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Hodge theory on almost-Hermitian manifolds

Abstract. This survey summarizes the results presented in a talk at the conference "Cohomology of Complex Manifolds and Special Structures, II" held in Levico Terme in July 2021. An important tool in the study of complex manifolds is provided by cohomology groups and the associated spaces of harmonic forms. However, when the integrability of the complex structure is not assumed it seems that cohomological theory and Hodge theory take two different directions. Here we will deal with the natural spaces of harmonic forms that arise in almost-Hermitian geometry and we will discuss their dependence on the metric. Most of the results are contained in [**17**] and in [**18**].

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1 - Introduction

Let (X, J) be a complex manifold, then the space of complex valued differential forms $A^\bullet(X, \mathbb{C})$ can be decomposed as the direct sum of the spaces of bi-graded forms $A^{\bullet,\bullet}(X)$. Similarly, the exterior derivative d splits as the sum of its $(1,0)$ component ∂ and its $(0,1)$-component $\overline{\partial}$. More precisely, ∂ and $\overline{\partial}$ are still differential operators and satisfy

$$\partial^2 = 0\,, \qquad \overline{\partial}^2 = 0\,, \qquad \partial\overline{\partial} + \overline{\partial}\partial = 0.$$

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Hence, $(A^{\bullet,\bullet}(X), \partial, \overline{\partial})$ has the structure of a double complex. It is therefore natural to consider the following cohomology groups that represent holomorphic invariant for (X, J): the Dolbeault cohomology and its conjugate

$$H^{\bullet,\bullet}_{\overline{\partial}}(X) := \frac{\operatorname{Ker} \overline{\partial}}{\operatorname{Im} \overline{\partial}}\,, \qquad H^{\bullet,\bullet}_{\partial}(X) := \frac{\operatorname{Ker} \partial}{\operatorname{Im} \partial}\,,$$

and the Bott-Chern and Aeppli cohomologies

$$H^{\bullet,\bullet}_{BC}(X) := \frac{\operatorname{Ker} \partial \cap \operatorname{Ker} \overline{\partial}}{\operatorname{Im} \partial\overline{\partial}}\,, \qquad H^{\bullet,\bullet}_{A}(X) := \frac{\operatorname{Ker} \partial\overline{\partial}}{\operatorname{Im} \partial + \operatorname{Im} \overline{\partial}}\,.$$

If X is assumed to be compact these cohomology groups are finite dimensional, and the easiest way to prove it is to use Hodge theory. Indeed, once fixed an Hermitian metric on (X, J), these spaces are isomorphic to the kernel of some suitable elliptic differential operators. More precisely,

$$H^{\bullet,\bullet}_{\overline{\partial}}(X) \simeq \operatorname{Ker} \Delta_{\overline{\partial}}\,, \quad H^{\bullet,\bullet}_{\partial}(X) \simeq \operatorname{Ker} \Delta_{\partial}\,,$$

$$H^{\bullet,\bullet}_{BC}(X) \simeq \operatorname{Ker} \Delta_{BC}\,, \quad H^{\bullet,\bullet}_{A}(X) \simeq \operatorname{Ker} \Delta_{A}\,,$$

where

$$\Delta_{\overline{\partial}} := \overline{\partial}\overline{\partial}^* + \overline{\partial}^*\overline{\partial}\,, \qquad \Delta_{\partial} := \partial\partial^* + \partial^*\partial\,,$$

$$\Delta_{BC} := (\partial\overline{\partial})(\partial\overline{\partial})^* + (\partial\overline{\partial})^*(\partial\overline{\partial}) + (\overline{\partial}^*\partial)(\overline{\partial}^*\partial)^* + (\overline{\partial}^*\partial)^*(\overline{\partial}^*\partial) + \overline{\partial}^*\overline{\partial} + \partial^*\partial\,,$$

$$\Delta_{A} := \partial\partial^* + \overline{\partial}\overline{\partial}^* + (\partial\overline{\partial})^*(\partial\overline{\partial}) + (\partial\overline{\partial})(\partial\overline{\partial})^* + (\overline{\partial}\partial^*)^*(\overline{\partial}\partial^*) + (\overline{\partial}\partial^*)(\overline{\partial}\partial^*)^*\,.$$

In particular, the dimensions of the kernels of these operators do not depend on the metric thanks to the isomorphisms with the respective cohomologies. Such dimensions are also very useful in the study of complex non-Kähler manifolds since they provide informations on the validity of the $\partial\overline{\partial}$-lemma (see [**3**], [**2**], [**4**]).

However, if we drop the integrability assumption on J, namely J is just an almost-complex structure, than the study of cohomology groups and the study of harmonic forms take two different roads.

This is essentially due to the fact that in the decomposition of the exterior derivative d there are two additional terms μ and $\bar{\mu}$ of degree respectively $(2, -1)$ and $(-1, 2)$. Hence, $\overline{\partial}^2 \neq 0$ and $\partial\overline{\partial} + \overline{\partial}\partial \neq 0$ and so the usual cohomology groups are not well defined. For this reason, in [**7**] and [**6**] natural generalizations of the Dolbeault, Bott-Chern and Aeppli cohomology groups have been introduced and studied. Such definitions just depend on the almost-complex structure and clearly take into account the presence of μ and $\bar{\mu}$. However, as shown in [**6**], these cohomologies behave very differently from the integrable case because they might be infinite dimensional; hence the divergence from Hodge theory appear. Indeed, if we fix an Hermitian metric on a compact almost-complex manifold (X, J), the Laplacian operators $\Delta_{\overline{\partial}}$, Δ_{∂}, Δ_{BC}, Δ_A and more (see [**17**]) are well defined and elliptic. In particular, one can study several symmetries for their kernels and these kernels are finite dimensional vector spaces. Since these spaces do not have a cohomological counterpart it is natural to ask whether, differently from the integrable case, their dimensions depend on the choice of the Hermitian metric (cf. Kodaira and Spencer's question in [**12**]). Holt and Zhang showed in [**14**] that on 4-dimensional almost-complex manifolds the dimension of $\operatorname{Ker}\Delta_{\overline{\partial}}$ for $(0,1)$-forms varies with the choice of the Hermitian metric and for $(1,1)$-forms with respect to almost-Kähler metrics it is always $b_- + 1$, namely it depend just on the topology of X. Later, in [**18**] studying on 4-dimensional almost-complex manifolds the dependence on the conformal class of Hermitian metrics of the dimension of $\operatorname{Ker}\Delta_{\overline{\partial}}$ on $(1,1)$-forms, we showed that with respect to a special class of metrics called *(strictly) locally conformally Kähler*, one obtains b_-. Namely, also the dimension of $\operatorname{Ker}\Delta_{\overline{\partial}}$ on $(1,1)$-forms depends on the metric. We point out that we construct an explicit almost-complex manifold admitting both almost-Kähler and (strictly) locally conformally Kähler metrics; these two kind of metrics cannot coexist on integrable complex manifolds as shown by [**20**].

The aim of this survey is to collect some of the results and the ideas behind the study of the spaces of harmonic forms and their dimensions on compact almost-Hermitian manifolds.

2 - Differential operators on almost-Hermitian manifolds

In this section we will focus on the definitions of several natural differential operators on almost-Hermitian manifolds as well as on their kernels and the induced symmetries. Most of the results are contained in [**17**]. Let X be a $2n$-dimensional smooth manifold endowed with an almost complex structure J, namely J is a $(1,1)$-tensor such that $J^2 = -\mathrm{Id}$. Then the space of complex valued differential forms on X, denoted with $A^\bullet(X, \mathbb{C})$, has a natural bi-grading induced by J,

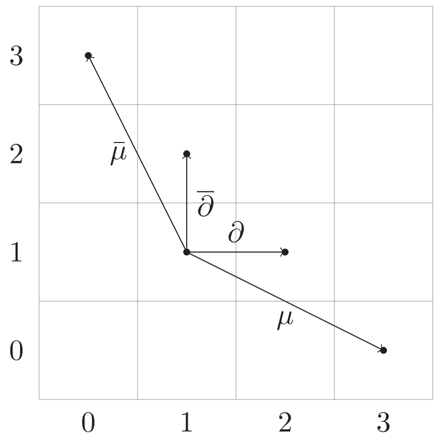
$$A^\bullet(X, \mathbb{C}) = \bigoplus_{p+q=\bullet} A^{p,q}(X).$$

According to this decomposition, the exterior derivative d splits into four components

$$d : A^{p,q}(X) \to A^{p+2,q-1}(X) \oplus A^{p+1,q}(X) \oplus A^{p,q+1}(X) \oplus A^{p-1,q+2}(X)\,,$$

$$d = \mu + \partial + \overline{\partial} + \bar{\mu}\,,$$

where μ and $\bar{\mu}$ are differential operators that are linear over functions.
For instance, in the following picture we can see in the 6-dimensional case how such operators behave on $(1,1)$-forms.



Using now that $d^2 = 0$, we have the following relations

$$\begin{cases} \mu^2 & = & 0 \\ \mu\partial + \partial\mu & = & 0 \\ \partial^2 + \mu\overline{\partial} + \overline{\partial}\mu & = & 0 \\ \partial\overline{\partial} + \overline{\partial}\partial + \mu\bar{\mu} + \bar{\mu}\mu & = & 0 \\ \overline{\partial}^2 + \bar{\mu}\partial + \partial\bar{\mu} & = & 0 \\ \bar{\mu}\overline{\partial} + \overline{\partial}\bar{\mu} & = & 0 \\ \bar{\mu}^2 & = & 0 \end{cases}.$$

It is natural to consider then the following operators (cf. [**9**])

$$\delta := \partial + \bar{\mu}\,, \qquad \bar{\delta} := \overline{\partial} + \mu$$

with $\delta : A^{\pm}(X) \to A^{\pm}(X)$ and $\bar{\delta} : A^{\pm}(X) \to A^{\mp}(X)$, where $A^{\pm}(X)$ are defined accordingly to the parity of q in the J-induced bigrading on $A^{\bullet}(X,\mathbb{C})$. In particular, the following relations hold

- $d = \delta + \bar{\delta}$,

- $\delta^2 + \bar{\delta}^2 = 0$,
- $\delta^2 = \partial^2 - \overline{\partial}^2$,
- $\delta\bar{\delta} + \bar{\delta}\delta = 0$.

We recall that J is integrable, namely (X, J) is a genuine complex manifold if and only if $\mu = \bar{\mu} = 0$. In particular, in such a case we recover the usual relations

$$\overline{\partial}^2 = 0\,, \qquad \partial^2 = 0\,, \qquad \partial\overline{\partial} + \overline{\partial}\partial = 0.$$

Moreover, on complex manifolds

$$\delta = \partial\,, \qquad \bar{\delta} = \overline{\partial}$$

and so one cannot appreciate, for instance, the distinctions between the associated spaces of harmonic forms. In the following we will not assume the integrability of J in general.

If $D = d, \partial, \delta, \bar{\delta}, \mu, \bar{\mu}$ we set $D^c := J^{-1}DJ$, then $\delta^c = -i\delta$ and $\bar{\delta}^c = i\bar{\delta}$ and

$$d^c = i(\bar{\delta} - \delta) = i(\overline{\partial} + \mu - \partial - \bar{\mu}).$$

Notice that in general if J is not integrable d and d^c do not anticommute, indeed we have

$$dd^c + d^c d = 2i(\bar{\delta}^2 - \delta^2) = 4i(\overline{\partial}^2 - \partial^2)\,.$$

Therefore, an almost-complex structure J is integrable if and only if $d^c = i(\overline{\partial} - \partial)$ if and only if d and d^c anticommute.

In the following we are going to recall the definitions of several natural operators on almost-Hermitian manifolds and discuss the relations among them together with several natural symmetries.

Let g be an Hermitian metric on (X, J), namely g is a Riemannian metric such that $g(J\cdot, J\cdot) = g(\cdot, \cdot)$. We will identify g with its associated fundamental $(1,1)$-form $\omega = g(J\cdot, \cdot)$. We denote with $*$ the associated $\mathbb{C}$-linear Hodge-$*$-operator, namely $* : A^{p,q}(X) \to A^{n-q,n-p}(X)$ is such that, for every $\alpha, \beta \in A^{p,q}(X)$,

$$\alpha \wedge *\bar{\beta} = g(\alpha, \beta)\,\frac{\omega^n}{n!}.$$

If $D = d, \partial, \overline{\partial}, \mu, \bar{\mu}, \bar{\delta}, \delta$ we set $D^* := - * \bar{D} *$ and it turns out that D^* is the adjoint of D with respect to the L^2-pairing induced on forms (cf. [**9**], [**7**], [**17**]). As usual one can consider the following differential operator

$$\Delta_D := DD^* + D^*D\,.$$

Proposition 2.1 (cf. [**12**], [**17**]). *Let (X, J, g) be a compact almost-Hermitian manifold. Then, the operators $\Delta_{\overline{\partial}}$, Δ_{∂}, $\Delta_{\bar{\delta}}$, Δ_{δ} are elliptic, and so the associated spaces of harmonic forms*

$$\mathcal{H}^{\bullet,\bullet}_{\overline{\partial}}(X) := \text{Ker } \Delta_{\overline{\partial}}\,, \quad \mathcal{H}^{\bullet,\bullet}_{\partial}(X) := \text{Ker } \Delta_{\partial}$$

$$\mathcal{H}^{\bullet}_{\bar{\delta}}(X) := \text{Ker}(\Delta_{\bar{\delta}_{|A^\bullet}}), \quad \mathcal{H}^{\bullet,\bullet}_{\bar{\delta}}(X) := \text{Ker}(\Delta_{\bar{\delta}_{|A^{\bullet,\bullet}}})$$

$$\mathcal{H}^{\bullet}_{\delta}(X) := \text{Ker}(\Delta_{\delta|A^\bullet}), \quad \mathcal{H}^{\bullet,\bullet}_{\delta}(X) := \text{Ker}(\Delta_{\delta|A^{\bullet,\bullet}})$$

are finite dimensional.

On the other side notice that in case of $\Delta_{\bar{\mu}}$, Δ_{μ} the spaces

$$\mathcal{H}^{\bullet,\bullet}_{\bar{\mu}}(X) := \text{Ker}\,\Delta_{\bar{\mu}} \quad \text{and} \quad \mathcal{H}^{\bullet,\bullet}_{\mu}(X) := \text{Ker}\,\Delta_{\mu}$$

are infinite-dimensional in general, since $\bar{\mu}$ and μ are linear over functions. Similarly, one can also consider the Bott-Chern and Aeppli Laplacians (cf. [**17**], [**16**])

$$\Delta_{BC} := (\partial\bar{\partial})(\partial\bar{\partial})^* + (\partial\bar{\partial})^*(\partial\bar{\partial}) + (\bar{\partial}^*\partial)(\bar{\partial}^*\partial)^* + (\bar{\partial}^*\partial)^*(\bar{\partial}^*\partial) + \bar{\partial}^*\bar{\partial} + \partial^*\partial\,,$$

$$\Delta_A \ := \ \partial\partial^* + \overline{\partial\partial}^* + (\partial\overline{\partial})^*(\partial\overline{\partial}) + (\partial\overline{\partial})(\partial\overline{\partial})^* + (\overline{\partial}\partial^*)^*(\overline{\partial}\partial^*) + (\overline{\partial}\partial^*)(\overline{\partial}\partial^*)^*\,,$$

$$\Delta_{BC(\delta,\bar{\delta})} := (\delta\bar{\delta})(\delta\bar{\delta})^* + (\delta\bar{\delta})^*(\delta\bar{\delta}) + (\bar{\delta}^*\delta)(\bar{\delta}^*\delta)^* + (\bar{\delta}^*\delta)^*(\bar{\delta}^*\delta) + \bar{\delta}^*\bar{\delta} + \delta^*\delta\,,$$

$$\Delta_{A(\delta,\bar{\delta})} \ := \ \delta\delta^* + \bar{\delta}\bar{\delta}^* + (\delta\bar{\delta})^*(\delta\bar{\delta}) + (\delta\bar{\delta})(\delta\bar{\delta})^* + (\bar{\delta}\delta^*)^*(\bar{\delta}\delta^*) + (\bar{\delta}\delta^*)(\bar{\delta}\delta^*)^*\,.$$

Notice that, since $\partial\overline{\partial} + \overline{\partial}\partial \neq 0$ a different choice of Δ_{BC} and Δ_A could have been done.

Proposition 2.2 (cf. [**16**], [**17**]). *Let (X, J, g) be a compact almost-Hermitian manifold. Then, the operators Δ_{BC}, Δ_A, $\Delta_{BC(\delta,\bar{\delta})}$, $\Delta_{A(\delta,\bar{\delta})}$ are elliptic differential operators of the fourth order, and so the associated spaces of harmonic forms*

$$\mathcal{H}^{\bullet,\bullet}_{BC}(X) := \text{Ker}\,(\Delta_{BC})\,, \quad \mathcal{H}^{\bullet,\bullet}_{A}(X) := \text{Ker}\,(\Delta_{A})$$

$$\mathcal{H}^{\bullet}_{BC(\delta,\bar{\delta})}(X) := \text{Ker}(\Delta_{BC(\delta,\bar{\delta})|A^\bullet}), \quad \mathcal{H}^{\bullet,\bullet}_{BC(\delta,\bar{\delta})}(X) := \text{Ker}(\Delta_{BC(\delta,\bar{\delta})|A^{\bullet,\bullet}})$$

$$\mathcal{H}^{\bullet}_{A(\delta,\bar{\delta})}(X) := \text{Ker}(\Delta_{A(\delta,\bar{\delta})|A^\bullet}), \quad \mathcal{H}^{\bullet,\bullet}_{A(\delta,\bar{\delta})}(X) := \text{Ker}(\Delta_{A(\delta,\bar{\delta})|A^{\bullet,\bullet}})$$

are finite dimensional.

2.1 - *Hodge decompositions*

As a consequence of the definitions and standard considerations using the L^2-product on forms we have the following decompositions on a compact almost-Hermitian manifold (X, J, g).

- Concerning the operators $\Delta_{\overline{\partial}}$ and Δ_{∂}
$$A^{p,q}(X) = \mathcal{H}^{p,q}_{\overline{\partial}}(X) \oplus \bar{\partial} A^{p,q-1}(X) \oplus \bar{\partial}^* A^{p,q+1}(X)$$
and
$$A^{p,q}(X) = \mathcal{H}^{p,q}_{\partial}(X) \oplus \partial A^{p-1,q}(X) \oplus \partial^* A^{p+1,q}(X).$$
Moreover, a (p,q)-form $\alpha \in \mathcal{H}^{p,q}_{\overline{\partial}}(X)$ if and only if $\overline{\partial}\alpha = 0$ and $\partial * \alpha = 0$. Similarly, a (p,q)-form $\alpha \in \mathcal{H}^{p,q}_{\partial}(X)$ if and only if $\partial\alpha = 0$ and $\overline{\partial} * \alpha = 0$.

- Concerning the operators $\Delta_{\bar{\delta}}$ and Δ_{δ}
$$A^k(X) = \mathcal{H}^k_{\bar{\delta}}(X) \oplus \bar{\delta} A^{k-1}(X) \oplus \bar{\delta}^* A^{k+1}(X)$$
and
$$A^k(X) = \mathcal{H}^k_{\delta}(X) \oplus \delta A^{k-1}(X) \oplus \delta^* A^{k+1}(X).$$
Moreover, for types reason, a (p,q)-form $\alpha \in \mathcal{H}^{p,q}_{\bar{\delta}}(X)$ if and only if $\alpha \in \mathcal{H}^{p,q}_{\overline{\partial}} \cap \mathcal{H}^{p,q}_{\mu}$. Similarly, a (p,q)-form $\alpha \in \mathcal{H}^{p,q}_{\delta}(X)$ if and only if $\alpha \in \mathcal{H}^{p,q}_{\partial} \cap \mathcal{H}^{p,q}_{\bar{\mu}}$. Namely,
$$\mathcal{H}^{p,q}_{\bar{\delta}}(X) = \mathcal{H}^{p,q}_{\overline{\partial}} \cap \mathcal{H}^{p,q}_{\mu} \quad \text{and} \quad \mathcal{H}^{p,q}_{\delta}(X) = \mathcal{H}^{p,q}_{\partial} \cap \mathcal{H}^{p,q}_{\bar{\mu}}.$$

- Concerning the operator Δ_{BC}
$$A^{p,q}(X) = \mathcal{H}^{p,q}_{BC}(X) \oplus \partial\bar{\partial} A^{p-1,q-1}(X) \oplus \left(\bar{\partial}^* A^{p,q+1}(X) + \partial^* A^{p+1,q}(X)\right) .$$
Moreover, a (p,q)-form $\alpha \in \mathcal{H}^{p,q}_{BC}(X)$ if and only if
$$\begin{cases} \partial\alpha & = & 0 \\ \overline{\partial}\alpha & = & 0 \\ \partial\overline{\partial} * \alpha & = & 0 \end{cases} .$$

- Concerning the operator Δ_A
$$A^{p,q}(X) = \mathcal{H}^{p,q}_{A}(X) \oplus \left(\partial A^{p-1,q}(X) + \overline{\partial} A^{p,q-1}(X)\right) \oplus \left(\partial\overline{\partial}\right)^* A^{p+1,q+1}(X) .$$
Moreover, a (p,q)-form $\alpha \in \mathcal{H}^{p,q}_{A}(X)$ if and only if
$$\begin{cases} \partial * \alpha & = & 0 \\ \overline{\partial} * \alpha & = & 0 \\ \partial\overline{\partial}\alpha & = & 0 \end{cases} .$$

- Concerning the operator $\Delta_{BC(\delta,\bar\delta)}$

$$A^k(X) = \mathcal{H}^k_{BC(\delta,\bar\delta)}(X) \overset{\perp}{\oplus} \left(\delta\bar\delta A^{k-2}(X) \oplus \left(\bar\delta^* A^{k+1}(X) + \delta^* A^{k+1}(X)\right)\right) .$$

Moreover, a (p,q)-form $\alpha \in \mathcal{H}^{p,q}_{BC(\delta,\bar\delta)}(X)$ if and only if

$$\begin{cases} \partial\alpha & = & 0 \\ \overline{\partial}\alpha & = & 0 \\ \mu\alpha & = & 0 \\ \bar\mu\alpha & = & 0 \\ (\partial\overline{\partial} + \bar\mu\mu)(*\alpha) & = & 0 \\ \partial\mu(*\alpha) & = & 0 \\ \bar\mu\overline{\partial}(*\alpha) & = & 0 \end{cases} .$$

- Concerning the operator $\Delta_{A(\delta,\bar\delta)}$

$$A^k(X) = \mathcal{H}^k_{A(\delta,\bar\delta)}(X) \overset{\perp}{\oplus} \left(\left(\delta A^{k-1}(X) + \bar\delta A^{k-1}(X)\right) \oplus \left(\delta\bar\delta\right)^* A^{k+2}(X)\right) .$$

Moreover, a (p,q)-form $\alpha \in \mathcal{H}^{p,q}_{A(\delta,\bar\delta)}(X)$ if and only if

$$\begin{cases} \partial^*\alpha & = & 0 \\ \overline{\partial}^*\alpha & = & 0 \\ \mu^*\alpha & = & 0 \\ \bar\mu^*\alpha & = & 0 \\ (\partial\overline{\partial} + \bar\mu\mu)\alpha & = & 0 \\ \partial\mu\alpha & = & 0 \\ \bar\mu\overline{\partial}\alpha & = & 0 \end{cases} .$$

2.2 - *Conjugation*

We now observe that conjugation provide several symmetries for the spaces of harmonic forms. In fact, by the very definitions one obtains

$$\overline{\Delta_\partial} = \Delta_{\overline{\partial}}\,, \qquad \overline{\Delta_\delta} = \Delta_{\bar\delta}\,, \qquad \overline{\Delta_{BC(\delta,\bar\delta)}} = \Delta_{BC(\delta,\bar\delta)}\,, \qquad \overline{\Delta_{A(\delta,\bar\delta)}} = \Delta_{A_(\delta,\bar\delta)}\,.$$

Hence for the spaces of harmonic forms

$$\overline{\mathcal{H}^{\bullet_1,\bullet_2}_{\overline{\partial}}(X)} = \mathcal{H}^{\bullet_2,\bullet_1}_{\partial}(X)\,, \qquad \overline{\mathcal{H}^{\bullet}_{\bar\delta}(X)} = \mathcal{H}^{\bullet}_{\delta}(X)\,,$$

$$\overline{\mathcal{H}^{\bullet}_{BC(\delta,\bar\delta)}(X)} = \mathcal{H}^{\bullet}_{BC(\delta,\bar\delta)}(X)\,, \qquad \overline{\mathcal{H}^{\bullet}_{A(\delta,\bar\delta)}(X)} = \mathcal{H}^{\bullet}_{A(\delta,\bar\delta)}(X)\,.$$

In particular, for any p, q

$$\overline{\mathcal{H}^{p,q}_{\bar\delta}(X)} = \mathcal{H}^{q,p}_{\delta}(X)\,,\ \ \overline{\mathcal{H}^{p,q}_{BC(\delta,\bar\delta)}(X)} = \mathcal{H}^{q,p}_{BC(\delta,\bar\delta)}(X)\,,\ \ \overline{\mathcal{H}^{p,q}_{A(\delta,\bar\delta)}(X)} = \mathcal{H}^{q,p}_{A(\delta,\bar\delta)}(X)\,.$$

Therefore, we have the following dimensional equalities for every k,

$$h^k_{\bar\partial}(X) = h^k_{\partial}(X)\,, \qquad h^k_{\bar\delta}(X) = h^k_{\delta}(X)$$

and for every p, q,

$$h^{p,q}_{\bar\delta}(X) = h^{q,p}_{\delta}(X)\,, \quad h^{p,q}_{BC(\delta,\bar\delta)}(X) = h^{q,p}_{BC(\delta,\bar\delta)}(X)\,, \quad h^{p,q}_{A(\delta,\bar\delta)}(X) = h^{q,p}_{A(\delta,\bar\delta)}(X)\,.$$

Where we denoted with h the dimension of the corresponding space of Harmonic forms $\mathcal{H}$.

2.3 - *Hodge duality*

Let (X, J, g) be a compact almost-Hermitian manifold of real dimension $2n$, then the $\mathbb{C}$-linear Hodge-$*$-operator induces duality isomorphisms. For every p, q

$$* : \mathcal{H}^{p,q}_{\bar\partial}(X) \to \mathcal{H}^{n-q,n-p}_{\partial}(X)\,, \quad * : \mathcal{H}^{p,q}_{\partial}(X) \to \mathcal{H}^{n-q,n-p}_{\bar\partial}(X)\,.$$

and for every k one has

$$* : \mathcal{H}^{k}_{\bar\delta}(X) \to \mathcal{H}^{2n-k}_{\delta}(X)\,, \quad * : \mathcal{H}^{k}_{\delta}(X) \to \mathcal{H}^{2n-k}_{\bar\delta}(X)\,.$$

In particular, for every p, q

$$* : \mathcal{H}^{p,q}_{\bar\delta}(X) \to \mathcal{H}^{n-q,n-p}_{\delta}(X)\,, \quad * : \mathcal{H}^{p,q}_{\delta}(X) \to \mathcal{H}^{n-q,n-p}_{\bar\delta}(X)\,.$$

In particular, we have the usual symmetries for the Hodge diamonds, namely for every p, q and for every k,

$$h^{p,q}_{\bar\partial}(X) = h^{n-q,n-p}_{\partial}(X)\,, \quad h^{p,q}_{\partial}(X) = h^{n-q,n-p}_{\bar\partial}(X)\,,$$

$$h^{k}_{\bar\delta}(X) = h^{2n-k}_{\delta}(X)\,, \quad h^{k}_{\delta}(X) = h^{2n-k}_{\bar\delta}(X)$$

$$h^{p,q}_{\bar\delta}(X) = h^{n-q,n-p}_{\delta}(X)\,, \quad h^{p,q}_{\delta}(X) = h^{n-q,n-p}_{\bar\delta}(X)\,.$$

Notice that one can combine this symmetries, as usual, with conjugation and obtain

$$h^{p,q}_{\bar\partial}(X) = h^{n-q,n-p}_{\partial}(X) = h^{n-p,n-q}_{\bar\partial}(X)$$

and so on.

Similarly, for every p, q

$$* : \mathcal{H}^{p,q}_{BC}(X) \to \mathcal{H}^{n-q,n-p}_{A}(X)\,.$$

And for every k, p, q

$$* : \mathcal{H}^{k}_{BC(\delta,\bar\delta)}(X) \to \mathcal{H}^{2n-k}_{A(\delta,\bar\delta)}(X)\,, \qquad * : \mathcal{H}^{p,q}_{BC(\delta,\bar\delta)}(X) \to \mathcal{H}^{n-q,n-p}_{A(\delta,\bar\delta)}(X)\,.$$

Therefore we have the usual symmetries for the Hodge diamonds, namely for every p, q, k

$$h^{p,q}_{BC}(X) = h^{n-q,n-p}_{A}(X)\,,$$

$$h^{k}_{BC(\delta,\bar\delta)}(X) = h^{2n-k}_{A(\delta,\bar\delta)}(X)\,, \quad h^{p,q}_{BC(\delta,\bar\delta)}(X) = h^{n-q,n-p}_{A(\delta,\bar\delta)}(X)\,.$$

Again, combining this with conjugation, one obtains for instance

$$h^{p,q}_{BC}(X) = h^{n-q,n-p}_{A}(X) = h^{n-p,n-q}_{A}(X)$$

and so on.

2.4 - *Relations among the Laplacians*

Let (X, J, g) be a compact almost-Hermitian manifold. As already observed if J is non-integrable we are able to appreciate the difference between the operators $\overline{\partial}$ and $\bar\delta$. In particular, the associated Laplacians will differ; more precisely it holds

$$\Delta_{\bar\delta} = \Delta_{\overline{\partial}} + \Delta_{\mu} + [\overline{\partial}, \mu^*] + [\mu, \overline{\partial}^*]$$

and

$$\Delta_{\delta} = \Delta_{\partial} + \Delta_{\bar\mu} + [\partial, \bar\mu^*] + [\bar\mu, \partial^*]\,.$$

In particular,

$$\mathcal{H}^{\bullet}_{\overline{\partial}}(X) \cap \mathcal{H}^{\bullet}_{\mu}(X) \subseteq \mathcal{H}^{\bullet}_{\bar\delta}(X)$$

even though on bi-graded forms we have the equality.

If we further assume that the almost-Hermitian metric is almost-Kähler then we have additional relations. Let (X, J, g, ω) be an almost-Kähler manifold, namely the fundamental form is symplectic, that is

$$d\omega = 0\,.$$

Then, as proved in [**8**] using the almost-Kähler identities, one has

$$\Delta_{\overline{\partial}} + \Delta_{\mu} = \Delta_{\partial} + \Delta_{\bar\mu}\,,$$

and
$$\Delta_d = 2\left(\Delta_{\bar\partial} + \Delta_\mu + [\bar\mu, \partial^*] + [\mu, \overline{\partial}^*] + [\partial, \overline{\partial}^*] + [\overline{\partial}, \partial^*]\right).$$

Similarly, in [**17**], using the almost-Kähler identities, it was proved that $\Delta_{\bar\delta}$ and Δ_δ are related by
$$\Delta_{\bar\delta} = \Delta_\delta$$
and
$$\Delta_d = \Delta_{\bar\delta} + \Delta_\delta + [\delta, \bar\delta^*] + [\bar\delta, \delta^*].$$
In particular, their spaces of harmonic forms coincide, i.e. $\mathcal{H}^\bullet_\delta(X) = \mathcal{H}^\bullet_{\bar\delta}(X)$.

In fact, one can use this result to characterize Kähler manifolds among the almost-Kähler ones.

Theorem 2.3. *Let (X, J, g, ω) be a compact almost-Kähler manifold, then*
$$\Delta_d = 2\Delta_\delta \iff (X, J, g, \omega) \text{ is Kähler.}$$

As a consequence one has in general that
$$\mathcal{H}^\bullet_{\bar\delta}(X) \subseteq \mathcal{H}^\bullet_{dR}(X),$$
namely every $\bar\delta$-harmonic form is harmonic. In particular, this gives a topological upper bound
$$h^\bullet_{\bar\delta}(X) \leq b_\bullet(X),$$
where $b_\bullet(X)$ denotes the Betti numbers of X. In [**17**] it is shown that the inequality $h^\bullet_{\bar\delta}(X) \leq b_\bullet(X)$ does not hold for an arbitrary compact almost-Hermitian manifold.

Similarly, the Bott-Chern Laplacian $\Delta_{BC(\delta,\bar\delta)}$ is related to $\Delta_{\bar\delta}$ as follows

Theorem 2.4. *Let (X, J, g, ω) be an almost-Kähler manifold, then $\Delta_{BC(\delta,\bar\delta)}$, $\Delta_{\bar\delta}$ are related by*
$$\Delta_{BC(\delta,\bar\delta)} = \Delta^2_{\bar\delta} + \bar\delta^*\bar\delta + \delta^*\delta + F_J$$
where
$$F_J := [[\delta, \bar\delta^*], \delta^*\bar\delta] - \delta[\delta, \bar\delta^*]\bar\delta^* - \delta^*[\delta, \bar\delta^*]\bar\delta.$$

In particular, the spaces of harmonic forms, as expected, coincide.

Proposition 2.5. *Let (X, J, g, ω) be a compact almost-Kähler manifold, then*
$$\mathcal{H}^\bullet_{BC(\delta,\bar\delta)}(X) = \mathcal{H}^\bullet_{\bar\delta}(X) = \mathcal{H}^\bullet_{A(\delta,\bar\delta)}(X).$$
$$\mathcal{H}^{\bullet,\bullet}_{BC(\delta,\bar\delta)}(X) = \mathcal{H}^{\bullet,\bullet}_d(X).$$

We point out that a similar result does not hold for $\mathcal{H}^{\bullet,\bullet}_{BC}(X)$ and $\mathcal{H}^{\bullet,\bullet}_{\bar\partial}(X)$ indeed in [**16**, Corollary 5.2] it is proved with an explicit example the following

Proposition 2.6 ([**16**]). *There exists an almost Kähler 4-manifold such that for some bi-degree*

$$\mathcal{H}^{\bullet,\bullet}_{BC}(X) \neq \mathcal{H}^{\bullet,\bullet}_{\bar\partial}(X).$$

We conclude this sections recalling that on almost-Kähler manifolds the powers of the symplectic form induce symmetries for the space of Bott-Chern harmonic forms. Namely, an Hard-Lefschetz type theorem holds.

Theorem 2.7. *Let (X, J, g, ω) be a compact almost-Kähler $2n$-dimensional manifold, then, for any k, the maps*

$$\omega^k \wedge - : \mathcal{H}^{n-k}_{BC(\delta,\bar\delta)}(X) \to \mathcal{H}^{n+k}_{BC(\delta,\bar\delta)}(X)$$

are isomorphisms.

3 - Harmonic forms on 4-dimensional almost-Hermitian manifolds

In this section we are going to describe the dependence of the dimensions of the spaces of harmonic forms considered above in the particular case of 4 dimensions. Let (X, J) be an almost-complex manifold and g an Hermitian metric on it. Then, differently from complex manifolds, the spaces of harmonic forms previously considered do not have a cohomological counterpart. In particular, the dimensions of such spaces might depend on the choice of the metric. This motivated the following question, raised by Kodaira and Spencer and inserted in a list of open problems by Hirzebruch in [**12**, Problem 20],

Question. *Let (X, J) be an almost-complex manifold. Choose an Hermitian metric on (X, J) and consider the numbers $h^{p,q}_{\bar\partial}$. Is $h^{p,q}_{\bar\partial}$ independent of the choice of the Hermitian metric?*

Clearly, the answer to this question is yes in special bi-degrees $(p, 0)$ and, by duality, (n, p). However, the general problem remained opened until 2020, when Holt and Zhang in [**14**] answered negatively to this question, showing that there exist almost-complex structures on the Kodaira-Thurston manifold such that the Hodge number $h^{0,1}_{\bar\partial}$ varies with different choices of Hermitian metrics. Later, in [**15**] they showed that $h^{0,1}_{\bar\partial}$ varies also with different choices of almost-Kähler metrics.

Furthermore, in [**14**, Proposition 6.1] the authors showed that for a compact 4-dimensional almost-Kähler manifold $h^{1,1}_{\bar\partial}$ is independent of the metric, and more precisely $h^{1,1}_{\overline{\partial}} = b_- + 1$, where b_- denotes the dimension of the space of the anti-self-dual harmonic 2-forms.

More precisely, in dimension 4 the hodge diamond for $\overline{\partial}$-harmonic forms is the following

$$\begin{array}{ccccc} & & h^{2,2}_{\overline{\partial}} & & \\ & h^{2,1}_{\overline{\partial}} & & h^{1,2}_{\overline{\partial}} & \\ h^{2,0}_{\overline{\partial}} & & h^{1,1}_{\overline{\partial}} & & h^{0,2}_{\overline{\partial}} \\ & h^{1,0}_{\overline{\partial}} & & h^{0,1}_{\overline{\partial}} & \\ & & h^{0,0}_{\overline{\partial}} & & \end{array}$$

where

- $h^{2,2}_{\overline{\partial}}$, $h^{1,2}_{\overline{\partial}}$, $h^{2,0}_{\overline{\partial}}$, $h^{0,2}_{\overline{\partial}}$, $h^{1,0}_{\overline{\partial}}$, $h^{0,0}_{\overline{\partial}}$ are metric independent
- $h^{0,1}_{\overline{\partial}} = h^{2,1}_{\overline{\partial}}$ depends on the metric and it is essentially unknown
- $h^{1,1}_{\overline{\partial}}$ deserves particular attention.

In [**18**] we studied the behavior of $h^{1,1}_{\overline{\partial}}$ on compact almost-Hermitian manifolds. More precisely, we proved the following

Main Theorem. *Let (X^4, J) be a compact almost-complex manifold of dimension 4 and let ω be an Hermitian metric, then if ω is globally conformally Kähler (in particular if it is almost-Kähler), it holds*

$$h^{1,1}_{\overline{\partial}} = b_- + 1.$$

If ω is (strictly) locally conformally Kähler,

$$h^{1,1}_{\overline{\partial}} = b_-.$$

Where, by (*strictly*) *locally conformally Kähler metric*, we mean an Hermitian metric ω, such that

$$d\omega = \theta \wedge \omega$$

with θ a d-closed, non d-exact, differential 1-form. The form θ is also called the *Lee form* of ω. The metric ω is called *globally conformally Kähler* if the Lee form θ is d-exact. Indeed, if $\theta = df$, for some smooth function f, then the conformal metric $e^{-f}\omega$ is almost-Kähler.

By the Main Theorem it follows that for locally conformally Kähler and globally conformally Kähler metrics on compact 4-dimensional almost-complex manifolds, $h^{1,1}_{\overline{\partial}}$ is a topological invariant. Notice that this was already known in the integrable case, by [**11**, Proposition II.6]. In fact, on compact complex surfaces

- $h^{1,1}_{\bar\partial} = b_- + 1$ if and only if there exists a Kähler metric
- $h^{1,1}_{\bar\partial} = b_-$ if and only if there exist no Kähler metrics.

In particular, recall that in the integrable case $h^{1,1}_{\bar\partial}$ does not depend on the metric but just on the complex structure. However, in the non integrable case in [**18**] we constructed explicitly on the Kodaira-Thurston manifold X a family of almost-complex structures J_a, with $a \in \mathbb{R}\backslash\{0\}$, $a^2 < 1$, on X that admit both almost-Kähler and (strictly) locally conformally Kähler metrics. Notice that in view of [**20**] a similar example cannot be constructed on complex manifolds.

More precisely, let

$$\mathbb{H}_3(\mathbb{R}) := \left\{ \begin{bmatrix} 1 & x_1 & x_3 \\ 0 & 1 & x_2 \\ 0 & 0 & 1 \end{bmatrix} \mid x_1, x_2, x_3 \in \mathbb{R} \right\}$$

be the 3-dimensional Heisenberg group and let Γ be the subgroup of $\mathbb{H}_3(\mathbb{R})$ of the matrices with integral entries. Then,

$$X := \frac{\mathbb{H}_3(\mathbb{R})}{\Gamma} \times \mathbb{S}^1$$

is a compact 4-dimensional nilmanifold admitting both complex and symplectic structures but no Kähler structures. Denoting with x_4 the coordinate on $\mathbb{S}^1$, a global coframe on X is given by

$$e^1 := dx_1\,, \quad e^2 := dx_2\,, \quad e^3 := dx_3 - x_1 dx_2\,, \quad e^4 := dx_4\,,$$

and so the structure equations become

$$de^1 = de^2 = de^4 = 0\,, \quad de^3 = -e^1 \wedge e^2\,.$$

Now we construct the following family of almost-complex structures J_a on X, with $a \in \mathbb{R}$, setting as coframe of $(1,0)$-forms

$$\Phi^1_a := (e^1 + a e^4) + i e^3\,, \qquad \Phi^2_a := e^2 + i e^4\,.$$

For every $a \in \mathbb{R}$, we fix the following Hermitian metric

$$\omega_a := \frac{i}{2}\left(\Phi^1_a \wedge \bar\Phi^1_a + \Phi^2_a \wedge \bar\Phi^2_a\right).$$

and it is direct to show that for $a \neq 0$, ω_a is a strictly locally conformally Kähler metric. Notice that, for $a^2 < 1$, J_a admits a compatible almost-Kähler metric given by

$$\tilde\omega_a := \omega_a + \frac{a}{2}\Phi^{1\bar2}_a - \frac{a}{2}\Phi^{2\bar1}_a.$$

Therefore, we found a family (X, J_a) of compact almost-complex 4-dimensional manifolds which admit an almost-Kähler metric $\tilde{\omega}_a$ and a non almost-Kähler Hermitian metric ω_a such that

$$h^{1,1}_{\overline{\partial}} = b_- \neq b_- + 1\,.$$

Hence, this example answers affirmatively to [**14**, Question 6.2] in the case of the Kodaira-Thurston manifold endowed with the 1-parameter family of almost-complex structures J_a.

Moreover, this answers to Kodaira and Spencer's question, showing that also the Hodge number $h^{1,1}_{\overline{\partial}}$ depends on the Hermitian metric and not just on the almost-complex structure.

As a consequence, since in dimension 4 for type reason $\mathcal{H}^{1,1}_{\overline{\delta}} = \mathcal{H}^{1,1}_{\overline{\partial}} \cap \mathcal{H}^{1,1}_{\mu} = \mathcal{H}^{1,1}_{\overline{\partial}}$ we get the following

Corollary 3.1. *Let (X^4, J) be a compact almost-complex manifold of dimension 4, then $h^{1,1}_{\overline{\delta}}$ depends on the choice of the Hermitian metric.*

In the following we are going to summarize the steps in the proof of the Main Theorem.

Step 1. Let (X^{2n}, J) be a compact $2n$-dimensional almost-complex manifold. For $p+q=n$, $h^{p,q}_{\overline{\partial}}$ is a conformal invariant of Hermitian metrics.

Suppose to have two conformal Hermitian metrics $\tilde{\omega} = \Phi\,\omega$, with Φ smooth positive function on X. Then, on (p,q)-forms the associated Hodge-$*$-operators are related by,

$$*_{\tilde{\omega}} = \Phi^{n-p-q} *_{\omega}\,.$$

In particular, if $p+q=n$ we have that such operators coincide

$$*_{\tilde{\omega}} = *_{\omega}$$

and so the harmonic forms are the same $\mathcal{H}^{p,q}_{\overline{\partial},\Phi\omega} = \mathcal{H}^{p,q}_{\overline{\partial},\omega}$.

Step 2. On a compact 4-dimensional almost-complex manifold, $h^{1,1}_{\overline{\partial}}$ is a conformal invariant of Hermitian metrics.

This follows immediately from the previous step, taking $n=2$ and $(p,q) = (1,1)$.

These first two steps show that we can always choose a suitable representative in the conformal class of an Hermitian metric in order to compute $h^{1,1}_{\overline{\partial}}$. Hence we recall that a *Gauduchon metric* is an Hermitian metric ω on a $2n$-dimensional almost-complex manifold (X, J) such that $dd^c\omega^{n-1} = 0$ or equivalently the Lee form is co-closed. These metrics are a very useful tool in conformal and almost-Hermitian geometry, in view of the celebrated result by Gauduchon, [**10**, Théorème 1], which states that if (X, J) is a $2n$-dimensional compact almost-complex manifold with $n > 1$, then there exists a unique (up to multiplication with positive constants) Gauduchon metric in every conformal class. In view of this, it is natural to study $\overline{\partial}$-harmonic $(1,1)$-forms with respect to Gauduchon metrics. This motivates the following step.

Step 3. Let (X^4, J) be a compact 4-dimensional almost-complex manifold and let ω be a Gauduchon metric, then $\psi \in \mathcal{H}^{1,1}_{\overline{\partial}}$ can be written as

$$\psi = \mathrm{const} \cdot \omega + \gamma \qquad \text{with } * \gamma = -\gamma\,.$$

This follows from the fact that a $(1,1)$-form $\psi \in A^{1,1}(X)$ can be decomposed as

$$\psi = f\omega + \gamma$$

with f smooth function and γ anti-self dual $(1,1)$-form, namely $*\gamma = -\gamma$.

Now, imposing that ψ is $\overline{\partial}$-harmonic, namely

$$\overline{\partial}\psi = 0 \quad \text{and} \quad \partial * \psi = 0$$

one directly obtains that

$$dd^c(f\omega) = 0.$$

Now, since ω is a Gauduchon metric this implies that f is constant (cf. [**11**], [**1**]).

Step 4. Let (X^4, J) be a 4-dimensional compact almost-complex manifold.

- if there exists a (strictly) locally conformally Kähler metric then, with respect to it, $h^{1,1}_{\overline{\partial}} = b_-$,
- if there exists a globally conformally Kähler metric then, with respect to it, $h^{1,1}_{\overline{\partial}} = b_- + 1$.

Suppose to have a (strictly) locally conformally Kähler metric then by [**11**] there exists a Gauduchon metric ω in the same conformal class. Now, since $h^{1,1}_{\overline{\partial}}$ is a conformal invariant, we can compute it using such representative ω that is now both a Gauduchon and a (strictly) locally conformally Kähler metric. This follows from the fact that the Hermitian metrics conformal to a (strictly) locally conformally Kähler are still (strictly) locally conformally Kähler. Indeed, if $\tilde{\omega} = \Phi\omega$, with $\Phi \in \mathcal{C}^\infty(X, \mathbb{R})$, $\Phi > 0$, are two conformal Hermitian metrics, then the associated Lee forms are related by

$$\theta_{\tilde{\omega}} = \theta_\omega + d \log \Phi\,,$$

in particular, $d\theta_{\tilde{\omega}} = d\theta_{\omega}$.

Now, let $\psi \in \mathcal{H}^{1,1}_{\overline{\partial}}$ then

$$\psi = \mathrm{c} \cdot \omega + \gamma \qquad \text{with } * \gamma = -\gamma \,,$$

and c constant. Then, one can show that

$$c \cdot \theta_\omega \in \operatorname{Im} d^*,$$

and this implies $c = 0$. Otherwise, one would have that $\theta_\omega \in \operatorname{Im} d^* \cap \operatorname{Ker} d$ namely $\theta_\omega = 0$ but this is absurd since $d\omega \neq 0$.

On the other side, suppose now to have a globally conformally Kähler metric then this means that there exists an almost-Kähler metric in the same conformal class. Since $h^{1,1}_{\overline{\partial}}$ is a conformal invariant, it can be computed using such almost-Kähler metric and by [**14**], $h^{1,1}_{\overline{\partial}} = b_- + 1$, and this concludes the proof.

Finally, Holt in [**13**, Theorem 3.1] closed the picture showing that b_- and $b_- + 1$ are the only two options for $h^{1,1}_{\overline{\partial}}$. More precisely,

Theorem 3.2 ([**13**]). *Let (X^4, J, ω) be a compact almost-Hermitian 4-dimensional manifold, then $h^{1,1}_{\overline{\partial}}$ is either b_- or $b_- + 1$.*

With similar techniques Piovani and Tomassini proved in [**16**, Theorem 4.3] that the same holds for Bott-Chern harmonic forms.

Theorem 3.3 ([**16**]). *Let (X^4, J, ω) be a compact almost-Hermitian 4-dimensional manifold, then $h^{1,1}_{BC}$ is either b_- or $b_- + 1$.*

This result was further improved in [**13**, Theorem 4.2] where Holt proves the following

Theorem 3.4 ([**13**]). *Let (X^4, J, ω) be a compact almost-Hermitian 4-dimensional manifold, then $h^{1,1}_{BC} = b_- + 1$.*

In particular, $h^{1,1}_{BC}$ does not depend on the metric but just on the topology on compact almost-Hermitian 4-dimensional manifolds.

Remark 3.5. We remark that, as soon as we consider almost-Hermitian manifolds of dimension 6 or higher, not much is known about the Hodge numbers. In [**19**] some explicit computations are performed on families of almost-Kähler and almost-Hermitian 6-dimensional solvmanifolds. In particular, we show that in dimension 6 the Hodge numbers can vary when the almost-complex structures are almost-Kähler and vary continuously.

Moreover, in [**5**] we show that on a compact $2n$-dimensional almost-Kähler manifold (X, J, g, ω) one can provide natural decompositions for the spaces $\mathcal{H}^{1,1}_{\overline{\partial}}$ and, dually $\mathcal{H}^{n-1,n-1}_{\overline{\partial}}$, that generalize the one in dimension 4 that was used in the proof of the Main Theorem, namely

$$\mathcal{H}^{1,1}_{\overline{\partial}} = \mathbb{C} \cdot \omega \oplus \left(\mathcal{H}^{1,1}_{\overline{\partial}} \cap P^{1,1} \right)$$

where $P^{1,1}$ denotes the space of primitive $(1,1)$-forms, namely $\alpha \in P^{1,1}$ if and only if $\omega^{n-1} \wedge \alpha = 0$.

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