde{S} is a weak del Pezzo surface, there does not exist a smooth rational curve on \tilde{S} whose self-intersection number is less than (-2) . From these, we can readily show that no curve intersecting \tilde{l} is contracted by ϕ . Hence, in a neighborhood of \tilde{l} , ϕ is exactly the blow down of the (-1) -curve \tilde{l} . This implies that for any line $\ell \subset \mathbb{C}\mathbb{P}_2$ through the point $\phi(\tilde{l})$, the preimage $\phi^{-1}(\ell)$ contains the curve \tilde{l} by multiplicity precisely one. Hence no member of the system $|L|$ contains \tilde{l} with multiplicity strictly greater than one. Therefore no member of the system $|K_S^{-1}|$ contains \tilde{l} by multiplicity strictly greater than two. So we may write the unique member of the pencil $(T_p S)^*|_S$ which has a triple point at the point p as $2l + C$, where C is a conic which goes through p . Because we are assuming non-existence of a line through p , the conic C cannot be reducible. If C is tangent to l at the point p , then the curve $l + C$ would lie on the 2-plane which is spanned by C , and this contradicts Proposition 3.3. Hence the conic C intersects l transversely at a unique point. This proves the first item in the proposition.

Next suppose that one of the two singularities on l is an A_2 -point as in the second item of the proposition. We may suppose that the fundamental cycle Z_1 is over the A_2 -point, and we write it as $Z_1 = E_1 + E_2$, where E_2 is the component which intersects \tilde{l} . The cycle Z_2 is a single (-2) -curve. We readily have

$$L.\tilde{l} = L.E_2 = 0, \quad L.E_1 = L.Z_2 = 1.$$

These mean that not only \tilde{l} but also E_2 are contracted to a point by the birational morphism $\phi : \tilde{S} \rightarrow \mathbb{C}\mathbb{P}_2$, and E_1 and Z_2 are mapped to lines by ϕ . Further we have $\phi(E_1) \neq \phi(Z_2)$ because ϕ is birational. Furthermore, in a similar way to the case of $\{A_1, A_1\}$, using $\tilde{l}^2 = -1$ and $E_2^2 = -2$, we can show that in a neighborhood of the chain $\tilde{l} + E_2$, the birational morphism ϕ contracts exactly these two curves. From the above self-intersection numbers, ϕ blows down \tilde{l} first and next E_2 . This implies that the member $\phi^{-1}(\phi(Z_2)) \in |L|$ contains \tilde{l} by multiplicity precisely two. Adding $\tilde{l} + Z_1 + Z_2$ to it and taking the image by $\mu : \tilde{S} \rightarrow S$, we obtain a hyperplane section of S which is of the form $3l + l'$, where l' is a line different from l . Thus we have obtained the second item in the proposition.

Finally, suppose that one of the two singularities on l is an A_3 -point, and assume that the chain $Z_1 = E_1 + E_2 + E_3$ is the fundamental cycle over this singularity, with the component E_3 (resp. E_2) intersecting \tilde{l} (resp. E_3). Then this time we have

$$L.\tilde{l} = L.E_2 = L.E_3 = 0, \quad L.E_1 = L.Z_2 = 1.$$

These mean that the chain $\tilde{l} + E_3 + E_2$ is contracted to a point by $\phi : \tilde{S} \rightarrow \mathbb{CP}_2$, and E_1 and Z_2 are mapped to lines by ϕ . Further these lines are mutually different by the same reason to the A_2 -case. Since $\tilde{l}^2 = -1$ and $E_3^2 = E_2^2 = -2$, the birational morphism ϕ blows down components of the chain $\tilde{l} + E_3 + E_2$ in this order. It follows that the member $\phi^{-1}(\phi(Z_2)) \in |L|$ contains \tilde{l} by multiplicity precisely three. Adding $\tilde{l} + Z_1 + Z_2$ to it and taking the image by the minimal resolution $\mu : \tilde{S} \rightarrow S$, we obtain a hyperplane section of S which is of the form $4l$. Thus, we obtain the third item in the proposition. \square

Note that in the proof for the second and third items, absence of a line through the point p is not used.

By using Lemma 4.6, we prove the following result about the divisor \mathbf{D}_1 .

Proposition 4.8. *Let $S \subset \mathbb{CP}_4$ be a Segre surface and l a line on S . If $l \cap \text{Sing}(S)$ consists of two points, then $\pi_1^{-1}(l)$ is a component of the divisor \mathbf{D}_1 , and its multiplicity is at least two.*

Proof. Again an idea of the proof is similar to Proposition 4.3. Pick any point $p \in l \cap S_{\text{reg}}$. We choose homogeneous coordinates (x, y, z, w) on \mathbb{CP}_3 centered at p , such that the conditions (4.3) hold, and let $F(x, y)$ and $G(x, y)$ be holomorphic functions around p such that (4.4) holds. By Lemma 4.6, we can write $F = y^2 f$ and $G = y^2 g$ for some holomorphic functions f and g around p . Then by elementary calculations, we have

$$y^2 \mid \text{Hess}(F), \quad y^2 \mid \text{Hess}(G) \quad \text{and} \quad y^2 \mid F_{xx}G_{yy} + G_{xx}F_{yy} - 2F_{xy}G_{xy}.$$

Thus all coefficient functions of the defining function \mathbf{H} in (2.3) of the divisor \mathbf{D} on $I(S)$ are divisible by y^2 , and this implies that $\pi^{-1}(l)$ is contained in the divisor \mathbf{D}_1 by multiplicity at least two. \square

We remark that when both of the two singularities on the line l are A_1 -points as in the first item in Lemma 4.7, by making use of the unique hyperplane section which has a triple point at $p \in l \cap S_{\text{reg}}$, it is possible to show that the multiplicity of the component $\pi_1^{-1}(l)$ in \mathbf{D}_1 is precisely two, in a similar way to Propositions 4.3 and 4.5. But if one of the two singularities on the line l is not an A_1 -point, Lemma 4.7 does not give an element of the pencil $(T_p S)^*|_S$ whose equation can be assumed to be in a sufficiently informative form (like xy) as in the cases of Propositions 4.3 and 4.5, and by this reason we cannot determine the precise multiplicity of the component $\pi_1^{-1}(l)$ in \mathbf{D}_1 . By a similar reason, it seems more difficult to determine whether such a line l is really a branch divisor of the generically finite double covering $\mathbf{D}_S \rightarrow S$. In Section 6, we determine the precise multiplicity of $\pi^{-1}(l)$ by using explicit defining polynomials of Segre surfaces, and show that the lines in question are *never* branch divisors of the double covering.

5 - Conclusions, and singularity of the cuspidal locus

5.1 - Irreducibility of the cuspidal locus

Let S be a Segre surface. From Corollaries 3.19 and 3.20, we already know that the cuspidal locus in S^* is empty if the Segre symbol of S is either $[(11)(11)1]$ or $[(11)(12)]$, and that the cuspidal locus is irreducible and birational to S if the symbol contains exactly one of the four units (11), (12), (13) and (14). For the remaining Segre surfaces, it is already easy to show the following conclusion.

Proposition 5.1. *If the Segre symbol of a Segre surface S is among the seven ones listed in (5.1), then the divisor \mathbf{D}_S and the cuspidal locus in S^* are irreducible, and they are mutually birational. In particular, the cuspidal locus is birational to a double covering over S .*

$$(5.1) \quad [11111], [1112], [113], [122], [14], [23], [5].$$

Proof. The seven symbols (5.1) are characterized by the property that they do not contain any units among (11), (12), (13) and (14). By Proposition 3.17, this condition is equivalent to absence of a pencil of double conics on the Segre surface. From Proposition 3.12 and the results in Section 3.3, absence of a pencil of double conics means that whole of the divisor \mathbf{D}_S is mapped precisely to the cuspidal locus in S^* . Further, as presented in Table 1, the Segre surfaces having these seven symbols are exactly the ones which have a line that does not pass any singularity of the surfaces. By Proposition 4.3, these lines are simple branch divisors of the generically finite double covering $\mathbf{D}_S \rightarrow S$. Therefore the divisor \mathbf{D}_S is irreducible for these seven kinds of Segre surfaces.

It remains to show that the projection from \mathbf{D}_S to the cuspidal locus in S^* is birational. But this is immediate from the fact that for a generic point (p, H) of the divisor \mathbf{D}_S , the hyperplane section $H|_S$ has a single ordinary cusp as its all singularity. \square

Irreducibility of the divisor \mathbf{D}_S for these Segre surfaces are reflected in Table 1.

Combined with Corollaries 3.19 and 3.20, Proposition 5.1 gives the following result.

Corollary 5.2. *The cuspidal locus in the dual variety of any Segre surface is either empty or an irreducible surface.*

For the structure of the cuspidal locus beyond irreducibility for Segre surfaces in Proposition 5.1, the cuspidal locus is birational to the divisor \mathbf{D}_S in

$I(S)$, and D_S has a structure of generically finite double covering over S . From Proposition 4.1, any component of the branch divisor of the double covering is a line on S , and from Propositions 4.3 and 4.5, any line on S which passes at most one singularity of S is actually a branch divisor of the covering. So what is still missing is whether lines through two singular points of S are components of the branch divisor of the covering $D_S \rightarrow S$.

If S is a Segre surface whose symbol is among (5.1), and S has a line passing through two singular points of the surface, then from Table 1, the symbol of S is either [221] or [32]. In Appendix, we will show by explicit calculations using defining equations of Segre surfaces, that the lines through two singular points of the surfaces are *never* a component of a branch divisor of $D_S \rightarrow S$. This implies the following

Proposition 5.3. *Let S be a Segre surface whose symbol is among (5.1). Then the branch divisor of the generically finite double covering $D_S \rightarrow S$ consists of all the lines which pass through at most one singular point of S .*

Next, by using the results obtained so far, we explicitly construct a surface which is birational to the cuspidal locus in S^* for a smooth Segre surface S .

A smooth Segre surface is nothing but a smooth del Pezzo surface of degree four. It is well-known that any smooth del Pezzo surface of degree four can be realized as a 5 points blown up of \mathbb{CP}_2 , where 5 points are in a general position. In the following, instead of using this realization, we make use of a realization of smooth Segre surfaces as 4 points blown up of $\mathbb{CP}_1 \times \mathbb{CP}_1$. This seems more economical than to work under the above realization.

Choose 4 points on $\mathbb{CP}_1 \times \mathbb{CP}_1$ such that no two points are on the same $(1, 0)$ -curve nor on the same $(0, 1)$ -curve, and such that there exists no $(1, 1)$ -curve to which all the 4 points belong. Let $S \rightarrow \mathbb{CP}_1 \times \mathbb{CP}_1$ be a blowing up at these 4 points. Then the anti-canonical system $|K_S^{-1}|$ gives a projective embedding $S \subset \mathbb{CP}_4$ which realizes S as a smooth complete intersection of two quadrics. Thus, S is a smooth Segre surface. If e_1, e_2, e_3 and e_4 are the exceptional curves of the blowup, the following 16 curves are all (-1) -curves on S :

$$(5.2) \quad e_1, e_2, e_3, e_4,$$

$$(5.3) \quad (1, 0) - e_1, (1, 0) - e_2, (1, 0) - e_3, (1, 0) - e_4,$$

$$(5.4) \quad (0, 1) - e_1, (0, 1) - e_2, (0, 1) - e_3, (0, 1) - e_4,$$

$$(5.5) \quad (1, 1) - e_{234}, (1, 1) - e_{134}, (1, 1) - e_{124}, (1, 1) - e_{123},$$

where in the last line, e_{234} means $e_2 + e_3 + e_4$ and so on. All these curves are mapped to lines in \mathbb{CP}_4 isomorphically by the anti-canonical system. By adding

up all these 16 classes, we obtain the class

$$(5.6) \quad (8, 8) - 4e_{1234}.$$

By Propositions 4.3 and 5.3, this is exactly the class of the branch divisor of the generically finite double covering $\mathbf{D}_S \rightarrow S$ for the smooth Segre surface S . The class (5.6) is exactly the class $4K_S^{-1}$. Moreover, since each component of the branch divisor is a line, all their intersections are transverse. Therefore, any singularity of the branch divisor is an A_1 -point. Hence, any singularity of the double covering branched along all lines on S is an A_1 -point. If π_1 denotes the covering map as before, the canonical class of this double covering is given by

$$\pi_1^* \left(K_S + \frac{1}{2} \cdot 4K_S^{-1} \right) \simeq \pi_1^* K_S^{-1}.$$

Since K_S^{-1} is ample, this means that the canonical class of the double covering is big. In particular, we obtain

Proposition 5.4. *The cuspidal locus in S^* of any smooth Segre surface S is an irreducible surface of general type.*

5.2 - Singularities of the cuspidal locus

Finally, we discuss singularities of the cuspidal locus in the dual variety S^* when the Segre surface S has a line that does not go through any singularity of S (namely when the divisor \mathbf{D}_S is irreducible). By Proposition 4.3, such a line l is always a simple branch divisor of the generically finite double covering $\mathbf{D}_S \rightarrow S$. In particular, the divisor \mathbf{D}_S is smooth at generic points of the ramification curve over l . In this subsection, we first identify the image of this ramification curve under the projection π_2 in a concrete form, and next show that the image $\pi_2(\mathbf{D}_S)$, namely the cuspidal locus in S^* , has ordinary cusps along the image of the ramification curve over l . For these purposes, we first prove a few lemmas.

Lemma 5.5. *If a line l on a Segre surface S does not pass any singularity of S , then $\pi_1^{-1}(l) \simeq l \times \mathbb{C}\mathbb{P}^1$ holomorphically.*

Proof. Take any $p \in S_{\text{reg}}$ and let $N_p \simeq \mathbb{C}^2$ denote the fiber of the normal bundle $N_{S_{\text{reg}}/\mathbb{C}\mathbb{P}^4} \rightarrow S_{\text{reg}}$ over the point p . The fiber of the projection $\pi_1 : I(S) \rightarrow S$ over p is identified with $\mathbb{P}(N_p) \simeq \mathbb{C}\mathbb{P}^1$. Therefore we have a natural isomorphism

$$\pi_1^{-1}(l) \simeq \mathbb{P}(N_{S_{\text{reg}}/\mathbb{C}\mathbb{P}^4}|_l).$$

Take a pair of quadrics Q_1 and Q_2 in $\mathbb{C}P_4$ which satisfy $Q_1 \cap Q_2 = S$. Then we have an isomorphism $N_{S_{\text{reg}}/\mathbb{C}P_4} \simeq N_{S_{\text{reg}}/Q_1} \oplus N_{S_{\text{reg}}/Q_2}$. Further, we have $N_{S_{\text{reg}}/Q_1}|_l \simeq [Q_2]|_l \simeq \mathcal{O}(2)$ and similarly $N_{S_{\text{reg}}/Q_2}|_l \simeq \mathcal{O}(2)$. Hence, $N_{S_{\text{reg}}/\mathbb{C}P_4}|_l \simeq \mathcal{O}(2)^{\oplus 2}$. These mean the assertion of the lemma. \square

Lemma 5.6. *Let S and l be as in the previous lemma, and l^* the 2-plane in $\mathbb{C}P_4^*$ formed by hyperplanes which contain the line l . Then we have $\pi_2(\pi_1^{-1}(l)) = l^*$.*

Proof. Suppose $(p, H) \in \pi_1^{-1}(l)$. Then $p \in l$ and $T_p S \subset H$. As $l \subset T_p S$, these mean $l \subset H$. Namely $H \in l^*$. Hence $\pi_2(\pi_1^{-1}(l)) \subset l^*$. To show the reverse inclusion, it suffices to show that the image $\pi_2(\pi_1^{-1}(l))$ is 2-dimensional because l^* is irreducible. If p is any point of l , then the fiber $\pi_1^{-1}(p)$ is identified with a line in $\mathbb{C}P_4^*$ formed by hyperplanes containing $T_p S$. So, if the image $\pi_2(\pi_1^{-1}(l))$ would be 1-dimensional, the tangent plane $T_p S$ has to be independent of a choice of a point $p \in l$. If this is actually the case, any H containing the common 2-plane has to satisfy $H|_S \geq 2l$. But by Lemma 4.2, such a hyperplane does not exist. Hence the tangent plane $T_p S$ really varies as the point p moves on l . \square

The next lemma identifies the map $\pi_1^{-1}(l) \rightarrow l^*$ in a concrete form:

Lemma 5.7. *Let $l \subset S$ be as in Lemma 5.5, so that $\pi_1^{-1}(l) \simeq \mathbb{C}P_1 \times \mathbb{C}P_1$. Then the restriction of the projection $\pi_2 : I(S) \rightarrow S^*$ to the divisor $\pi_1^{-1}(l)$ can be identified with a quotient map from $\pi_1^{-1}(l)$ under the involution on $\pi_1^{-1}(l) \simeq \mathbb{C}P_1 \times \mathbb{C}P_1$ given by the reflection $(x, y) \mapsto (y, x)$.*

Proof. By the previous lemma, the image $\pi_2(\pi_1^{-1}(l))$ is a 2-plane. It is well-known that any degree-two morphism from $\mathbb{C}P_1 \times \mathbb{C}P_1$ to $\mathbb{C}P_2$ is identified with the quotient map under the reflection as in the lemma. Hence it is enough to show that for a generic point $H \in l^*$, there exist precisely two points $p, q \in S_{\text{reg}}$ which satisfy $(p, H) \in \pi_1^{-1}(l)$ and $(q, H) \in \pi_1^{-1}(l)$. The condition $(p, H) \in \pi_1^{-1}(l)$ is equivalent to the conditions $p \in l$ and $T_p S \subset H$, and therefore, the presence of the above two points (p, H) and (q, H) is equivalent to the condition that the hyperplane H is tangent to S at p and q . By [7, Proposition 3.12], if H is generic in the 2-plane l^* , then $H|_S$ is of the form $l + C$, where C is a smooth rational cubic curve intersecting l transversely at exactly two points. Hence H is indeed tangent to S at two points on l . \square

By Lemma 5.7, the ramification divisor of the projection $\pi_1^{-1}(l) \rightarrow l^*$ can be identified with the diagonal of $\pi_1^{-1}(l) \simeq \mathbb{C}P_1 \times \mathbb{C}P_1$, and the branch divisor is a smooth conic in the 2-plane l^* . In the sequel, we denote these ramification divisor and branch divisor by Δ and $\bar{\Delta}$ respectively. The divisor D_S is smooth

at generic points of the ramification divisor Δ . For a generic point (p, H) belonging to Δ , the structure of the hyperplane section $H|_S$ is described as follows.

Lemma 5.8. *Let $(p, H) \in \Delta$ be generic. Then the section $H|_S$ is of the form $l + C$, where C is a rational normal cubic curve which intersects l at a unique point, and which is tangent to l at the point with order two.*

Proof. By the two projections π_1 and π_2 , there are natural isomorphisms $l \simeq \Delta \simeq \overline{\Delta}$. As the conic $\overline{\Delta}$ is the branch divisor of the double covering $\pi_1^{-1}(l) \rightarrow l^*$, a hyperplane H belongs to $\overline{\Delta}$ iff the residual cubic curve $C = H|_S - l$ satisfies the property that $C \cap l$ consists of a single point. Let p be this point. Of course this is a smooth point of S , and $p \in l$ and $H \in \overline{\Delta}$ are identified under the above isomorphism $l \simeq \overline{\Delta}$. To show the lemma, it is enough to show that if the point p does not belong to any other line on S , then the residual curve C satisfies the properties in the proposition.

For intersection numbers, we have

$$(5.7) \quad C.l = (H|_S - l).l = H.l - l.l = 1 - (-1) = 2,$$

where $H.l$ means the intersection number taken in \mathbb{CP}_4 . Since $C \cap l = \{p\}$ as above, this means that C and l do not intersect transversely at p . If C would be reducible, then from the above assumption on the point p , C has to have a smooth conic D as a component and D has to be tangent to l at p . Then the curve $l + D$ lies on a 2-plane, and this contradicts Proposition 3.3. Hence C is an irreducible cubic curve through p which does not intersect transversely at p . Moreover, again by Proposition 3.3, C has to be non-planar. Therefore, C is a non-degenerate irreducible cubic curve in \mathbb{CP}_3 . Hence it has to be a rational cubic curve, and by (5.7), it has to be tangent to l at p with order two. \square

Thus, the singularity of hyperplane sections of S given by generic points of the conic $\overline{\Delta}$ is not an ordinary cusp but a tacnode. So it would be natural to expect that the cuspidal locus is singular along the conic $\overline{\Delta}$. The next result shows that this is actually the case and identify the singularity along $\overline{\Delta}$.

Proposition 5.9. *Let $S \subset \mathbb{CP}_4$ be a Segre surface and $l \subset S$ a line satisfying $l \cap \text{Sing}(S) = \emptyset$. Then the cuspidal locus in S^* has ordinary cusps along generic points of the conic $\overline{\Delta}$ in the 2-plane l^* .*

Proof. Let $H \subset \mathbb{CP}_4$ be a generic hyperplane belonging to the conic $\overline{\Delta}$, so that by the previous proposition, the section $H|_S$ is of the form $l + C$, where l is a line and C is a rational normal cubic that is tangent to l at a point. Let p be the tangent point. Similarly to the situation in Proposition 3.15, the curve

$l + C$ is singular only at p , and the type of the singularity is a tacnode. On this reducible curve, we still have an exact sequence

$$(5.8) \quad 0 \longrightarrow N'_{l+C} \longrightarrow N_{l+C} \longrightarrow T^1_{l+C} \longrightarrow 0.$$

By Proposition 3.1 and Lemma 5.8, we have $l^2 = -1$, $C^2 = 1$, and $l \cdot C = 2$. Hence we obtain

$$(5.9) \quad N_{l+C}|_l \simeq [l + C]|_l \simeq \mathcal{O}_l(1) \quad \text{and} \quad N_{l+C}|_C \simeq [l + C]|_C \simeq \mathcal{O}_C(3).$$

Let \mathcal{J} be the Jacobian ideal sheaf of the curve $l + C$, and write \mathcal{J}_{l+C} for $\mathcal{J}|_{l+C}$. Let $\nu : l \sqcup C \rightarrow l + C$ be the normalization of $l + C$. Then the calculations in the proof of Proposition 3.15 works without any change, and as in (3.6), we obtain

$$(\nu^* \mathcal{J}_{l+C})|_C \simeq \mathcal{O}_C(-2) \quad \text{and} \quad (\nu^* \mathcal{J}_{l+C})|_l \simeq \mathcal{O}_l(-2).$$

From these and (5.9), we obtain

$$\nu^* N'_{l+C} \simeq \mathcal{O}_l(-1) \sqcup \mathcal{O}_C(1),$$

and hence, also an exact sequence

$$(5.10) \quad 0 \longrightarrow N'_{l+C} \longrightarrow \nu_*(\mathcal{O}_l(-1) \sqcup \mathcal{O}_C(1)) \longrightarrow \mathbb{C}_p \longrightarrow 0.$$

From this, we again have

$$H^0(N'_{l+C}) \simeq \mathbb{C} \quad \text{and} \quad H^1(N'_{l+C}) = 0.$$

Hence we again obtain that any first order displacement of the curve $l + C$ in S is unobstructed, and that the versal family of equi-singular displacements of $l + C$ in S is 1-dimensional. From (5.8), we also obtain an exact sequence

$$(5.11) \quad 0 \longrightarrow H^0(N'_{l+C}) \longrightarrow H^0(N_{l+C}) \longrightarrow H^0(T^1_{l+C}) \longrightarrow 0.$$

For the versal deformation of the tacnode, as discussed right before the proof of Theorem 2.3, the versal family is smooth and 3-dimensional. In the notation and argument there, the fiber of the lift of the versal family by the Galois cover $(t_1, t_2, t_3, t_4) \mapsto (s_1, s_2, s_3)$ has an A_2 -singularity iff (t_1, t_2, t_3, t_4) is equal to

$$(t, t, t, -3t), (t, t, -3t, t), (t, -3t, t, t) \text{ or } (-3t, t, t, t)$$

for some $t \neq 0$. Therefore, the fiber of the versal family over a point (s_1, s_2, s_3) has an A_2 -singularity iff it is of the form $(-6t^2, -8t^3, -3t^4)$ for some $t \neq 0$. Similarly to the notations we used in Section 2.2, let $\mathcal{A}_2 \subset \mathbb{C}^3$ be the locus

formed by points in this form. Then the closure of \mathcal{A}_2 has an A_2 -singularity at the origin, since, up to a non-zero constant multiple to each factor, the map $(z_1, z_2) \mapsto (z_1, z_2, z_1^2)$ gives an isomorphism from a curve with an ordinary cusp to the above closure.

As in the proof of Theorem 2.3, take a small neighborhood B of the point $H \in \Delta$ in $\mathbb{C}\mathbb{P}_4^*$, and regard B as a parameter space for displacements of the hyperplane section $l + C$ in S . Then by versality, we have an induced holomorphic map $f : B \rightarrow \mathbb{C}^3$, and the differential $(df)_H$ is identified with the map $H^0(N_{l+C}) \rightarrow H^0(T_{l+C}^1)$ in (5.11). The intersection of B with the cuspidal locus in S^* is exactly $f^{-1}(\mathcal{A}_2 \cup \{H\})$. By the surjectivity of the map $H^0(N_{l+C}) \rightarrow H^0(T_{l+C}^1)$ and the above property about the closure of the locus \mathcal{A}_2 , the cuspidal locus has A_2 -singularities along $f^{-1}(0)$. Since sections of S by hyperplanes belonging to the conic $\bar{\Delta}$ have tacnode by Lemma 5.8, $f^{-1}(0)$ is equal to $\bar{\Delta} \cap B$. Thus, the cuspidal locus has A_2 -singularities along generic points of $\bar{\Delta}$. \square

Proposition 5.9 in particular means that, for a Segre surface S which has a line not passing any singularity of S , the cuspidal locus in S^* is non-normal, and that the projection $\pi_2 : \mathbf{D}_S \rightarrow \pi_2(\mathbf{D}_S)$ gives the normalization along the conic $\bar{\Delta}$.

6 - Appendix. Explicit calculations for the cuspidal locus

Here is a list of Segre symbols for all Segre surfaces that have a line which passes two singularities of the surfaces: A Segre surface with symbol [(11)111]

symbol	[(11)(11)1]	[2(11)1]	[221]	[3(11)]	[32]	[(21)(11)]	[2(21)]
singularities	$4A_1$	$3A_1$	$2A_1$	$A_2 + 2A_1$	$A_1 + A_2$	$A_3 + 2A_1$	$A_1 + A_3$

has two singularities, but it is not included in this list because the line through these points does not lie on the surface.

In this subsection, to each surface in the above list, by making use of the common tangent plane to the surface along the line l connecting two singularities of S (see Lemma 4.6), we calculate the two functions F and G in concrete forms to some extent. Next, we determine the precise multiplicity of the divisor $\pi_1^{-1}(l)$ as a component of \mathbf{D}_1 , and show that the line l is *never* a branch divisor of the generically finite double covering $\mathbf{D}_S \rightarrow S$. Applying this to the cases where the Segre symbol is [32] or [221], this in particular proves Proposition 5.3 about the branch divisor of the covering $\mathbf{D}_S \rightarrow S$.

In the following argument, $(X_0, X_1, X_2, X_3, X_4)$ are homogeneous coordinates on $\mathbb{C}\mathbb{P}_4$. For brevity, we use the symbol e_i ($0 \leq i \leq 4$) to mean the point

of \mathbb{CP}_4 whose unique non-zero entry is the i -th one. For two given functions $F(x, y)$ and $G(x, y)$, we put

$$K := F_{xx}G_{yy} + F_{yy}G_{xx} - 2F_{xy}G_{xy}$$

for simplicity, so that the defining equation of the divisor $\mathbf{D} = \mathbf{D}_1 + \mathbf{D}_S$ in the incidence variety $I(S)$ is given by

$$\mathbf{H} = \text{Hess}(F)\lambda^2 + K\lambda\mu + \text{Hess}(G)\mu^2.$$

The way how one can derive the normal forms for the pair of the defining quadratic polynomials for each type of Segre surface is briefly explained in [7, Section 3.2].

6.1 - Appendix

First we discuss a Segre surface S whose symbol is $[(11)(11)1]$. This would be the easiest case in actual calculations. Let α, β, γ be mutually different complex numbers. The normal forms for the equations of S on \mathbb{CP}_4 are given by

$$\alpha X_0^2 + \alpha X_1^2 + \beta X_2^2 + \beta X_3^2 + \gamma X_4^2 = X_0^2 + X_1^2 + X_2^2 + X_3^2 + X_4^2 = 0.$$

It is more convenient to make an obvious coordinate change and rewrite the equations as

$$\alpha X_0 X_1 + \beta X_2 X_3 + \gamma X_4^2 = X_0 X_1 + X_2 X_3 + X_4^2 = 0.$$

The surface S has exactly four singularities, and all of them are A_1 -points. They are concretely given by e_i with $i = 0, 1, 2, 3$. The involution of \mathbb{CP}_4 which exchanges X_0 and X_1 keeps S invariant, and it exchanges the two A_1 -points e_0 and e_1 . Similarly, the involution of \mathbb{CP}_4 which exchanges X_2 and X_3 keeps S invariant, and it exchanges the two points e_2 and e_3 . Among six lines connecting the four A_1 -points, the four lines $\overline{e_0 e_2}$, $\overline{e_0 e_3}$, $\overline{e_1 e_2}$ and $\overline{e_1 e_3}$ lie on S . The group generated by the above two involutions on S acts transitively on the set of these four lines. So in the following, we choose a line $l := \overline{e_0 e_2} = \{X_1 = X_3 = X_4 = 0\}$ and calculate an equation of the divisor \mathbf{D} in the incidence variety $I(S)$, in a neighborhood of points of $\pi_1^{-1}(l)$. The common tangent plane to S along points on $l \setminus \{e_0, e_2\}$ is given by

$$(6.1) \quad T = \{X_1 = X_3 = 0\}.$$

In the sequel, we work on the open subset $\{X_0 \neq 0\} = \mathbb{C}^4$ and use $x_i = X_i/X_0$, $1 \leq i \leq 4$, as coordinates on it. Then from (6.1), we may use $(x, y) :=$

(x_2, x_4) as coordinates on S around any point of $l \setminus \{e_0, e_2\}$. The two points e_0 and e_2 are excluded since these are singular points of S . On S , we have $l = \{y = 0\}$. From the second equation of S , we obtain

$$(6.2) \quad x_1 = -xx_3 - y^2.$$

Substituting this into the affine form of the first equation of S , we obtain

$$(6.3) \quad (\beta - \alpha)xx_3 + (\gamma - \alpha)y^2 = 0.$$

We put

$$F = x_3 \quad \text{and} \quad G = x_1.$$

From (6.3) and (6.2), we obtain

$$F = \frac{\alpha - \gamma}{\beta - \alpha} \frac{y^2}{x}, \quad G = \frac{\gamma - \beta}{\beta - \alpha} y^2.$$

Thus we have obtained the two functions F and G in explicit forms. (The divisibility of F and G by y^2 follows from Lemma 4.6 from the beginning.) From these, we can compute

$$\text{Hess}(F) = \text{Hess}(G) = 0, \quad K = F_{xx}G_{yy} = \frac{4(\alpha - \gamma)(\gamma - \beta)}{(\beta - \alpha)^2} \frac{y^2}{x^3}.$$

Hence, we obtain

$$\mathbf{H} = K\lambda\mu = \frac{4(\alpha - \gamma)(\gamma - \beta)}{(\beta - \alpha)^2} \frac{y^2}{x^3} \lambda\mu.$$

Since this is divisible precisely by y^2 , we obtain that *the component $\pi_1^{-1}(l)$ is included in \mathbf{D} and \mathbf{D}_1 with multiplicity precisely two*. Also we obtain that the equation of the divisor $\mathbf{D}_S = \mathbf{D} - \mathbf{D}_1$ is simply $\lambda\mu = 0$. This implies that *the generically finite double covering $\mathbf{D}_S \rightarrow S$ does not have the line l as a branch divisor*.

6.2 - Appendix

Next, we discuss a Segre surface S whose symbol is $[2(11)1]$. After making a simple coordinate change as in the previous case, equations of S are given by

$$2\alpha X_0 X_1 + X_1^2 + \beta X_2 X_3 + \gamma X_4^2 = 2X_0 X_1 + X_2 X_3 + X_4^2 = 0,$$

where α, β, γ are again mutually different complex numbers. This surface has exactly three singularities, and they are the points e_0, e_2 and e_3 . All of them are

A_1 -points. The involution of $\mathbb{C}\mathbb{P}_4$ which exchanges X_2 and X_3 keeps S invariant, and it exchanges e_2 and e_3 . The two lines $\overline{e_0e_2}$ and $\overline{e_0e_3}$ are contained in S , and the line $\overline{e_2e_3}$ is not. The first two lines are exchanged by the above involution. In the following, we choose the line $l := \overline{e_0e_2} = \{X_1 = X_3 = X_4 = 0\}$ and calculate an equation of the divisor D around points over $l \setminus \{e_0, e_2\}$. The common tangent plane to S along points on $l \setminus \{e_0, e_2\}$ is given by

$$(6.4) \quad T = \{X_1 = X_3 = 0\}.$$

In the sequel, we again work on the open subset $\{X_0 \neq 0\} = \mathbb{C}^4$ and use $x_i = X_i/X_0$, $1 \leq i \leq 4$, as coordinates on it. We put $(x, y) := (x_2, x_4)$ and use these as local coordinates on S around any point on $l \setminus \{e_0, e_2\}$. On S , the line l is again defined by $y = 0$. From the affine form of the second equation, we obtain

$$(6.5) \quad x_1 = -\frac{1}{2}(xx_3 + y^2).$$

Substituting this into the affine form of the first equation, we obtain

$$(6.6) \quad x^2x_3^2 + 2x\{2(\beta - \alpha) + y^2\}x_3 + y^2\{y^2 + 4(\gamma - \alpha)\} = 0.$$

We put

$$F = x_3, \quad G = -2x_1.$$

Then (6.6) and (6.5) can be written respectively as

$$(6.7) \quad x^2F^2 + 2x\{2(\beta - \alpha) + y^2\}F + y^2\{y^2 + 4(\gamma - \alpha)\} = 0$$

and

$$(6.8) \quad G = xF + y^2.$$

Any hyperplane section of S containing the common tangent plane (6.4) includes $2l$ as a subdivisor, so we may put $F = y^2f$ and $G = y^2g$ for some holomorphic functions f and g . Substituting these into (6.7) and (6.8) and dividing by y^2 , we obtain

$$(6.9) \quad x^2y^2f^2 + 2x\{2(\beta - \alpha) + y^2\}f + \{y^2 + 4(\gamma - \alpha)\} = 0$$

and

$$(6.10) \quad g = xf + 1.$$

Differentiating (6.9) by x and using $\alpha \neq \beta$, we readily obtain a divisibility

$$(6.11) \quad y \mid (f + xf_x).$$

If we write f as $f = u(x) + yv(x, y)$, we have

$$f + xf_x = (u + xu') + y(v + xv_x).$$

From (6.11), this in particular means $u + xu' = 0$. Since the function $f(x, y)$ is known to be holomorphic only in a neighborhood of a point on l which is different from singularities of S , and since the singularity is the origin $(x, y) = (0, 0)$ in the present coordinates, we have to allow a pole for $u = u(x)$ at $x = 0$. From the last equation $u + xu' = 0$, this means $u(x) = \frac{c}{x}$ for some constant c . So we have

$$(6.12) \quad f = \frac{c}{x} + yv(x, y).$$

If $c = 0$, we would obtain $y \mid f$, which means $y^3 \mid F$. This implies that the section of S by the hyperplane $H = \{X_3 = 0\}$ has a triple line $3l$ as component. But this cannot happen because both of the singularities on l are A_1 -points, and from Lemma 4.7, no hyperplane section contains the triple line $3l$. Hence, $c \neq 0$ holds. Differentiating (6.12) twice by x , we obtain

$$(6.13) \quad f_{xx} = \frac{2c}{x^3} + yv_{xx}.$$

This means that f_{xx} is not divisible by y . We will soon use these to determine exact multiplicity of the component $\pi_1^{-1}(l)$ in the divisor D_1 .

From $F = y^2 f$, we easily obtain

$$(6.14) \quad \text{Hess}(F) = y^2 \{f_{xx}(2f + 4yf_y + y^2 f_{yy}) - (2f_x + yf_{xy})^2\},$$

and similarly for $\text{Hess}(G)$, meaning $y^2 \mid \text{Hess}(F)$ and $y^2 \mid \text{Hess}(G)$. From (6.14), it follows that

$$y^3 \mid \text{Hess}(F) \iff y \mid (ff_{xx} - 2f_x^2).$$

From (6.12) and (6.13), we readily obtain

$$ff_{xx} \equiv \frac{2c^2}{x^4} \pmod{y} \quad \text{and} \quad f_x^2 \equiv \frac{c^2}{x^4} \pmod{y}.$$

These mean $y \mid (ff_{xx} - 2f_x^2)$. Hence we obtain $y^3 \mid \text{Hess}(F)$.

Recall that we have written $F_{xx}G_{yy} + F_{yy}G_{xx} - 2F_{xy}G_{xy}$ as K . Next we show $y^2 \mid K$ and $y^3 \nmid K$. We have $F_{xx} = y^2 f_{xx}$, and since $y \nmid f_{xx}$ from (6.13), we have $y^2 \mid F_{xx}$ and $y^3 \nmid F_{xx}$. From (6.10) and (6.12), we readily obtain

$$(6.15) \quad g = c + xyv + 1.$$

If $c = -1$, then $y \mid g$ follows and hence $y^3 \mid G$. Again this contradicts Lemma 4.7. Hence $c \neq -1$. The constant term of G_{yy} is easily seen to be $2(c + 1) \neq 0$. From these, we obtain $y^2 \mid F_{xx}G_{yy}$ and $y^3 \nmid F_{xx}G_{yy}$. Also, from (6.15), we easily obtain $y \mid g_{xx}$. This means $y^3 \mid G_{xx}$ and hence $y^3 \mid F_{yy}G_{xx}$. Further, it is immediate to see $y \mid F_{xy}$, and from (6.15) we also have $y^2 \mid G_{xy}$. Hence $y^3 \mid F_{xy}G_{xy}$. So $y \nmid G_{yy}$. From these, we obtain $y^2 \mid K$ and $y^3 \nmid K$.

Thus, we obtain $y^2 \mid \mathbf{H}$ and $y^3 \nmid \mathbf{H}$. Hence, *the multiplicity of the divisor $\pi_1^{-1}(l)$ in \mathbf{D}_1 is precisely two*. The discriminant of the quadratic polynomial \mathbf{H}/y^2 is equal to

$$(6.16) \quad \left(\frac{K}{y^2}\right)^2 - 4\frac{\text{Hess}(F)}{y^2}\frac{\text{Hess}(G)}{y^2}.$$

The second term of this is divisible by y since $y^3 \mid \text{Hess}(F)$ as above. But the first term is not divisible by y since $y^3 \nmid K$ as above. Hence (6.16) is not divisible by y . Thus we can conclude that *the line l is not a component of the branch locus of the generically finite double covering $\mathbf{D}_S \rightarrow S$* .

6.3 - Appendix

Next we discuss the case where the Segre symbol of S is [221]. The normal forms of the equations of S are given by

$$2\alpha X_0X_1 + X_1^2 + 2\beta X_2X_3 + X_3^2 + \gamma X_4^2 = 2X_0X_1 + 2X_2X_3 + X_4^2 = 0,$$

where α, β, γ are mutually different complex numbers. The two points e_0 and e_2 are A_1 -singularities of S , and these are all singularities of S . The line connecting these two singularities is

$$l = \{X_1 = X_3 = X_4 = 0\},$$

and this is contained in S . The common tangent plane to S along points on $l \setminus \{e_0, e_2\}$ is again given by

$$T = \{X_1 = X_3 = 0\}.$$

In the sequel, we again work on the open subset $\{X_0 \neq 0\} = \mathbb{C}^4$ and use $x_i = X_i/X_0$, $1 \leq i \leq 4$, as coordinates on it. From the equations of T , we may use $(x, y) := (x_2, x_4)$ as local coordinates on S around any point of $l \setminus \{e_0, e_2\}$. On S , the line l is defined by $y = 0$. From the affine form of the second equation of S , we obtain

$$(6.17) \quad x_1 = -xx_3 - \frac{1}{2}y^2.$$

Substituting this into the affine form of the first equation, we readily obtain

$$(6.18) \quad (x^2 + 1)x_3^2 + x\{y^2 - 2(\alpha - \beta)\}x_3 + \frac{1}{4}y^4 + (\gamma - \alpha)y^2 = 0.$$

We put

$$F = x_3 \quad \text{and} \quad G = x_1$$

for the normal directions to S . As in the last case of [2(11)1], we may put $F = y^2f$ and $G = y^2g$. Then (6.18) is divisible by y^2 , and we obtain that the function f is subject to the equation

$$(6.19) \quad (x^2 + 1)y^2f^2 + x\{y^2 - 2(\alpha - \beta)\}f + \frac{1}{4}y^2 + (\gamma - \alpha) = 0.$$

Differentiating this by x and using $\alpha \neq \beta$, we readily obtain the divisibility

$$y^2 \mid (f + xf_x).$$

In the same way to the last case of [2(11)1], this means

$$(6.20) \quad f = \frac{c}{x} + yv(x, y),$$

for some constant c and a holomorphic function v . If $c = 0$, we obtain $y^3 \mid F$, which contradicts Lemma 4.7. So $c \neq 0$. In the same way to the last case of [2(11)1], we obtain from $F = y^2f$ that $y^2 \mid \text{Hess}(F)$, and further,

$$y^3 \mid \text{Hess}(F) \iff y \mid (ff_{xx} - 2f_x^2).$$

Again from (6.20), we obtain that f satisfies the latter condition. Therefore $y^3 \mid \text{Hess}(F)$ holds. On the other hand, from (6.17), we have

$$G = -xF - \frac{1}{2}y^2,$$

which means

$$(6.21) \quad g = -\left(c + \frac{1}{2}\right) - xyv.$$

This implies $y \mid g_x$ and $y \mid g_{xx}$. Hence $y \mid (gg_{xx} - 2g_x^2)$ also holds. So, in the same way for F , we have $y^3 \mid \text{Hess}(G)$.

From $F = y^2f$ we have $F_{xx} = y^2f_{xx}$, and $f_{xx} \equiv \frac{2c}{x^3} \pmod{y}$ from (6.20). Also, from $G = y^2g$, we have $G_{yy} = 2g + 4yg_y + y^2g_{yy}$. From these, we have

$$F_{xx}G_{yy} \equiv \frac{4cg}{x^3}y^2 \pmod{y^3}.$$

In particular, $y^2 \mid F_{xx}G_{yy}$ and $y^3 \nmid F_{xx}G_{yy}$ as $c \neq 0$. Next, from $G = y^2g$, we have $G_{xx} = y^2g_{xx}$, and since $y \mid g_{xx}$ as above, we have $y^3 \mid G_{xx}$. Therefore $y^3 \mid F_{yy}G_{xx}$. Next, $F_{xy} = y(2f_x + yf_{xy})$ and therefore $y \mid F_{xy}$. Further, $G = y^2g$ means $G_x = y^2g_x$ and $y \mid g_x$ as above. So $y^3 \mid G_x$, which means $y^2 \mid G_{xy}$. Therefore $y^3 \mid F_{xy}G_{xy}$. From these, we conclude $y^2 \mid K$ and $y^3 \nmid K$.

These imply that the defining function \mathbf{H} of the divisor \mathbf{D} satisfies $y^2 \mid \mathbf{H}$ and $y^3 \nmid \mathbf{H}$. Hence, *the multiplicity of the divisor $\pi_1^{-1}(l)$ in \mathbf{D}_1 is precisely two.* The discriminant of the quadratic polynomial \mathbf{H}/y^2 is

$$(6.22) \quad \left(\frac{K}{y^2}\right)^2 - 4\frac{\text{Hess}(F)}{y^2}\frac{\text{Hess}(G)}{y^2}.$$

The second term is divisible by y as $y^3 \mid \text{Hess}(F)$. But the first term is not divisible by y as K is not divisible by y^3 as above. Hence (6.22) is not divisible by y . Thus we can conclude that *the line l is not a branch divisor of the generically finite double covering $\mathbf{D}_S \rightarrow S$.*

6.4 - Appendix

Next, we discuss a Segre surface S whose symbol is [3(11)]. After making a simple coordinate change, equations of S are given by

$$2\alpha X_0X_2 + \alpha X_1^2 + 2X_1X_2 + \beta X_3X_4 = 2X_0X_2 + X_1^2 + X_3X_4 = 0,$$

where $\alpha \neq \beta$. This surface has exactly three singularities, and they are e_0, e_3 and e_4 . The point e_0 is an A_2 -point of S , and both e_3 and e_4 are A_1 -points of S . The two lines $\overline{e_0e_3}$ and $\overline{e_0e_4}$ are contained in S , and $\overline{e_3e_4}$ is not. The involution of $\mathbb{C}\mathbb{P}_4$ which exchanges X_3 and X_4 keeps S invariant. It fixes the point e_0 , and exchanges e_3 and e_4 . Hence it exchanges the two lines $\overline{e_0e_3}$ and $\overline{e_0e_4}$. So in the following, we only consider the line

$$l := \overline{e_0e_3} = \{X_1 = X_2 = X_4 = 0\}.$$

The common tangent plane to S at smooth points on this line is given by

$$(6.23) \quad T = \{X_2 = X_4 = 0\}.$$

In the sequel, we again work on the affine open subset $\{X_0 \neq 0\} = \mathbb{C}^4$ in $\mathbb{C}\mathbb{P}_4$, and use $x_i = X_i/X_0$, $1 \leq i \leq 4$, as coordinates on the open subset. From (6.23), we may use $(x, y) := (x_3, x_4)$ as coordinates on S around any point of $l \setminus \{e_0, e_3\}$. On S , the line l is defined by $y = 0$. The affine form of the second defining equation of S is given by

$$(6.24) \quad x_2 = -\frac{1}{2}(y^2 + xx_4).$$

Substituting this into the affine form of the first defining equation of S , we obtain

$$(6.25) \quad y^3 + (y + \alpha - \beta)xx_4 = 0.$$

Thus the surface S is locally identified with the surface in \mathbb{C}^3 defined by this cubic equation. If we substitute $x_4 = 0$ to this equation, we obtain $y^3 = 0$. This means that if H is the hyperplane defined by $X_4 = 0$, then the section $H|_S$ contains the triple line $3l$. This is the unique hyperplane given in Lemma 4.7 for the present surface S . We put

$$F = x_4 \quad \text{and} \quad G = -2x_2,$$

and we also write $\delta = \alpha - \beta$ for short. Note that $\delta \neq 0$ since $\alpha \neq \beta$. Then from (6.25), the function F is subject to the equation

$$y^3 + (y + \delta)xF = 0 \quad \text{i.e.} \quad F = -\frac{y^3}{x(y + \delta)}.$$

Hence from (6.24), we also obtain

$$(6.26) \quad G = \frac{\delta y^2}{y + \delta}.$$

Thus we were able to obtain the functions F and G in explicit forms. From this, we obtain

$$F_{xx} = -\frac{2y^3}{x^3(y + \delta)}, \quad y^2 | F_{xy}, \quad y | F_{yy}.$$

These imply that $\text{Hess}(F)$ is divisible by y^4 . (We just need $y^3 | F$ for obtaining this.) On the other hand, we obtain from (6.26) that

$$G_{xx} = G_{xy} = 0 \quad \text{and} \quad G_{yy} = \frac{2\delta^3}{(y + \delta)^3}.$$

Hence $\text{Hess}(G) = 0$. Further we obtain

$$K = F_{xx}G_{yy} + F_{yy}G_{xx} - 2F_{xy}G_{xy} = -\frac{4\delta^3 y^3}{x^3(y + \delta)^4}.$$

Using $\delta \neq 0$, these mean $y^3 | K$ and $y^4 \nmid K$. Therefore, we obtain

$$\mathbf{H} = y^3 \lambda \left(\frac{\text{Hess}(F)}{y^3} \lambda - \frac{4\delta^3}{x^3(y + \delta)^4} \mu \right).$$

Noting that $y^4 | \text{Hess}(F)$ and $\delta \neq 0$ as above, it follows that *the divisor $\pi_1^{-1}(l)$ is included as a component of \mathbf{D}_1 with multiplicity precisely three, and the line l is not contained in the branch divisor of the generically finite double covering $\mathbf{D}_S \rightarrow S$.*

6.5 - Appendix

Next we discuss the case where the symbol of a Segre surface S is [32]. The normal forms of defining equations of S are given by

$$2\alpha X_0 X_2 + \alpha X_1^2 + 2X_1 X_2 + 2\beta X_3 X_4 + X_4^2 = 2X_0 X_2 + X_1^2 + 2X_3 X_4 = 0.$$

Singularities of S are the point e_0 , which is an A_2 -point, and e_3 , which is an A_1 -point. The line

$$l := \{X_1 = X_2 = X_4 = 0\}$$

passes e_0 and e_3 and is contained in S . The common tangent plane to S along points of $l \setminus \{e_0, e_3\}$ is given by

$$(6.27) \quad T = \{X_2 = X_4 = 0\}.$$

Again we work on the affine open subset $\{X_0 \neq 0\}$ and use $x_i = X_i/X_0$ as coordinates on it. The singular point e_0 is the origin. From (6.27), we may use $(x, y) := (x_3, x_1)$ as coordinates on S around points on $l \setminus \{e_0, e_3\}$. On S we have $l = \{y = 0\}$. The second equation for S becomes

$$(6.28) \quad 2x_2 = -y^2 - 2xx_4.$$

Substituting this into the affine form of the first equation of S , we obtain an equation

$$(6.29) \quad x_4^2 - 2x(y + \delta)x_4 - y^3 = 0,$$

where we again put $\delta = \alpha - \beta \neq 0$. The surface S is locally identified with this surface in \mathbb{C}^3 . If we let $x_4 = 0$ in this equation, we obtain $y^3 = 0$. This shows that if H is the hyperplane defined by $X_4 = 0$, then the section $H|_S$ contains the triple line $3l$. This is the unique hyperplane given in Lemma 4.7 in the present surface. By putting

$$F = x_4, \quad G = -2x_2,$$

the equations (6.29) and (6.28) can be written respectively as

$$(6.30) \quad F^2 - 2x(y + \delta)F - y^3 = 0 \quad \text{and} \quad G = y^2 + 2xF.$$

These represent the functions F and G in implicit forms. Around any point on the line l except the two singularities of S , we may put $F = y^3 f$ and $G = y^2 g$ for some holomorphic functions f and g . Then from (6.29) and (6.28) we obtain

$$(6.31) \quad y^3 f^2 - 2x(y + \delta)f - 1 = 0 \quad \text{and} \quad g = 1 + 2xyf.$$

Differentiating the former equation by x , we obtain

$$(6.32) \quad y^3 f f_x - 2(y + \delta)(f + x f_x) = 0.$$

Since $\delta \neq 0$, this means $y^3 \mid (f + x f_x)$. Then in the same way to (6.11)–(6.13), by putting $f = u(x) + yv(x, y)$, we obtain

$$(6.33) \quad f = \frac{c}{x} + yv, \quad f_{xx} = \frac{2c}{x^3} + yv_{xx}.$$

If $c = 0$, we would obtain $y \mid f$, which means $y^4 \mid F$. This implies $\{X_4 = 0\}|_S = 4l$, which contradicts Lemma 4.7. Hence $c \neq 0$. Then the latter of (6.33) means that f_{xx} is not divisible by y .

From $F = y^3 f$, we obtain

$$(6.34) \quad y^3 \mid F_{xx}, \quad y^2 \mid F_{xy} \quad \text{and} \quad y \mid F_{yy}.$$

So $y^4 \mid \text{Hess}(F)$. Also, in the same way to (6.14), from $G = y^2 g$, we obtain

$$(6.35) \quad \text{Hess}(G) = y^2 \{2(gg_{xx} - 2g_x^2) + yh(x, y)\}$$

for some function $h(x, y)$ whose explicit form is not needed. From the second equation of (6.31), we obtain $y \mid g_x$ and $y \mid g_{xx}$. So we have

$$y \mid (gg_{xx} - 2g_x^2).$$

Therefore from (6.35) we obtain $y^3 \mid \text{Hess}(G)$. We have $F_{xx} = y^3 f_{xx}$ from $F = y^3 f$, and $G_{yy} \equiv 2g \pmod{y}$ from $G = y^2 g$. Hence $F_{xx} G_{yy} \equiv 2f_{xx} g y^3 \pmod{y^4}$. As $y \mid F_{yy}$ from (6.34) and $G_{xx} = y^2 g_{xx}$ which is divisible by y^3 as $y \mid g_{xx}$, we obtain $y^4 \mid F_{yy} G_{xx}$. Also, we have $y^2 \mid F_{xy}$ from (6.34), and $y^2 \mid G_{xy}$ from $G_x = y^2 g_x$, $G_{xy} = y(2g_x + y g_{xy})$ and $y \mid g_x$. Hence $y^4 \mid F_{xy} G_{xy}$. From these, we obtain

$$K \equiv 2f_{xx} g y^3 \pmod{y^4}.$$

We have $y \nmid f_{xx}$ from the latter of (6.33) and $c \neq 0$. If g would be divisible by y , we have $y^3 \mid G$, which means that the hyperplane section of S by $\{X_2 = 0\}$ also contains the triple line $3l$. This means that for any hyperplane H which contains the common tangent 2-plane T , the section $H|_S$ contains $3l$. This cannot happen from Lemma 4.7. So $y \nmid g$. It follows from these that $y^4 \nmid K$. Therefore, we obtain

$$y^3 \mid \mathbf{H} \quad \text{and} \quad y^4 \nmid \mathbf{H}.$$

Hence, the divisor $\pi_1^{-1}(l)$ is included as a component of \mathbf{D}_1 with multiplicity precisely three.

The discriminant of the quadratic polynomial \mathbf{H}/y^3 is equal to

$$(6.36) \quad \left(\frac{K}{y^3}\right)^2 - 4\frac{\text{Hess}(F)}{y^3}\frac{\text{Hess}(G)}{y^3}.$$

Now we have $y \mid \frac{\text{Hess}(F)}{y^3}$ as $y^4 \mid \text{Hess}(F)$. So the second term is divisible by y . But the first term is not divisible by y as K is not divisible by y^4 . Hence (6.36) is not divisible by y . Thus we can again conclude that *the line l is not contained in the branch locus of the generically finite double covering $\mathbf{D}_S \rightarrow S$.*

6.6 - Appendix

Next we consider the case where the symbol of a Segre surface S is [(21)(11)]. After making a simple coordinate change to the normal forms, the defining equations of S are given by

$$2\alpha X_0 X_1 + X_1^2 + \alpha X_2^2 + \beta X_3 X_4 = 2X_0 X_1 + X_2^2 + X_3 X_4 = 0,$$

where $\alpha \neq \beta$. Singularities of S are the point e_0 , which is an A_3 -point, and e_3 and e_4 , which are A_1 -points. The two lines $\overline{e_0 e_3}$ and $\overline{e_0 e_4}$ are contained in S , and the line $\overline{e_3 e_4}$ is not. The involution on $\mathbb{C}\mathbb{P}_4$ which exchanges X_3 and X_4 preserves S and it fixes the A_3 -point e_0 . The A_1 -points e_3 and e_4 are exchanged by this involution. Therefore, the two lines $\overline{e_0 e_3}$ and $\overline{e_0 e_4}$ are exchanged by the involution. So in the following, we choose the line

$$l := \overline{e_0 e_3} = \{X_1 = X_2 = X_4 = 0\},$$

and work around points on $l \setminus \{e_0, e_3\}$. The common tangent plane to S along these points is given by

$$(6.37) \quad T = \{X_1 = X_4 = 0\}.$$

Again we work on the affine open subset $\{X_0 \neq 0\}$ and use $x_i = X_i/X_0$ as coordinates on it. The singular point e_0 is the origin. From (6.37), we may use $(x, y) := (x_3, x_2)$ as coordinates on S around points on $l \setminus \{e_0, e_3\}$. We then have $l = \{y = 0\}$ on S . The second equation for S becomes

$$(6.38) \quad 2x_1 = -y^2 - x_4 x.$$

Substituting this into an affine form of the first defining equation of S and again putting $\delta := \alpha - \beta \neq 0$, we obtain

$$(6.39) \quad (y^2 + x x_4)^2 - 4\delta x x_4 = 0.$$

The surface S is locally identified with this surface in \mathbb{C}^3 . If we put $x_4 = 0$ in this equation, one obtains $y^4 = 0$. This shows that the hyperplane section $\{X_4 = 0\}|_S$ contains $4l$. This is the unique hyperplane given in Lemma 4.7 in the present surface, and we have the coincidence $\{X_4 = 0\}|_S = 4l$. By putting

$$F = x_4, \quad G = -2x_1,$$

the equations (6.39) and (6.38) can be written respectively as

$$(6.40) \quad x^2 F^2 + 2x(y^2 - 2\delta)F + y^4 = 0 \quad \text{and} \quad G = y^2 + xF.$$

By putting $F = y^4 f$ and substituting it into the first equation of (6.40), we obtain that the function f is subject to the equation

$$(6.41) \quad x^2 y^4 f^2 + 2x(y^2 - 2\delta)f + 1 = 0.$$

From $F = y^4 f$, it is immediate to obtain

$$(6.42) \quad y^4 | F_{xx}, \quad y^3 | F_{xy} \quad \text{and} \quad y^2 | F_{yy}.$$

These mean $y^6 | \text{Hess}(F)$. From the second equation of (6.40), we obtain

$$(6.43) \quad G = y^2(1 + xy^2 f),$$

and from this, we readily have

$$(6.44) \quad y^4 | G_{xx}, \quad y^3 | G_{xy} \quad \text{and} \quad y \nmid G_{yy}.$$

These mean $y^4 | \text{Hess}(G)$. Also, from (6.42) and (6.44), we readily obtain $y^4 | K$. In order to show $y^5 \nmid K$, differentiating (6.41) by x , we obtain

$$y^4(xf^2 + x^2 f f_x) + (y^2 - 2\delta)(f + x f_x) = 0.$$

As $\delta \neq 0$, this means $y^4 | (f + x f_x)$. Writing $f = u(x) + yv(x, y)$ as before, this implies

$$(6.45) \quad f_{xx} = \frac{2c}{x^3} + yv_{xx}$$

for some constant c . If $c = 0$, we obtain $y | f$, which cannot happen since no hyperplane section of S can contain $5l$. Hence from (6.45), we obtain $y \nmid f_{xx}$. So $y^5 \nmid y^4 f_{xx} = F_{xx}$. As $y \nmid G_{yy}$ as in (6.44), we conclude $y^5 \nmid F_{xx} G_{yy}$. This means $y^5 \nmid K$. Therefore, $y^4 | \mathbf{H}$ and $y^5 \nmid \mathbf{H}$ hold. Hence, *the divisor D_1*

includes the component $\pi_1^{-1}(l)$ with multiplicity precisely four. The discriminant of the quadratic polynomial \mathbf{H}/y^4 is given by

$$(6.46) \quad \left(\frac{K}{y^4}\right)^2 - 4\frac{\text{Hess}(F)}{y^4}\frac{\text{Hess}(G)}{y^4}.$$

Since $y^5 \nmid K$ as above, K/y^4 is not divisible by y . So $(K/y^4)^2$ is also not divisible by y . On the other hand, we have $y^6 \mid \text{Hess}(F)$ as above. These mean that the discriminant (6.46) is not divisible by y . Therefore, *the line l is not a branch divisor of the generically double covering $D_S \rightarrow S$.*

6.7 - Appendix

Finally in this section, we consider a Segre surface S whose symbol is $[2(21)]$. The normal forms of the defining equations of S are given by

$$2\alpha X_0 X_1 + X_1^2 + 2\beta X_2 X_3 + X_3^2 + \beta X_4^2 = 2X_0 X_1 + 2X_2 X_3 + X_4^2 = 0,$$

where $\alpha \neq \beta$. Singularities of S are the point e_0 , an A_1 -point, and e_2 , an A_3 -point. The line $l := \{X_1 = X_3 = X_4 = 0\}$ passes these singularities, and it is contained in S . The common tangent plane along $l \setminus \{e_0, e_2\}$ is given by

$$(6.47) \quad T = \{X_1 = X_3 = 0\}.$$

This time, we work on the affine open subset $\{X_2 \neq 0\}$ in \mathbb{CP}_4 and use $x_i = X_i/X_2$ as coordinates on it. The A_2 -point e_2 is the origin. From (6.47), we may use $(x, y) := (x_0, x_4)$ as coordinates on S around points of $l \setminus \{e_0, e_2\}$. On S , we again have $l = \{y = 0\}$. The second equation for S becomes

$$(6.48) \quad 2x_3 = -(2xx_1 + y^2).$$

Substituting this into an affine form of the first equation of S , we obtain

$$(6.49) \quad 4(1 + x^2)x_1^2 + 4x(2\delta + y^2)x_1 + y^4 = 0.$$

If we let $x_1 = 0$ in this equation, we obtain $y^4 = 0$. This means that the hyperplane section $\{X_1 = 0\}|_S$ is exactly $4l$. The hyperplane $\{X_1 = 0\}$ is the unique one given in Lemma 4.7. We put

$$(6.50) \quad F = x_1, \quad G = -2x_3,$$

and define f and g by $F = y^4 f$ and $G = y^2 g$. From (6.49), we obtain that the function f is subject to the equation

$$(6.51) \quad 4(1 + x^2)y^4 f^2 + 4x(2\delta + y^2)f + 1 = 0.$$

Also, from (6.48), we have

$$G = y^2(2xy^2f + 1).$$

From this, we readily obtain

$$(6.52) \quad y^4 \mid G_{xx}, \quad y^3 \mid G_{xy} \quad \text{and} \quad y \nmid G_{yy}.$$

On the other hand, in the same way to the last case, from $F = y^4f$, we obtain

$$(6.53) \quad y^4 \mid F_{xx}, \quad y^3 \mid F_{xy} \quad \text{and} \quad y^2 \mid F_{yy}.$$

From (6.52) and (6.53), we obtain

$$y^6 \mid \text{Hess}(F), \quad y^4 \mid \text{Hess}(G) \quad \text{and} \quad y^4 \mid K.$$

Also, differentiating (6.51) by x , we obtain

$$2y^4\{xf^2 + (1+x^2)ff_x\} + (2\delta + y^2)(f + xf_x) = 0.$$

Since $\delta \neq 0$, this means $y^4 \mid (f + xf_x)$. The rest is completely the same as the previous case of [(21)(11)], and we obtain that *the divisor \mathbf{D}_1 includes the component $\pi_1^{-1}(l)$ with multiplicity precisely four, and that the line l is not a branch divisor of $\mathbf{D}_S \rightarrow S$.*

6.8 - Appendix

Summing up the results in this section, as promised, we obtain the following conclusion.

Proposition 6.1. *Let $S \subset \mathbb{C}\mathbb{P}_4$ be a Segre surface and l a line which passes two singularities of S . Then l is not a branch divisor of the generically finite double covering $\mathbf{D}_S \rightarrow S$.*

Finally, we make a remark about multiplicities of the lines as branch divisor of the generically finite double covering $\mathbf{D}_S \rightarrow S$. If a line on a Segre surface S does not pass any singularity of S , then the multiplicity of the line is one (Proposition 4.3). If a line passes two singularities of S , it is not a branch divisor (Proposition 6.1). When a line passes exactly one singularity of S , we know from Proposition 4.5 that the multiplicity of the line is at least two. The exact multiplicities of these lines might be determined by making the following heuristic argument solid.

Take a Segre surface S whose symbol is [1112] for example. S has exactly one singularity and 12 lines. Precisely 4 of the lines pass the singularity. (See

Table 1.) By realizing S as a degeneration of smooth Segre surfaces, each of the 4 lines can be obtained by unifying (or gathering) two lines on smooth Segre surfaces through the degeneration. By continuity, this would mean that the multiplicities of the 4 lines as branch divisor are precisely two. Similarly, a Segre surface S with symbol [113] has exactly one singularity and 8 lines. Precisely 4 of the lines pass the singularity. Again by realizing S as a degeneration of smooth Segre surfaces, each of the 4 lines can be obtained by unifying (or gathering) three lines on smooth Segre surfaces through the degeneration. This would mean that the multiplicities of the 4 lines are precisely three. In Table 1, to each line which passes at least one singularity, the multiplicity of the line counted by using degeneration from smooth Segre surfaces in this way is presented in the y -column. These multiplicities would be equal to the multiplicities as branch divisors of the covering $D_S \rightarrow S$.

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NOBUHIRO HONDA
Department of Mathematics
Tokyo Institute of Technology
2-12-1, O-okayama, Meguro, 152-8551, Japan
e-mail: honda@math.titech.ac.jp

AYATO MINAGAWA
Department of Mathematics
Tokyo Institute of Technology
2-12-1, O-okayama, Meguro, 152-8551, Japan
Current affiliation: Internet Initiative Japan Inc.
e-mail: benkyo-gatizei@akane.waseda.jp