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Compact homogeneous locally conformally Kähler manifolds are Vaisman. A new proof

Abstract. An LCK manifold with potential is a complex manifold with a Kähler potential on its cover, such that any deck transformation multiplies the Kähler potential by a constant multiplier. We prove that any homogeneous LCK manifold admits a metric with LCK potential. This is used to give a new proof that any compact homogeneous LCK manifold is Vaisman.

Keywords. Locally conformally Kähler, homogeneous, LCK potential, Vaisman manifold, holomorphic action.

Mathematics Subject Classification: 53C55, 53C30.

1 - Introduction

A locally conformally Kähler (LCK) manifold is a complex manifold (M, I) equipped with a Hermitian metric g (LCK metric) which is locally conformally equivalent to a Kähler metric. Then the Hermitian form $\omega(x, y) := g(Ix, y)$ satisfies $d\omega = \theta \wedge \omega$, where θ is a 1-form, called *the Lee form* (see [**DO**] for an introduction to the subject).

An LCK metric g is called Vaisman if the Lee form is not only closed, but also *parallel* with respect to the Levi Civita connection of g.

The LCK condition is conformally invariant: if g is an LCK metric on M, then $e^f g$ is LCK for all smooth functions f on M. By contrast, the Vaisman

condition is not conformally invariant. Moreover, on a compact complex manifold, a Vaisman metric (if it exists) is unique up to homothety in its conformal class and coincides with the Gauduchon metric of that conformal class.

One of the most interesting problems in LCK geometry is to find sufficient conditions for an LCK manifold to admit Vaisman metrics. In this paper we prove that homogeneity is such a sufficient condition.

An LCK manifold (M, I, g) is *homogeneous* if it admits a transitive and effective action of a Lie group by holomorphic isometries of the LCK metric. In this case, $M = G/H$, where G is a connected Lie group, and H is the stabilizer subgroup of G.

R e m a r k 1.1. (i) A group G as above also preserves ω and θ.

(ii) Recall that the group $\mathrm{Aut}(M)$ of biholomorphic conformalities of a compact LCK manifold is compact. Indeed, any holomorphic conformality preserves the corresponding Gauduchon metric which is unique up to a constant. We can therefore assume that M is homogeneous under the group of holomorphic conformalities.

The following result was announced in [**HK1**], see also [**HK2**], [**HK3**] and [**Gu**] for a subsequent discussion. A short, complete and self-contained proof appeared in [**GMO**]. Here we present a new proof, based on the notion of LCK structure with potential.

T h e o r e m 1.2. *A compact homogeneous locally conformally Kähler manifold is Vaisman.*

The new proof amounts in showing that the homogeneity implies the existence of a holomorphic circle action whose lift to the Kähler cover does not contain only homotheties of the Kähler metric. This will imply that the manifold is LCK with potential. We then observe that the Lee form of this LCK structure with potential has constant length, which characterizes the Vaisman metrics among the LCK metrics with potential.

Section 2 of this paper gathers the necessary background on LCK geometry, inlcuding Vaisman and LCK with potential metrics. In Section 3 we present the proof of Theorem 1.2. In the last section, we give new proofs for two classical results about homogeneous Vaisman manifolds.

2 - Preliminaries

2.1 - *Locally conformally Kähler manifolds*

D e f i n i t i o n 2.1. Let (M, I) be a complex manifold, $\dim_{\mathbb{C}} M \geqslant 2$. It is called *locally conformally Kähler* (LCK) if it admits a Hermitian metric g whose

fundamental 2-form $\omega(\cdot,\cdot) := g(\cdot, I\cdot)$ satisfies

(2.1)
$$d\omega = \theta \wedge \omega, \quad d\theta = 0,$$

for a certain closed 1-form θ called *the Lee form*.

R e m a r k 2.2. Definition 2.1 is equivalent to the existence of a covering $\tilde{M}$ endowed with a Kähler metric Ω which is acted on by the deck group $\mathrm{Aut}_M(\tilde{M})$ by homotheties.

R e m a r k 2.3. The operator $d_\theta := d - \theta \wedge$ is called *twisted de Rham operator*. It obviously satisfies $d_\theta^2 = 0$ and hence $(\Lambda^* M, d_\theta)$ produces a cohomology called *Morse-Novikov* or *twisted*.

The following fundamental result of I. Vaisman shows that on compact complex manifolds, Kähler and LCK metrics cannot coexist.

T h e o r e m 2.4 ([**Va1**]). *Let (M, ω, θ) be a compact LCK manifold, not globally conformally Kähler (i.e. with non-exact Lee form). Then M does not admit a Kähler metric.*

2.2 - *Vaisman manifolds*

D e f i n i t i o n 2.5. An LCK manifold (M, ω, θ) is called *Vaisman* if $\nabla \theta = 0$, where ∇ is the Levi-Civita connection of g.

E x a m p l e 2.6. All diagonal Hopf manifolds are Vaisman ([**OV4**]). The Vaisman compact complex surfaces are classified in [**B**], see also [**VVO**].

There exist compact LCK manifolds which do not admit Vaisman metrics. Such are the LCK Inoue surfaces, [**B**], the Oeljeklaus-Toma manifolds, [**OT**], [**O**], and the non-diagonal Hopf manifolds, [**OV4**], [**VVO**].

R e m a r k 2.7. On a Vaisman manifold, the Lee field $\theta^\sharp$ and the anti-Lee field $I\theta^\sharp$ are real holomorphic ($\mathrm{Lie}_{\theta^\sharp} I = \mathrm{Lie}_{I\theta^\sharp} I = 0$) and Killing ($\mathrm{Lie}_{\theta^\sharp} g = \mathrm{Lie}_{I\theta^\sharp} g = 0$), see [**DO**]. Moreover $[\theta^\sharp, I\theta^\sharp] = 0$. The 2-dimensional foliation Σ they generate is called *the canonical foliation*.

According to the above Remark, the Lee field generates a flow of holomorphic isometries. The following characterization is a partial converse:

T h e o r e m 2.8 ([**KO**]). *Let (M, ω, θ) be an LCK manifold equipped with a holomorphic and conformal $\mathbb{C}$-action ρ without fixed points, which lifts to non-isometric homotheties on the Kähler cover $\tilde{M}$. Then (M, ω, θ) is conformally equivalent with a Vaisman manifold.*

R e m a r k 2.9. Since θ is parallel, it has constant norm and thus we can always scale the LCK metric such that $|\theta| = 1$. In this assumption, the following formula holds, [**DO**]:

$$(2.2) \qquad d\theta^c = \omega - \theta \wedge \theta^c, \quad \text{where} \quad \theta^c(X) = -\theta(IX).$$

Moreover, one can see, [**Ve**], that the (1,1)-form $\omega_0 := d^c\theta$ is semi-positive definite, having all eigenvalues[1] positive, except one which is 0.

2.3 - *LCK manifolds with potential*

D e f i n i t i o n 2.10. An LCK manifold has *LCK potential* if it admits a Kähler covering on which the Kähler metric has a global and positive potential function ψ such that the deck group multiplies ψ by a constant. In this case, M is called *LCK manifold with potential*.

P r o p o s i t i o n 2.11. *The LCK manifold (M, I, g, θ) is LCK with potential if and only if equation* (2.2) *is satisfied.*

D e f i n i t i o n 2.12. A function $\varphi \in C^\infty(M)$ is called $d_\theta d_\theta^c$-*plurisubharmonic* if $\omega = d_\theta d_\theta^c(\varphi)$, where d_θ is the twisted de Rham operator (Remark 2.3) and $d_\theta^c := I d_\theta I^{-1}$.

Equation (2.2) (and hence the definition of LCK manifolds with potential) can be translated on the LCK manifold itself:

T h e o r e m 2.13 ([**OV3**], Claim 2.8]). *(M, I, θ, ω) is LCK with potential if and only if $\omega = d_\theta d_\theta^c(\psi)$ for a strictly positive $d_\theta d_\theta^c$-plurisubharmonic function ψ on M.*

R e m a r k 2.14. All Vaisman manifolds are LCK manifolds with potential: on their Kähler covering, the automorphic potential is represented by the squared length of the pull-back of the Lee form with respect to the Kähler metric. Among the non-Vaisman examples, we mention the non-diagonal Hopf manifolds, [**OV3**].

We shall need the following characterizations of the Vaisman metrics among the LCK metrics with potential:

P r o p o s i t i o n 2.15 ([**OV4**]). *Let (M, ω, θ) be a compact LCK manifold with potential. Then the LCK metric is Gauduchon if and only if $\omega_0 = d^c\theta$ is semi-positive definite, and then it is Vaisman. Equivalently, a compact LCK manifold with potential and with constant norm of θ is Vaisman.*

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[1]The eigenvalues of a Hermitian form η are the eigenvalues of the symmetric operator L_η defined by the equation $\eta(x, Iy) = g(L_\eta x, y)$.

In the proof that we shall present, we make use of the following sufficient condition for a compact LCK manifold to admit an LCK metric with potential (compare with Theorem 2.8).

T h e o r e m   2.16 ([**OV2**], [**I**, Theorem 0.4]). *Let M be a compact complex manifold, equipped with a holomorphic S^1-action and an LCK metric g (not necessarily S^1-invariant). Suppose that this S^1 action does not lift to an isometric action on the Kähler cover of M. Then g admits an LCK potential.*

P r o o f.   The automorphic Kähler potential was obtained in [**OV2**] by cohomological arguments, and it is not necessarily positive. In our terminology, "LCK potential" is always positive. The existence of a positive potential is implied a posteriori using a result from [**OV5**]. N. Istrati ([**I**, Theorem 0.4]) produced an explicit form of this argument, showing that any S^1-invariant LCK metric, in assumptions of Theorem 2.16, admits an LCK potential.   □

3 - Homogeneous LCK manifolds admit LCK potential

The idea is to show that the homogeneity of G/H implies the existence of a holomorphic circle action which is lifted to non-isometric homotheties of the universal cover. This will imply that the manifold is LCK with potential (Theorem 2.16). We translate this potential on the manifold itself, we average it on G and obtain an invariant LCK metric with potential whose Lee form has constant length, which characterizes the Vaisman metrics among the LCK metrics with potential (Proposition 2.15). The manifold is thus of Vaisman type. Finally, we prove that the initial homogeneous LCK metric itself is Vaisman.

We start with the following lemma.

L e m m a   3.1. *Let (M, ω, θ) be an LCK manifold, and A a vector field acting on M by holomorphic isometries. Assume that the function $\theta(A)$ is not identically zero. Then A does not act by isometries when lifted to a Kähler cover of M.*

P r o o f.   Let $\tilde{M} \xrightarrow{\pi} M$ be a Kähler cover of M. Then $\theta_1 := \pi^*\theta$ is exact: $\theta_1 = d\varphi$, and the corresponding Kähler form on $\tilde{M}$ can be written as $\tilde{\omega} = e^\varphi \pi^* \omega$. Denote by A_1 the lift of A to $\tilde{M}$. Then $\mathrm{Lie}_{A_1}(\tilde{\omega}) = (\mathrm{Lie}_{A_1} \varphi)\tilde{\omega}$, in other words, A_1 acts by isometries if and only if $\mathrm{Lie}_{A_1} \varphi = 0$. However, Cartan's formula gives

$$\mathrm{Lie}_{A_1} \varphi = A_1 \lrcorner\, d\varphi = d\varphi(A_1) = \theta_1(A_1).$$

Since, by assumption, $\theta(A) \neq 0$ somewhere on M, we have $\theta_1(A_1) \neq 0$ somewhere on $\tilde{M}$. $\qquad\square$

We can prove now that M is of Vaisman type. Every homogeneous manifold M can be obtained as $M = G/H$, where G acts on M transitively by automorphisms, and H is the stabilizer of the point. In our case, M is LCK, and G acts on M by holomorphic LCK isometries. Since the group of isometries of a compact manifold is compact, we may freely assume that G is compact. This will be our running assumption from now on.

P r o p o s i t i o n   3.2. *Let (M, ω, θ) be a compact, homogeneous LCK manifold, $M = G/H$, where G is a compact Lie group. Then M admits a G-homogeneous Vaisman metric with the same Lee form.*

P r o o f.   S t e p 1:   Let $\mathfrak{g} = \mathrm{Lie}(G)$. Choose $A \in \mathfrak{g}$ such that $\theta(A) \neq 0$ somewhere on M, and let $T \subset G$ be the closure of the one-parametric subgroup of G generated by e^{tA}. Then T is a compact torus, hence A can be obtained as a limit of vector fields A_i such that each one-parametric subgroup of G generated by e^{tA_i} is a circle. Then for some i the vector field A_i satisfies $\theta(A_1) \neq 0$ somewhere on M. By Lemma 3.1, A_i it cannot act by isometries on the Kähler cover of $\tilde{M}$. By Theorem 2.16, M admits an LCK metric with potential.

S t e p 2:   By Theorem 2.13, the fundamental form of an LCK metric with potential is written as $\omega_\psi = d_\theta d_\theta^c(\psi)$, for a positive function ψ on M. Averaging on the compact group G, we obtain a metric g_1 with the fundamental form

$$\omega_1 = \mathrm{Av}_G(\omega_\psi) = d_\theta d_\theta^c(\mathrm{Av}_G(\psi)).$$

Then g_1 is still LCK with potential, but the potential is now G-invariant, and the metric is also G-invariant. Since θ is already G-invariant, the averaging process does not change it. Then θ is also the Lee form of g_1, and its norm in this metric is constant: $|\theta|_{\omega_1} = \mathsf{const}$. Then Proposition 2.15 implies that g_1 is Vaisman. $\qquad\square$

Finally, we can prove that the initial homogeneous LCK metric is itself Vaisman.

T h e o r e m   3.3. *Let $(M, g, I, \omega, \theta)$ be a compact, homogeneous LCK manifold, $M = G/H$. Then (M, ω, θ) is Vaisman.*

P r o o f.   By Proposition 3.2, M admits a G-homogeneous Vaisman metric ω_1 with the same Lee form. By Theorem 2.16, we can assume that ω is an LCK metric with potential. Then Theorem 3.3 would follow if we prove that a

G-homogeneous metric with potential and a given Lee form is unique up to a constant multiplier. $\square$

P r o p o s i t i o n 3.4. *Let $M = G/H$ be a G-homogeneous complex manifold, and ω_1, ω_2 two G-homogeneous LCK metrics with potential and the same Lee form. Then ω_1 is proportional to ω_2.*

P r o o f. By Theorem 2.13, $\omega_i = d_\theta d_\theta^c(\varphi_i)$. Averaging φ_i with G if necessary, we can assume that it is a G-invariant function, hence constant. Then $\omega_i = d_\theta d_\theta^c(a_i)$, where a_i are constant functions, hence these two forms are proportional. $\square$

4 - Homogeneous Vaisman manifolds

For the sake of completion, we give here new proofs of some old important results concerning homogeneous Vaisman manifolds.

T h e o r e m 4.1 ([**Va2**]). *The canonical foliation of a compact homogeneous Vaisman manifold $(M = G/H, I, g)$ is regular.*[2]

P r o o f. Since M is compact, Σ has at least one compact leaf (this was proven in [**T**], but follows also by results of Kato (in [**K**]) or [**OV4**]). Then, by homogeneity, all leaves are compact, and hence Σ is quasi-regular. Consider the elliptic fibration $\pi : M \to M/\Sigma$. This map has at least one smooth fibre. To see this, just consider π with values in the smooth part of M/Σ and apply Sard-Brown's theorem. But then, again by homogeneity, all fibers are smooth. $\square$

R e m a r k 4.2. On homogeneous Vaisman manifolds the foliation Σ is regular, but the foliation $\langle \theta^\sharp \rangle \subset \Sigma$ generated by the Lee field is not necessarily regular. Indeed, consider the Hopf surface $H = \frac{(\mathbb{C}^2 \backslash 0)}{\langle A \rangle}$, where $A = \alpha \,\mathrm{Id}$ and α is a complex number such that $\frac{\alpha}{|\alpha|}$ is not a root of unity. In this case the Lee field is a radial vector field on $\mathbb{C}^2$, and its trajectory applied to a 3-dimensional sphere gives a diffeomorphism $v \longrightarrow \frac{\alpha}{|\alpha|}$ which has infinite order.

A consequence of Theorem 4.1 is the following.

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[2]A foliation on a manifold is *regular* if its leaf space is a manifold; and it is *quasi-regular* if all its leaves are compact, in which case the leaf space is an orbifold.

Corollary 4.3. *A compact homogeneous Vaisman manifold M has $b_1 = 1$.*

Proof. Since the canonical foliation Σ is regular, M is a T^2-fibration over a homogeneous projective manifold P. Since any algebraic group is rational, homogeneous projective manifolds are also rational. Using [**D**, Chapter 4], we obtain that P is simply connected.

Write the exact sequence

$$0 \longrightarrow H^1(P) \longrightarrow H^1(M) \longrightarrow H^1(T^2) \stackrel{\gamma}{\longrightarrow} H^2(P)$$

and observe that the rank of γ is 1 (see [**OV1**]). Therefore, $b_1(P) = 0$ implies that $b_1(M) = 1$. $\qquad\qquad\square$

Remark 4.4. One can construct homogeneous Vaisman manifolds as in [**Va2**], starting from a compact homogeneous projective manifold $P = G/H$ such that the action of G on P can be linearized (*id est P* admits an equivariant ample line bundle). Note that the structure of compact homogeneous Kähler manifolds is clarified in [**M**]: up to biholomorphisms, they are products of flag manifolds.

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