Riv. Mat. Univ. Parma, Vol. **13** (2022), 353-372

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On finitely Levi non degenerate homogeneous CR manifolds

Abstract. A CR manifold M is a differentiable manifold together with a complex subbundle of the complexification of its tangent bundle, which is formally integrable and has zero intersection with its conjugate bundle. A fundamental invariant of a CR manifold M is its vector-valued Levi form. A Levi non degenerate CR manifold of order $k\geq 1$ has non degenerate Levi form in a higher order sense. For a (locally) homogeneous manifold Levi non degeneracy of order k can be described in terms of its CR algebra, i.e. a pair of Lie algebras encoding the structure of (locally) homogeneous CR manifolds. I will give an introduction to these topics presenting some recent results.

Keywords. Lie pair, CR algebra, Lie algebra extension, Levi degeneracy.

Mathematics Subject Classification: Primary: 32V35, 32V40, Secondary: 17B22, 17B10.

Introduction

Cauchy–Riemann manifolds, in brief CR manifolds, are the abstract models of real hypersurfaces in complex manifolds. A natural invariant of a CR manifold is its Levi form, i.e. an hermitian symmetric form on the space of tangent holomorphic vector fields, which, when the CR codimension is larger than one, is vector valued. The focus of this survey is to present some recent results for Levi non degenerate of order $k\geq 1$ [**9**, **10**] homogeneous CR manifolds. This notion of Levi non degeneracy in a higher order sense provides for example an obstruction from having an infinite dimensional group of local CR automorphisms [**14**]. In the case of homogeneous CR manifolds this notion of non degeneracy can be rephrased in terms of their associated CR

Received: November 26, 2021; accepted in revised form: April 22, 2022.

algebras [**14**, **15**, **17**, **21**]. In fact iterations of the Levi forms can be described by building descending chains of algebras of vector fields, whose lengths can be taken as a measure of Levi non degeneracy order. The real submanifolds M of a complex flag manifold which are real orbits, form an interesting class of homogeneous CR manifolds [**1**, **3**, **17**, **18**, **19**]. In [**7**] G. Fels showed that when the isotropy $\mathbf{Q}$ of X is a maximal parabolic subgroup, and M is Levi non degenerate of order k, then k is at most 2. In §2.4 we prove that the bound of $k \leq 3$ is valid for general Levi non degenerate of order k real orbits, dropping the maximality assumption on $\mathbf{Q}$. Moreover in the same paper G. Fels posed the question of the existence of Levi non degenerate homogeneous CR manifold with order larger than 3. In §2.5 we exhibit, by constructing some CR vector bundles over $\mathbb{CP}^1$, a Levi non degenerate homogeneous CR manifold of order k for every positive integer $k \geq 1$.

1 - Preliminaries on CR manifolds

In this section we discuss some notions of non degeneracy for general smooth abstract CR manifolds of type (n, k). We will eventually be interested in the locally homogeneous case and therefore, in the rest of this section, in their reformulation in the framework of Lie algebras theory.

1.1 - *CR manifolds*

A CR manifold of type (n, k) is defined as the pair $(\mathsf{M}, T^{0,1}\mathsf{M})$, of a smooth manifold M of real dimension $2n+k$, and a rank n smooth complex linear subbundle $T^{0,1}\mathsf{M}$ of its complexified tangent bundle $T^{\mathbb{C}}\mathsf{M}$, satisfying

(1) $T^{0,1}\mathsf{M} \cap \overline{T^{0,1}\mathsf{M}} = \{0\}$;

(2) $[\Gamma^\infty(\mathsf{M}, T^{0,1}\mathsf{M}), \Gamma^\infty(\mathsf{M}, T^{0,1}\mathsf{M})] \subseteq \Gamma^\infty(\mathsf{M}, T^{0,1}\mathsf{M})$;

The (2) is called *formal integrability condition.* Here n and k are called *CR dimension* and *CR codimension* respectively. We use the following notations:

- $T^{1,0}\mathsf{M} \doteq \overline{T^{0,1}\mathsf{M}}$;
- $H^{\mathbb{C}}\mathsf{M} \doteq T^{1,0}\mathsf{M} \oplus T^{0,1}\mathsf{M}$;
- $H\mathsf{M} \doteq H^{\mathbb{C}}\mathsf{M} \cap T\mathsf{M}$;

where the rank $2n$ real subbundle $H\mathsf{M}$ of $T\mathsf{M}$ is the *real contact distribution* underlying the CR structure of M. A smooth $\mathbb{R}$-linear bundle endomorphism

$\mathsf{J} : H\mathsf{M} \to H\mathsf{M}$ is defined by the equation $T^{0,1}\mathsf{M} = \{X + i\mathsf{J}X \mid X \in H\mathsf{M}\}$. The map J squares to $-Id$ and it is the *partial complex structure* of M. An equivalent definition of the CR structure can be given by assigning first an even dimensional real distribution $H\mathsf{M}$ and then a smooth partial complex structure J on $H\mathsf{M}$ in such a way that the complex distribution $T^{0,1}\mathsf{M}$ satisfies (1) and (2).

Notation 1.1. With an abuse of notation, to facilitate the reader, we will still use the notation $T\mathsf{M}$, $H\mathsf{M}$,, $H^{\mathbb{C}}\mathsf{M}$, $T^{0,1}\mathsf{M}$, $T^{1,0}\mathsf{M}$ for the sheaf of germs of smooth sections.

We finish giving the following definition:

Definition 1.2. A CR manifold M is called *fundamental* at its point x if $H\mathsf{M}$ bracket generates the Lie algebra $T\mathsf{M}$.

1.2 - *Finitely Levi non degenerate CR manifolds*

Let M be a CR manifold of type (n, k), then the *complex Levi form* of M at x is the Hermitian symmetric map

$$\mathcal{L}_x : T^{0,1}_x\mathsf{M} \times T^{1,0}_x\mathsf{M} \to T_x\mathsf{M}^{\mathbb{C}} \big/ H^{\mathbb{C}}_x\mathsf{M},$$

defined by $\mathcal{L}_x(Z, \overline{W}) = \frac{1}{2i}\hat{\pi}_x([Z, \overline{W}])$, where $\hat{\pi}_x$ is the canonical projection $T_x\mathsf{M} \otimes \mathbb{C} \to T_x\mathsf{M}^{\mathbb{C}} \big/ H^{\mathbb{C}}_x\mathsf{M}$, and $Z, W \in T^{0,1}\mathsf{M}$ are smooth sections. Similary by the standard isomorphism from $T^{0,1}\mathsf{M}$ onto $H\mathsf{M}$, we can define the *real vector Levi form* as the real bilnear form

$$\mathcal{L}^{\mathbb{R}}_x : H_x\mathsf{M} \times H_x\mathsf{M} \to T_x\mathsf{M} \big/ H_x\mathsf{M}$$

defined by $\mathcal{L}^{\mathbb{R}}_x(X, Y) = \pi_x([\mathsf{J}X, Y] - [X, \mathsf{J}Y])$ for $X, Y \in H\mathsf{M}$. Observe that the *Levi form* measures whether the subbundles $H\mathsf{M}$ of $T\mathsf{M}$ is integrable and how the complex structure J interplays with the integrability on the respective fibers. We define the *Levi Kernel* at $x \in M$ as the null space of the Levi form:

$$Null(\mathcal{L}) = \{Z \in T^{1,0}\mathsf{M} \mid \mathcal{L}(Z, W) = 0,\ \forall W \in T^{1,0}\mathsf{M}\}.$$

Definition 1.3. A CR manifold $(\mathsf{M}, T^{0,1}\mathsf{M})$ is *(strictly) Levi non degenerate* in $x \in \mathsf{M}$ if the Levi form has $Null_x(\mathcal{L}) = \{0\}$.

A first generalization of this definition can be obtained considering itered bracket, checking at x what is the smallest k for which, given any nonzero germ $\bar{Z} \in T^{0,1}_x\mathsf{M}$ we can find a $k' \leq k$ and $Z_1, \ldots, Z_{k'} \in T^{1,0}_x\mathsf{M}$ such that

$$[Z_1, [Z_2, \ldots, [Z_{k'}, \bar{Z}]]] \notin H^{\mathbb{C}}_x\mathsf{M}.$$

For this purpose we define recursively a nested sequence of sheaves of germs of smooth complex valued vector fields on M

$$\mathrm{F}^0 \supseteq \mathrm{F}^1 \supseteq \cdots \supseteq \mathrm{F}^{\mathrm{k}} \supseteq \mathrm{F}^{\mathrm{k}+1} \supseteq \cdots \tag{1.1}$$

by setting

$$\begin{cases} \mathrm{F}^0 = T^{0,1}\mathsf{M}, \\ \mathrm{F}^{\mathrm{k}} = \bigsqcup_{x\in\mathsf{M}} \left\{ Z \in \mathrm{F}^{k-1}_x \;\middle|\; [Z, T^{1,0}_x\mathsf{M}] \subseteq \mathrm{F}^{k-1}_x + T^{1,0}_x\mathsf{M} \right\}, \text{ for } k \geq 1. \end{cases}$$

This sequence was considered by Freeman in [**9**, Thm.3.1].

Definition 1.4. The CR manifold M is, at its point x, *Levi non degenerate* of order k if $\mathrm{F}^{k-1}_x \supsetneq \mathrm{F}^k_x = \{0\}$, otherwise we will say that M is *holomorphically degenerate*.

1.3 - *Homogeneous CR manifolds*

Let $\mathbf{G}_{\mathbb{R}}$ be a real Lie group of CR diffeomorphisms, i.e. that leave the CR structure stable, acting transitively on a CR manifold M. Fix a point x of M and let $\pi : \mathbf{G}_{\mathbb{R}} \ni g \to g{\cdot}x \in \mathsf{M}$ be the natural projection. The differential at x defines a map $\pi_* : \mathfrak{g}_{\mathbb{R}} \to T_x\mathsf{M}$ of the Lie algebra $\mathfrak{g}_{\mathbb{R}}$ of $\mathbf{G}_{\mathbb{R}}$ onto the tangent space to M at x. By the *formal integrability* condition of $T^{0,1}_x\mathsf{M}$, the pullback $\mathfrak{q} \doteq (\pi^{\mathbb{C}}_*)^{-1}(T^{0,1}_x\mathsf{M})$ by the complexification of the differential is a complex Lie subalgebra $\mathfrak{q}$ of the complexification $\mathfrak{g} \doteq \mathbb{C} \otimes_{\mathbb{R}} \mathfrak{g}_{\mathbb{R}}$. Vice versa, the assignment of a complex Lie subalgebra $\mathfrak{q}$ of $\mathfrak{g}$ yields a formally integrable $\mathbf{G}_{\mathbb{R}}$-equivariant partial complex structure on a (locally) $\mathbf{G}_{\mathbb{R}}$-homogeneous space M requiring that $T^{0,1}_x\mathsf{M} \doteq \pi^{\mathbb{C}}_*(\mathfrak{q})$ (see e.g. [**1**, **21**]). We give the following definition:

Definition 1.5. A *CR algebra* is a pair $(\mathfrak{g}_{\mathbb{R}}, \mathfrak{q})$, consisting of a real Lie algebra $\mathfrak{g}_{\mathbb{R}}$ and a complex Lie subalgebra $\mathfrak{q}$ of its complexification $\mathfrak{g} = \mathbb{C} \otimes_{\mathbb{R}} \mathfrak{g}_{\mathbb{R}}$, such that the quotient $\mathfrak{g}_{\mathbb{R}}/(\mathfrak{g}_{\mathbb{R}} \cap \mathfrak{q})$ is a finite dimensional real vector space.

Roughly speaking the CR algebra $(\mathfrak{g}_{\mathbb{R}}, \mathfrak{q})$, is a pair of a real algebra $\mathfrak{g}_{\mathbb{R}}$ enconding the group of CR diffeomorphisms and a complex Lie algebra $\mathfrak{q}$ encoding the CR structure. We call the intersection $\mathfrak{q} \cap \mathfrak{g}_{\mathbb{R}}$ its *isotropy subalgebra*, which can be seen as the Lie algebra of the stabilizer of the point x by the action of $\mathbf{G}_{\mathbb{R}}$, and say that $(\mathfrak{g}_{\mathbb{R}}, \mathfrak{q})$ is *effective* when $\mathfrak{g}_{\mathbb{R}} \cap \mathfrak{q}$ does not contain any nontrivial ideal of $\mathfrak{g}_{\mathbb{R}}$.

Following the standard definition of fundamental distribution, we have:

Definition 1.6. We call *fundamental* a CR algebra $(\mathfrak{g}_{\mathbb{R}}, \mathfrak{q})$ such that $\mathfrak{q} + \bar{\mathfrak{q}}$ generates $\mathfrak{g}$ as a Lie algebra.

2 - Finitely Levi non degenerate homogeneous CR manifolds

In this section we reharse the definition of Levi non degeneracy of order $k\geq 1$ in the context of homogeneous CR manifolds. After that we present a bound result for the order of Levi non degeneracy for real orbits in a complex flag manifold. We conclude then by presenting an example of homogeneous CR manifold with an arbitrary integer order of Levi non degeneracy.

2.1 - *Levi non degeneracy of order k in the homogeneous context*

For a Levi non degenerate of order $k\geq 1$ (locally) homogeneous CR manifold M, we compute k by using its associated CR algebra $(\mathfrak{g}_{\mathbb{R}}, \mathfrak{q})$. Observe that the complexification of the isotropy subalgebra $\mathfrak{g}_{\mathbb{R}}\cap\mathfrak{q}$ equals $\mathfrak{q}\cap\bar{\mathfrak{q}}$, then the (strongly) non degeneracy of the Levi form (1.3) can be stated by

$$\forall Z \in \mathfrak{q}\backslash\bar{\mathfrak{q}}, \quad \exists\, Z' \in \bar{\mathfrak{q}} \quad \text{such that} \quad [Z, Z'] \notin \mathfrak{q} + \bar{\mathfrak{q}},$$

this is equivalent to

$$\mathfrak{q}^{(1)} \doteq \{Z \in \mathfrak{q} \mid [Z, \bar{\mathfrak{q}}] \subseteq \mathfrak{q} + \bar{\mathfrak{q}}\} = \mathfrak{q} \cap \bar{\mathfrak{q}}.$$

Following the generalization of non degeneracy given by the Freeeman sequence (1.1), in the homogeneous case one can consider a $Z\in\mathfrak{q}\backslash(\mathfrak{q}\cap\bar{\mathfrak{q}})$, to seek whether it is possible to find $\overline{Z}_1, \ldots, \overline{Z}_k \in \bar{\mathfrak{q}}$ such that

$$[\overline{Z}_1, \ldots, \overline{Z}_k, Z] \notin \mathfrak{q}+\bar{\mathfrak{q}}.$$

To this aim, it is convenient to consider the descending chain (see e.g. [**7**, **9**, **10**, **14**, **21**])

$$\mathfrak{q}^{(0)} \supseteq \mathfrak{q}^{(1)} \supseteq \cdots \supseteq \mathfrak{q}^{(k-1)} \supseteq \mathfrak{q}^{(k)} \supseteq \mathfrak{q}^{(k+1)} \supseteq \cdots, \tag{2.1}$$

with

$$\begin{cases} \mathfrak{q}^{(0)} = \mathfrak{q}, \\ \mathfrak{q}^{(k)} = \{Z \in \mathfrak{q}^{(k-1)} \mid [Z, \bar{\mathfrak{q}}] \subseteq \mathfrak{q}^{(k-1)} + \bar{\mathfrak{q}}\} \quad \text{for } p\geq 1. \end{cases}$$

Note that $\mathfrak{q}\cap\bar{\mathfrak{q}}\subseteq\mathfrak{q}^{(k)}$ for all integers $k\geq 0$. Since by assumption $\mathfrak{q}/(\mathfrak{q}\cap\bar{\mathfrak{q}})$ is finite dimensional, there exists a smallest nonnegative integer k such that $\mathfrak{q}^{(k')}=\mathfrak{q}^{(k)}$ for all $k'\geq k$. We call (2.1) the *descending Levi chain* of $(\mathfrak{g}_{\mathbb{R}}, \mathfrak{q})$.

Definition 2.1. Let k be a positive integer. The CR algebra $(\mathfrak{g}_{\mathbb{R}}, \mathfrak{q})$ is said to be *Levi non degenerate* of *order k* if $\mathfrak{q}^{(k-1)}\supsetneqq\mathfrak{q}^{(k)}=\mathfrak{q}\cap\bar{\mathfrak{q}}$. Otherwise we say that $(\mathfrak{g}_{\mathbb{R}}, \mathfrak{q})$ is *holomorphically degenerate* if $\mathfrak{q}^{(k)}\neq\mathfrak{q}\cap\bar{\mathfrak{q}}$ for all integers $k>0$.

Proposition 2.2. *The terms* $\mathfrak{q}^{(k)}$ *of* (2.1) *are Lie subalgebras of* $\mathfrak{q}$.

Proof. By definition, $\mathfrak{q}^{(0)} = \mathfrak{q}$ is a Lie subalgebra of $\mathfrak{q}$. If $Z_1, Z_2 \in \mathfrak{q}^{(1)}$, then

$$[[Z_1, Z_2], \bar{\mathfrak{q}}] \subseteq [Z_1, [Z_2, \bar{\mathfrak{q}}]] + [Z_2, [Z_1, \bar{\mathfrak{q}}]] \subseteq [Z_1 + Z_2, \mathfrak{q} + \bar{\mathfrak{q}}] \subseteq \mathfrak{q} + \bar{\mathfrak{q}}$$

because $[Z_i, \mathfrak{q}] \subseteq [\mathfrak{q}, \mathfrak{q}] \subseteq \mathfrak{q}$, and $[Z_i, \bar{\mathfrak{q}}] \subseteq \mathfrak{q} + \bar{\mathfrak{q}}$ by the definition of $\mathfrak{q}^{(1)}$. This shows that $\mathfrak{q}^{(1)}$ is a Lie subalgebra of $\mathfrak{q}$. Next we argue by recurrence. Let $k \geq 1$ and assume that $\mathfrak{q}^{(k)}$ is a Lie subalgebra of $\mathfrak{q}$. If $Z_1, Z_2 \in \mathfrak{q}^{(k+1)}$, then $[Z_1, Z_2] \in \mathfrak{q}^{(k)}$ by the inductive assumption that $\mathfrak{q}^{(k)}$ is a Lie subalgebra and

$$[[Z_1, Z_2], \bar{\mathfrak{q}}] \subseteq [Z_1, [Z_2, \bar{\mathfrak{q}}]] + [Z_2, [Z_1, \bar{\mathfrak{q}}]] \subseteq [Z_1 + Z_2, \mathfrak{q}^{(k)} + \bar{\mathfrak{q}}] \subseteq \mathfrak{q}^{(k)} + \bar{\mathfrak{q}},$$

showing that also $[Z_1, Z_2] \in \mathfrak{q}^{(k+1)}$. This completes the proof. $\square$

Remark 2.3. We point out that the *weak non degeneracy* defined in [**21**], is equivalent to Definition (2.1) consisting in the requirement that, for a complex Lie subalgebra $\mathfrak{q}'$ of $\mathfrak{g}$,

$$\mathfrak{q} \subseteq \mathfrak{q}' \subseteq \mathfrak{q} + \bar{\mathfrak{q}} \implies \mathfrak{q}' = \mathfrak{q}.$$

Indeed, it easily follows from [**21**, Lemma 6.1] that

$$\mathfrak{q}' = \mathfrak{q} + \bar{\mathfrak{q}}^{(\infty)}, \quad \text{with} \quad \mathfrak{q}^{(\infty)} = \bigcap\nolimits_{k \geq 0} \mathfrak{q}^{(k)}$$

is the largest complex Lie subalgebra $\mathfrak{q}'$ of $\mathfrak{g}$ with $\mathfrak{q} \subseteq \mathfrak{q}' \subseteq \mathfrak{q} + \bar{\mathfrak{q}}$.

2.2 - *Real orbits in Complex flag manifolds*

We reharse that a real Lie algebra $\mathfrak{g}_{\mathbb{R}}$ is called a *real form* of a complex Lie algebra $\mathfrak{g}$ if $\mathfrak{g} \simeq \mathfrak{g}_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$. A complex flag manifold X is a smooth compact algebraic variety that can be described as the quotient of a complex semisimple Lie group $\mathbf{G}$, i.e. its Lie algebra been semisimple, by a parabolic subgroup $\mathbf{Q}$, i.e. a subgroup containing a maximal solvable subgroup named Borel subgroup. In [**25**] A. J. Wolf shows that a real form $\mathbf{G}_{\mathbb{R}}$ of $\mathbf{G}$ has as a finite number of orbits in X, with only one of them, being compact and having minimal dimension. With the partial complex structures induced by X, these orbits make a class of homogeneous CR manifolds that were studied by many authors (see e.g. [**1**,**2**,**3**, **6**,**7**,**8**,**11**,**13**,**17**,**18**,**19**]). Being connected and simply connected, a complex flag manifold $\mathsf{X} = \mathbf{G}/\mathbf{Q}$ is completely described by the Lie pair $(\mathfrak{g}, \mathfrak{q})$ consisting of the Lie algebras of $\mathbf{G}$ and of $\mathbf{Q}$ and vice versa to any Lie pair $(\mathfrak{g}, \mathfrak{q})$ of a complex semisimple Lie algebra and a parabolic subalgebra $\mathfrak{q}$ corresponds a unique flag manifold X. Therefore the classification of complex flag manifolds reduces to that of parabolic subalgebras of semisimple complex Lie algebras. Parabolic subalgebras $\mathfrak{q}$ of $\mathfrak{g}$ are classified, modulo automorphisms, by a finite set of parameters as follow. Fix any Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$, i.e. a selfnormalizing nilpotent subalgebra, then a set of *roots*, denoted by $\mathcal{R}(\mathfrak{g}, \mathfrak{h})$, are the nonzero elements α of $\mathfrak{h}^*$ such that $\mathfrak{g}_\alpha \doteq \{Z \in \mathfrak{g} \mid [H, Z] = \alpha(H)\, Z,\ \forall H \in \mathfrak{h}\} \neq \{0\}$, which is called the weightspace. For each root $\alpha \in \mathcal{R}$, $\mathfrak{g}_\alpha$ is a one-dimensional complex vector space, and $\mathfrak{g} = \bigoplus_{\alpha \in \mathcal{R}(\mathfrak{g},\mathfrak{h})} \mathfrak{g}_\alpha$. A root $\alpha \in \mathcal{R}$ is called *simple* if it can't be written as sum of other two roots in $\mathcal{R}$. The equivalence classes of root system are in one to one correspondence with the subsets of a basis $\mathcal{B}$ of simple roots of the root system $\mathcal{R}$ of $(\mathfrak{g}, \mathfrak{h})$ (see e.g. [**5**, Ch.VIII,§3.4]). We now recall the definition of Dynkin diagram $\Delta_{\mathcal{B}}$: this is a graph with no loops, whose nodes are the roots in $\mathcal{B}$ and in which two nodes may be joined by at most 3 edges. Each root β in $\mathcal{R}$ can be written in a unique way as a notrivial linear combination $\beta = \sum_{\alpha \in \mathcal{B}} k_{\beta,\alpha}\alpha$, with integral coefficients $k_{\beta,\alpha}$ which are either all ≥ 0, or all ≤ 0. Set $\operatorname{supp}(\beta) = \{\alpha \in \mathcal{B} \mid k_{\beta,\alpha} \neq 0\}$; then the parabolic subalgebras $\mathfrak{q}$ are parametrized modulo isomorphisms, by subsets Φ of $\mathcal{B}$ as follows: to any $\Phi \subseteq \mathcal{B}$ we associate

$$\mathsf{Q}_\Phi = \{\beta \in \mathcal{R} \mid k_{\beta,\alpha} \leq 0,\ \forall \alpha \in \Phi\} \subseteq \mathcal{R}, \tag{2.2}$$

with $\mathfrak{q}_\phi = \mathfrak{h} \oplus \sum_{\beta \in \mathsf{Q}_\Phi} \mathfrak{g}_\beta$. The set Q_Φ is a called a *parabolic* set of roots, i.e. $(\mathsf{Q}_\Phi + \mathsf{Q}_\Phi) \cap \mathcal{R} \subseteq \mathsf{Q}_\Phi$ *and* $\mathsf{Q}_\Phi \cup (-\mathsf{Q}_\Phi) = \mathcal{R}$ and $\mathfrak{q}_\phi$ is a the Lie algebra of parabolic subgroup in the above sense. To specify the $\mathfrak{q}_\Phi$ of (2.2) we can cross the nodes corresponding to the roots in Φ. In this way each cross-marked Dynkin diagram encodes a specific complex flag manifold X_Φ.

Notation 2.4. Let ξ_Φ be the linear functional on the linear span of $\mathcal{R}$ which equals one on the roots in Φ and zero on those in $\mathcal{B} \setminus \Phi$. Then

$$\mathsf{Q}_\Phi = \{\beta \in \mathcal{R} \mid \xi_\Phi(\beta) \leq 0\} \tag{2.3}$$

and we get partitions

$$\mathsf{Q}_\Phi = \mathsf{Q}^r_\Phi \cup \mathsf{Q}^n_\Phi, \quad \mathcal{R} = \mathsf{Q}^r_\Phi \cup \mathsf{Q}^n_\Phi \cup \mathsf{Q}^c_\Phi, \tag{2.4}$$

with,

- $\mathsf{Q}^r_\Phi \doteq \{\beta \in \mathsf{Q}_\Phi \mid -\beta \in \mathsf{Q}_\Phi\} = \{\beta \in \mathcal{R} \mid \xi_\Phi(\beta) = 0\}$,
- $\mathsf{Q}^n_\Phi \doteq \{\beta \in \mathsf{Q}_\Phi \mid -\beta \notin \mathsf{Q}_\Phi\} = \{\beta \in \mathcal{R} \mid \xi_\Phi(\beta) < 0\}$,

- $\mathsf{Q}^c_\Phi \doteq \{\beta \in \mathcal{R} \mid -\beta \in \mathsf{Q}^n_\Phi\} = \{\beta \in \mathcal{R} \mid \xi_\Phi(\beta) > 0\}$.

We recall (see e.g. [**5**, Ch.VIII,§3]):

- $\mathfrak{q}^r_\Phi = \mathfrak{h} \oplus \sum_{\beta \in \mathsf{Q}^r_\Phi} \mathfrak{s}^\beta$ is a reductive complex Lie algebra;
- $\mathfrak{q}^n_\Phi = \sum_{\beta \in \mathsf{Q}^n_\Phi} \mathfrak{s}^\beta$ is the nilradical of $\mathfrak{q}_\Phi$;
- $\mathfrak{q}_\Phi = \mathfrak{q}^r_\Phi \oplus \mathfrak{q}^n_\Phi$ is the Levi-Chevalley decomposition of $\mathfrak{q}_\Phi$;
- $\mathfrak{q}^c_\Phi = \sum_{\beta \in \mathsf{Q}^c_\Phi} \mathfrak{s}^\beta$ is a Lie subalgebra of $\mathfrak{g}$ consisting of $\mathrm{ad}_{\mathfrak{s}}$-nilpotent elements;
- $\mathfrak{q}^\vee_\Phi = \mathfrak{q}^r_\Phi \oplus \mathfrak{q}^c_\Phi$ is the parabolic Lie subalgebra of $\mathfrak{s}$ *opposite of* $\mathfrak{q}_\Phi$, decomposed into the direct sum of its reductive subalgebra $\mathfrak{q}^r_\Phi$ and its nilradical $\mathfrak{q}^c_\Phi$.

Let us take, as we can, $\mathbf{G}$ connected and simply connected. Then real automorphisms of its Lie algebra $\mathfrak{g}$ lift to automorphisms of the Lie group $\mathbf{G}$, so that real forms $\mathbf{G}_\mathbb{R}$ of $\mathbf{G}$ are in one-to-one correspondence with the anti-$\mathbb{C}$-linear involutions σ of $\mathfrak{g}$. We will denote by $\mathfrak{g}^\sigma$ the real Lie subalgebra consisting of the fixed points of σ, i.e. it is the Lie algebra of the real form $\mathbf{G}^\sigma$ of fixed points of the lift $\tilde{\sigma}$ of σ to $\mathbf{G}$. Its orbits are CR submanifolds $\mathsf{M}_{\Phi,\sigma}$ of X_Φ whose CR algebra at the base point is the pair $(\mathfrak{g}^\sigma, \mathfrak{q}_\Phi)$.

Definition 2.5 (cf. [**1**, §5]). A *parabolic CR algebra* is a pair $(\mathfrak{g}^\sigma, \mathfrak{q}_\Phi)$ consisting of a real semisimple Lie algebra $\mathfrak{g}^\sigma$ and a parabolic complex Lie subalgebra $\mathfrak{q}_\Phi$ of its complexification $\mathfrak{g}$.

To list all the orbits of a real form, one can use the fact that the *isotropy subalgebra* $\mathfrak{g}^\sigma \cap \mathfrak{q}$ contains a Cartan subalgebra $\mathfrak{h}_\mathbb{R}$ of $\mathfrak{g}^\sigma$ (see e.g. [**3**]). By choosing $\mathfrak{h}$ equal to its complexification, we obtain on $\mathcal{R}$ a conjugation which is compatible with the one defined on $\mathfrak{g}$ by its real form $\mathfrak{g}^\sigma$ (and which, for simplicity, we still denote by σ). Vice versa, an orthogonal involution σ of $\mathcal{R}$ lifts, although in general not in a unique way, to a conjugation of $\mathfrak{g}$. The subalgebras $\mathfrak{q}_\Phi \cap \bar{\mathfrak{q}}_\Phi$, $\mathfrak{q}_\Phi$ and $\bar{\mathfrak{q}}_\Phi$ turn out to be direct sums of $\mathfrak{h}$ and root subspaces $\mathfrak{g}_\alpha$; in particular $\mathfrak{q}_\Phi \cap \bar{\mathfrak{q}}_\Phi$ is the direct sum of $\mathfrak{h}$ and the root subspaces $\mathfrak{g}_\alpha$ with $\mathfrak{g}_\alpha + \mathfrak{g}_{\bar{\alpha}} \subset \mathfrak{q}_\Phi$. We note that $\mathfrak{q}_\Phi \cap \bar{\mathfrak{q}}_\Phi$ is a Lie subalgebra of $\mathfrak{g}$ and $(\mathfrak{q}_\Phi + \bar{\mathfrak{q}}_\Phi)$ is a $(\mathfrak{q}_\Phi \cap \bar{\mathfrak{q}}_\Phi)$-module. We point out that different choices of σ may yield the same CR submanifold $\mathsf{M}_{\Phi,\sigma}$.

2.3 - *Conditions for Levi non degeneracy of order k*

We have

Proposition 2.6. *A real orbit $M_{\Phi,\sigma}$ is fundamental iff its CR algebra $(\mathfrak{g}^\sigma, \mathfrak{q}_\Phi)$ is fundamental. Let $(\mathfrak{g}^\sigma, \mathfrak{q}_\Phi)$ be a parabolic CR algebra and set*

$$\Phi_\circ^\sigma = \{\alpha \in \Phi \mid \sigma(\alpha) \succ 0\}.$$

If $\Phi_\circ^\sigma = \emptyset$, then $(\mathfrak{g}^\sigma, \mathfrak{q}_\Phi)$ is fundamental. When $\Phi_\circ^\sigma \neq \emptyset$, we have

- *$(\mathfrak{g}^\sigma, \mathfrak{q}_\Phi)$ is fundamental if and only if $(\mathfrak{g}^\sigma, \mathfrak{q}_{\Phi_\circ^\sigma})$ is fundamental;*
- *$(\mathfrak{g}^\sigma, \mathfrak{q}_\Phi)$ and $(\mathfrak{g}^\sigma, \mathfrak{q}_{\Phi_\circ^\sigma})$ are fundamental if and only if*

$$\bar{\mathcal{Q}}^c_{\Phi_\circ^\sigma} \cap \Phi_\circ^\sigma = \emptyset. \tag{2.5}$$

Proof. If $\Phi_\circ^\sigma = \emptyset$, then $\mathcal{B} \subseteq \mathsf{Q}_\Phi \cup \bar{\mathsf{Q}}_\Phi$ and hence $(\mathfrak{g}^\sigma, \mathfrak{q}_\Phi)$ is trivially fundamental. Let us consider next the case where $\Phi_\circ^\sigma \neq \emptyset$. Since $\Phi_\circ^\sigma \subseteq \Phi$, we have $\mathfrak{q}_\Phi \subseteq \mathfrak{q}_{\Phi_\circ^\sigma}$ and therefore $(\mathfrak{g}^\sigma, \mathfrak{q}_{\Phi_\circ^\sigma})$ is fundamental when $(\mathfrak{g}^\sigma, \mathfrak{q}_\Phi)$ is fundamental. To show the vice versa, we note that any Lie subalgebra of $\mathfrak{g}$ containing $\mathfrak{q}_\Phi$ is of the form $\mathfrak{q}_\Psi$ for some $\Psi \subseteq \Phi$. If it contains $\mathfrak{q}_\Phi + \bar{\mathfrak{q}}_\Phi$, then $\Psi \subseteq \Phi_\circ^\sigma$. This proves the first item. It suffices to prove the second item in the case where $\Phi = \Phi_\circ^\sigma$. Then condition (2.5) is equivalent to the fact that each $\alpha \in \mathcal{B}$ belongs either to Q_Φ or to $\bar{\mathsf{Q}}_\Phi$ and is therefore clearly sufficient for $(\mathfrak{g}^\sigma, \mathfrak{q}_\Phi)$ being fundamental. Vice versa, when this condition is not satisfied, we can pick $\alpha \in \bar{\mathsf{Q}}^c_\Phi \cap \Phi$. Then $\mathfrak{q}_{\{\alpha\}}$ is a proper parabolic subalgebra of $\mathfrak{g}$ containing both $\mathfrak{q}_\Phi$ and $\bar{\mathfrak{q}}_\Phi$. Therefore $\mathfrak{q}_\Phi + \bar{\mathfrak{q}}_\Phi$ generates a proper Lie subalgebra of $\mathfrak{g}$ and hence $(\mathfrak{g}^\sigma, \mathfrak{q}_\Phi)$ is not fundamental. This completes the proof. $\square$

To discuss the Levi non degeneracy of some order, we observe that the terms of the chain (2.1) for $(\mathfrak{g}^\sigma, \mathfrak{q}_\Phi)$ can be described by the combinatorics of the root system. Let us set

$$\mathsf{Q}^k_\Phi = \{\alpha \in \mathcal{R} \mid \mathfrak{g}_\alpha \subseteq \mathfrak{q}^{(k)}_\Phi\}, \text{ so that } \mathfrak{q}^{(k)}_\Phi = \mathfrak{h} \oplus \sum\nolimits_{\alpha \in \mathsf{Q}^k_\Phi} \mathfrak{g}_\alpha. \tag{2.6}$$

With the notation of §2.2, we have $\mathsf{Q}^0_\Phi = \mathsf{Q}_\Phi$ and

$$\begin{cases} \mathsf{Q}^1_\Phi = \{\alpha \in \mathsf{Q}_\Phi \mid (\alpha + \bar{\mathsf{Q}}_\Phi) \cap \mathcal{R} \subseteq \mathsf{Q}_\Phi + \bar{\mathsf{Q}}_\Phi\}, \\ \mathsf{Q}^k_\Phi = \{\alpha \in \mathsf{Q}^{k-1}_\Phi \mid (\alpha + \bar{\mathsf{Q}}_\Phi) \cap \mathcal{R} \subseteq \mathsf{Q}^{k-1}_\Phi + \bar{\mathsf{Q}}_\Phi\}, \quad \text{for } k>1. \end{cases} \tag{2.7}$$

This yields a characterization of the order of Levi non degeneracy in terms of roots:

Proposition 2.7. *A necessary and sufficient condition for* $(\mathfrak{g}^\sigma, \mathfrak{q}_\Phi)$ *being Levi non degenerate of order* q *is that* $\mathcal{Q}_\Phi^{k-1} \supsetneq \mathcal{Q}_\Phi^k = \mathcal{Q}_\Phi \cap \bar{\mathcal{Q}}_\Phi$. □

Remark 2.8. The necessary and sufficient condition for $(\mathfrak{g}^\sigma, \mathfrak{q}_\Phi)$ being Levi non degenerate of some order is that (cf. [**1**, Lemma 12.1])

$$\begin{cases} \forall \beta \in \mathcal{Q}_\Phi \setminus \bar{\mathcal{Q}}_\Phi,\ \exists\, k \in \mathbb{Z}_+,\ \exists\, \alpha_1, \dots, \alpha_k \in \bar{\mathcal{Q}}_\Phi \ \text{ s.t.} \\ \gamma_h = \beta + \sum_{i=1}^h \alpha_i \in \mathcal{R},\ \forall 1 \le h \le k \ \ \gamma_k \notin \mathcal{Q}_\Phi \cup \bar{\mathcal{Q}}_\Phi. \end{cases} \tag{2.8}$$

Definition 2.9. For any root $\beta \in \mathcal{Q}_\Phi \setminus \bar{\mathcal{Q}}_\Phi$ we denote by $k_\Phi^\sigma(\beta)$ and call its *Levi order* the smallest number k for which (2.8) is valid. We put $k_\Phi^\sigma(\beta) = +\infty$ when (2.8) is not valid for any positive integer k.

Lemma 2.10. *Assume that* $\beta \in \mathcal{Q}_\Phi \setminus \bar{\mathcal{Q}}_\Phi$ *has finite Levi order* $k_\Phi^\sigma(\beta) = k$ *and* (2.8) *is satisfied for a sequence* $\alpha_1, \dots, \alpha_k$. *Then*

(*i*) $\alpha_i \in \bar{\mathcal{Q}}_\Phi \setminus \mathcal{Q}_\Phi$ *for all* $1 \le i \le k$;

(*ii*) $\beta + \sum_{i \le h} \alpha_i \in \mathcal{Q}_\Phi \setminus \bar{\mathcal{Q}}_\Phi$ *for all* $h < k$;

(*iii*) (2.8) *is satisfied by all permutations of* $\alpha_1, \dots, \alpha_k$;

(*iv*) $\alpha_i + \alpha_j \notin \mathcal{R}$ *for all* $1 \le i < j \le k$.

Proof. Let us first prove (*ii*). With the notation in (2.8), we observe that $\gamma_h \notin \bar{\mathcal{Q}}_\Phi$ for $h < k$, because, otherwise, $\gamma_k \in \bar{\mathcal{Q}}_\Phi$.

Next we prove (*iii*). Let $\{Z_\alpha\}_{\alpha \in \mathcal{R}} \cup \{H_i \in \mathfrak{h} \mid 1 \le i \le \ell\}$ be a Chevalley basis for $(\mathfrak{s}, \mathfrak{h})$. Then (2.8) is equivalent to the fact that

$$[Z_{\alpha_k}, Z_{\alpha_{k-1}}, \dots, Z_{\alpha_1}, Z_\beta] := [Z_{\alpha_k}, [Z_{\alpha_{k-1}}, [\dots, [Z_{\alpha_1}, Z_\beta] \dots]]] \notin \mathfrak{q}_\Phi + \bar{\mathfrak{q}}_\Phi.$$

The item (*iii*) follows because

$$\begin{aligned} &[Z_{\alpha_k}, \dots, Z_{\alpha_{i+1}}, Z_{\alpha_i}, \dots, Z_{\alpha_1}, Z_\beta] - [Z_{\alpha_k}, \dots, Z_{\alpha_i}, Z_{\alpha_{i+1}}, \dots, Z_{\alpha_1}, Z_\beta] \\ =& [Z_{\alpha_k}, \dots, [Z_{\alpha_{i+1}}, Z_{\alpha_i}], \dots, Z_{\alpha_1}, Z_\beta] \end{aligned}$$

and, by the minimality assumption, the right hand side belongs to $\mathfrak{q}_\Phi + \bar{\mathfrak{q}}_\Phi$.

Let us prove (*i*) by contradiction. If $\alpha_i \in \mathcal{Q}_\Phi \cap \bar{\mathcal{Q}}_\Phi$ for some $1 \le i \le k$, then we could assume by (*iii*) that it was α_k. Then

$$[Z_{\alpha_{k-1}}, \dots, Z_{\alpha_1}, Z_\beta] \in \mathfrak{q}_\Phi + \bar{\mathfrak{q}}_\Phi \Longrightarrow [Z_{\alpha_k}, Z_{\alpha_{k-1}}, \dots, Z_{\alpha_1}, Z_\beta] \in \mathfrak{q}_\Phi + \bar{\mathfrak{q}}_\Phi$$

yields the contradiction. Also (*iv*) is an easy consequence of (*iii*), because if $\alpha_i + \alpha_j$ $(1 \le i, j \le k)$ is a root, than it would belong to $\bar{\mathcal{Q}}_\Phi \cap \mathcal{Q}_\Phi^c$ and, by substituting to the two roots α_i, α_j the single root $\alpha_i + \alpha_j$ we would obtain a sequence satisfying (2.8) and containing $k-1$ terms.

The proof is complete. $\square$

Remark 2.11. Since $\xi_\Phi(\alpha) \geq 1$ for all $\alpha \in \mathsf{Q}^c_\Phi$, if $\beta \in \mathsf{Q}_\Phi \cap \bar{\mathsf{Q}}^c_\Phi$ and $k^\sigma_\Phi(\beta) < +\infty$, then

$$k^\sigma_\Phi(\beta) \leq 1 - \xi_\Phi(\beta). \tag{2.9}$$

Corollary 2.12. *If $\beta \in \mathsf{Q}^r_\Phi \setminus \bar{\mathsf{Q}}_\Phi$, then its order of Levi non degeneracy is either one or $+\infty$.* $\square$

We obtain also a useful criterion of Levi non degeneracy of some order (cf. [**3**, Thm.6.4])

Proposition 2.13. *The parabolic CR algebra $(\mathfrak{g}^\sigma, \mathfrak{q}_\Phi)$ is Levi non degenerate of order $k \geq 1$ if and only if*

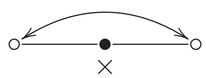
$$\forall \beta \in \mathsf{Q}_\Phi \cap \bar{\mathsf{Q}}^c_\Phi \;\; \exists \alpha \in \bar{\mathsf{Q}}_\Phi \cap \mathsf{Q}^c_\Phi \;\; \textit{such that} \;\; \beta + \alpha \in \bar{\mathsf{Q}}^c_\Phi. \tag{2.10}$$

Proof. By Lemma 2.10 the condition is necessary. To prove that it is also sufficient, we can argue by contradiction: if we could find $\beta \in \mathsf{Q}_\Phi \cap \bar{\mathsf{Q}}^c_\Phi$ with $k^\sigma_\Phi(\beta) = +\infty$, then by (2.10) we could construct an infinite sequence $(\alpha_i)_{i \geq 1}$ in $\bar{\mathsf{Q}}_\Phi \cap \mathsf{Q}^c_\Phi$ with

$$\gamma_h = \beta + \sum\nolimits_{i=1}^{h} \alpha_i \in \mathsf{Q}_\Phi \cap \bar{\mathsf{Q}}^c_\Phi, \;\; \forall h = 1, 2, \ldots$$

Since $\xi_\Phi(\gamma_h) \geq \xi_\Phi(\beta) + h$ and ξ_Φ is bounded, we get a contradiction. $\square$

Example 2.14. We present an example that summarizes all the builded theory so far. We remember that a Satake diagram is obtained from a Dynkin diagram by blackening some vertices, and connecting other vertices in pairs by arrows, according to certain rules; the Satake diagrams associated to the Dynkin diagram of a complex semisimple Lie algebra classify its real forms (see pp.531 [**12**]). Consider now the CR algebra $(\mathfrak{su}(1,3), \mathfrak{q}_\phi)$, described by the cross-marked Satake diagram



It is associated to the minimal orbit $\mathsf{M}_{\Phi,\sigma}$ of $\mathbf{SU}(1,3)$ in the Grassmannian of isotropic two-planes of $\mathbb{C}^4$ for an hermitian symmetric form of signature $(1,3)$. Here $\mathfrak{g} \simeq \mathfrak{sl}_4(\mathbb{C})$, with

$$\mathcal{R} = \{\pm(e_i - e_j) \mid 1 \leq i < j \leq 4\},$$

and

$$\mathcal{B}=\{e_1-e_2,\ e_2-e_3,\ e_3-e_4\}$$

for an orthonormal basis e_1,e_2,e_3,e_4 of $\mathbb{R}^4$, the subset of simple root defining $\mathfrak{q}_\phi$ is taken to be $\Phi=\{e_2-e_3\}$. The linear functional (2.4) is defined as follow

$$\xi(e_i)=\begin{cases}1, & i{=}1,2,\\ 0, & i{=}3,4,\end{cases}$$

and the conjugation defining the real form is taken to be

$$\begin{cases}\sigma(e_1)=-e_4, & \sigma(e_2)=-e_2,\\ \sigma(e_3)=-e_3, & \sigma(e_4)=-e_1.\end{cases}$$

We obtain

$$\begin{aligned}\mathsf{Q}^c_\Phi\cap\bar{\mathsf{Q}}^c_\Phi&=\{e_1-e_4\},\\ \bar{\mathsf{Q}}_\Phi\cap\mathsf{Q}^c_\Phi&=\{e_1-e_3,e_2-e_3,e_2-e_4\},\\ \mathsf{Q}_\Phi\cap\bar{\mathsf{Q}}^c_\Phi&=\{e_3-e_4,e_3-e_2,e_1-e_2\}.\end{aligned}$$

Since $\mathsf{Q}^c_\Phi\cap\bar{\mathsf{Q}}^c_\Phi$ is nonempty, $e_1-e_4=(e_3-e_4)+(e_1-e_3)$ and $\mathfrak{q}_\Phi$ is maximal, we obtain that $(\mathfrak{g}^\sigma,\mathfrak{q}_\Phi)$ is fundamental and Levi non degenerate, moreover $\xi_\Phi(e_3-e_2)$ $=-1$ and ξ_Φ is 1 on all the elements of $\bar{\mathsf{Q}}_\Phi\cap\mathsf{Q}^c_\Phi$, then the order of Levi non degeneracy is equal to 2, in fact

$$e_1-e_4=(e_3-e_2)+(e_1-e_3)+(e_2-e_4),\quad e_1-e_4=(e_1-e_2)+(e_2-e_4).$$

2.4 - *Finitely Levi non degenerate real orbits in complex flag manifolds*

To discuss order of Levi non degeneracy of a real orbits $\mathsf{M}_{\Phi,\sigma}$ in X_Φ by employing Lemma 2.10, we introduce:

Definition 2.15. If $\beta\in\mathcal{R}$, we denote by $k(\beta)$ the largest positive integer k for which there exists $\alpha_1,\dots,\alpha_k\in\mathcal{R}$ such that

- $\alpha_i+\alpha_j\notin\mathcal{R}\cup\{0\}$, $\forall 1\le i,j\le k$,
- $\gamma_{i_1,\dots,i_h}=\beta+\alpha_{i_1}+\dots+\alpha_{i_h}\in\mathcal{R}$, $\forall i_1,\dots,i_h\in\{1,\dots,k\}$.

Proposition 2.16. *Let $\beta\in\mathcal{R}$ belong to a simple root system containing more than two elements. Then $k(\beta)\le 4$ and, if $k(\beta)=4$ and $(\alpha_1,\alpha_2,\alpha_3,\alpha_4)$ is a sequence satisfying* (2.15)*, then*

$$\beta+\alpha_1+\alpha_2+\alpha_3+\alpha_4=-\beta. \tag{2.11}$$

More precisely we obtain:

(1) $k(\beta)=1$, *if* β *belongs to a root system of type* A_2 *or is a long root of a root system of type* B_2*;*

(2) $k(\beta)=2$, *if* β *belongs to a root system of type* $\mathrm{A}_{\geq 3}, \mathrm{C}$, *or is a short root of a system of type* B_2, G*;*

(3) $k(\beta)=3$, *if* β *is a short root of a root system of type* $\mathrm{B}_{\geq 3}, \mathrm{F}$*;*

(4) $k(\beta)=4$, *if* β *belongs to a root system of type* D, E, *or is a long root of a root system of type* $\mathrm{B}_{\geq 3}, \mathrm{F}, \mathrm{G}$*.*

Proof. For short we will call *admissible* a sequence (α_i) for which (2.15) is valid. Let us set

$$\mathcal{R}^{\mathrm{add}}(\beta) = \{\alpha \in \mathcal{R} \mid \beta+\alpha \in \mathcal{R}\}.$$

We consider the different cases using for root systems the notation of [**4**].

Type A. We have $\mathcal{R} = \{\pm(e_i - e_j) \mid 1 \leq i < j \leq n\}$ where $e_1, \ldots, e_n$ is an orthonormal basis of $\mathbb{R}^n$. We can take $\beta = e_2 - e_1$. Then

$$\mathcal{R}^{\mathrm{add}}(e_2 - e_1) = \{e_1 - e_i \mid i > 2\} \cup \{e_i - e_2 \mid i > 2\}. \tag{$*A$}$$

An admissible sequence (α_i) can contain at most one element from each of the two sets in the right hand side of $(*A)$.

If $n=3$, then $\mathcal{R}^{\mathrm{add}}(\beta) = \{e_3 - e_2,\ e_1 - e_3\}$ contains two elements, whose sum is still a root and therefore $k(\beta)=1$.

If $n>3$, then the only possible choice is that of a couple of roots $e_i - e_2$, $e_1 - e_j$ with $3 \leq i \neq j \leq n$ and hence $k(\beta)=2$.

Type B. We have $\mathcal{R} = \{\pm e_i \pm e_j \mid 1 \leq i < j \leq n\} \cup \{\pm e_i \mid 1 \leq i \leq n\}$, for an orthonormal basis $e_1, \ldots, e_n$ of $\mathbb{R}^n$ $(n \geq 2)$.

If β is a short root, we can take $\beta = -e_1$. Then

$$\mathcal{R}^{\mathrm{add}}(-e_1) = \{\pm e_i \mid 2 \leq i \leq n\} \cup \{e_1 \pm e_j \mid 2 \leq j \leq n\}. \tag{$*B$}$$

An admissible sequence contains at most one root from the first and two from the second set in the right hand side of $(*B)$. Thus $k(-e_1) \leq 3$. The sequence $e_1 - e_2$, $e_1 + e_2$ satisfies (2.15) and therefore $k(-e_1) \geq 2$.

We have equality if $n=2$, because in this case $\mathcal{R}^{\mathrm{add}}(-e_1) = \{\pm e_2,\ e_1 \pm e_2\}$ and the maximal admissible sequences are then (e_2), $(-e_2)$, $(e_1+e_2,\ e_1-e_2)$.

If $n>2$ the admissible sequence

$$(e_1+e_2,\ e_1-e_2, e_3)$$

shows that $k(-e_1)$=3. All admissible maximal sequences are of this form.

If β is a long root, we can assume that $\beta = -e_1 - e_2$. Then

$$(**B) \qquad \mathcal{R}^{\mathrm{add}}(-e_1-e_2)=\{e_1,e_2\}\cup\{e_1\pm e_j \mid j>2\}\cup\{e_2\pm e_j \mid j>2\}.$$

An admissible sequence contains at most two equal terms from the first and two from each of the second and third on the right hand side of $(**B)$. Moreover, if one term is taken from the first, we can take at most one from each one of the other two. This implies that $k(-e_1-e_2)\leq 4$ and in fact $k(-e_1-e_2)=4$, with maximal sequences isomorphic to one of

$$e_1+e_3,\ e_1-e_3,\ e_2+e_4,\ e_2-e_4,$$
$$e_1,\ e_1,\ e_1-e_3,\ e_1+e_3,$$

which, summed up to $(-e_1-e_2)$, gives e_1+e_2.

Type C. We can take $\mathcal{R}=\{\pm e_i\pm e_j \mid 1\leq i<j\leq n\}\cup\{\pm 2e_i \mid 1\leq i\leq n\}$, for an orthonormal basis $e_1,\dots,e_n$ of $\mathbb{R}^n$ $(n\geq 3)$.

If β is a short root, we can assume that $\beta=(-e_1-e_2)$. Then

$$(*C) \qquad \mathcal{R}^{\mathrm{add}}(-e_1-e_2)=\{2e_1,2e_2\}\cup\{e_1\pm e_j \mid j\geq 3\}\cup\{e_2\pm e_j \mid j\geq 3\}.$$

An admissible sequence may contain both roots of the first, but at most one root from each the second and third sets on the right hand side of $(*C)$. Moreover, a term in one of the last two forbids the corresponding term in the first one. This yields $k(-e_1-e_2)$=2, with maximal sequences isomorphic to (the third one should be omitted if n=3)

$$(2e_1,\ 2e_2),\quad (2e_1,\ e_2+e_3),\quad (e_1+e_3,\ e_2+e_4)$$

If β is a long root, we can assume that $\beta=-2e_1$. Then

$$(**\mathrm{C}) \qquad \mathcal{R}^{\mathrm{add}}(-2e_1)=\{e_1\pm e_i \mid i>1\}.$$

We note that $k(-2e_1)\leq 4$. We cannot take in an admissible sequence both the element e_1+e_i and e_1-e_i, because they add up to the root $2e_i$. Hence in fact $k(-2e_1)$=2, with maximal sequence isomorphic to

$$e_1+e_2,\ e_1+e_3.$$

Type D. We can take $\mathcal{R}=\{\pm e_i \pm e_j \mid 1\leq i<j\leq n\}$, where $e_1,\dots,e_n$ is an orthonormal basis of $\mathbb{R}^n$ $(n\geq 4)$.

We can assume that $\beta=-e_1-e_2$. We have

$$\mathcal{R}^{\mathrm{add}}(-e_1-e_2)=\{e_1\pm e_j \mid j\geq 3\}\cup\{e_2\pm e_j \mid j\geq 3\}. \tag{$*D$}$$

An admissible sequence contains at most two elements from each set in the right hand side of $(*D)$. Therefore $k(-e_1-e_2)\leq 4$ and in fact we have equality with maximal admissible sequences isomorphic to

$$e_1+e_3,\ e_1-e_3,\ e_2+e_4,\ e_2-e_4,$$

which, summed up to $(-e_1-e_2)$, give e_1+e_2.

Type E. Since the root systems E_6 and E_7 can be considered as subsystems of E_8, we will restrain to this case. We consider, for an orthonormal basis $e_1,\dots,e_8$ of $\mathbb{R}^8$,

$$\mathcal{R}=\{\pm e_i\pm e_j \mid 1\leq i<j\leq 8\}\cup\left\{\tfrac{1}{2}\sum\nolimits_{i=1}^{8}(-1)^{h_i}e_i \,\middle|\, h_i\in\mathbb{Z},\ \sum\nolimits_{i=1}^{8}h_i\in 2\,\mathbb{Z}\right\}.$$

We can take $\beta=(-e_1-e_2)$. Then

$$\begin{aligned}\mathcal{R}^{\mathrm{add}}(-e_1-e_2)&=\{e_1\pm e_i \mid 3\leq i\leq 8\}\cup\{e_2\pm e_i \mid 3\leq i\leq 8\}\\ &\cup\left\{\tfrac{1}{2}\left(e_1+e_2+\sum\nolimits_{i=3}^{8}(-1)^{h_i}e_i\right)\,\middle|\, h_i\in\mathbb{Z},\ \sum\nolimits_{i=3}^{8}h_i\in 2\,\mathbb{Z}\right\}.\end{aligned} \tag{$*E$}$$

An admissible sequence may contain at most two roots from each set on the right hand side of $(*E)$ and no more than four terms. Clearly we can take the maximal sequence

$$e_1+e_3,\ e_1-e_3,\ e_2+e_4,\ e_2-e_4,$$

showing that $k(-e_1-e_2)=4$. Moreover, any admissible sequence containing four terms sums up to $(-e_1-e_2)$ to yield e_1+e_2.

Type F. For an orthonormal basis e_1,e_2,e_3,e_4 of $\mathbb{R}^4$ we take

$$\mathcal{R}=\{\pm e_i \mid 1\leq i\leq 4\}\cup\{\pm e_1\pm e_j \mid 1\leq i<j\leq 4\}\cup\{\tfrac{1}{2}(\pm e_1\pm e_2\pm e_3\pm e_4)\}.$$

If β is a short root, we can take $\beta{=}-e_1$. Then

$$\mathcal{R}^{\mathrm{add}}(-e_1)=\{\pm e_i \mid 2\leq i\leq 4\}\cup\{e_1\pm e_i \mid 2\leq i\leq 4\}\cup\{\tfrac{1}{2}(e_1\pm e_2\pm e_3\pm e_4)\}. \tag{$*F$}$$

To build an admissible sequence we can take at most one element from the first, two from the second and from the third set in the right hand side of $(*F)$.

Indeed two roots of the form $\frac{1}{2}(\pm e_1 \pm e_2 \pm e_3 \pm e_4)$ do not add up to a root if and only if they differ by only one sign. Moreover, no root can be taken from the first if one is taken from the last set. These considerations imply that $k(-e_1) \leq 3$ and in fact equality holds, as $(-e_1)$ is contained in a subsystem of type B_3.

If β is a long root, we can assume $\beta = (-e_1 - e_2)$. We have

$$(**F) \qquad \begin{aligned} \mathcal{R}^{\mathrm{add}}(-e_1 - e_2) &= \{e_1, e_2\} \cup \{e_1 \pm e_i \mid 3 \leq i \leq 4\} \\ &\cup \{e_2 \pm e_i \mid 3 \leq i \leq 4\} \cup \{\tfrac{1}{2}(e_1 + e_2 \pm e_3 \pm e_4)\}. \end{aligned}$$

We note that the sum of four terms of $\mathcal{R}^{\mathrm{add}}(-e_1 - e_2)$ is a linear combination $\beta + k_1 e_1 + k_2 e_2 + k_3 e_3 + k_4 e_4$ with $k_1 + k_2 \geq 2$ and therefore, if they form an admissible sequence, is equal to $e_1 + e_2$. Since $\mathcal{R}$ contains subsystems of type B_3, there are indeed admissible sequences with four elements.

Type G. For an orthonormal basis e_1, e_2, e_3 of $\mathbb{R}^3$ we set

$$\mathcal{R} = \{\pm(e_i - e_j) \mid 1 \leq i < j \leq 3\} \cup \{\pm(2e_i - e_j - e_k) \mid \{i, j, k\} = \{1, 2, 3\}\}.$$

We consider first the case of a short root. We can take $\beta = e_2 - e_1$. Then

$$(*G) \qquad \mathcal{R}^{\mathrm{add}}(e_2 - e_1) = \{e_3 - e_2,\ e_1 - e_3\} \cup \{2e_1 - e_2 - e_3,\ e_1 + e_3 - 2e_2\}.$$

Maximal admissible sequences have a root from the first and one from the second set, hence $k(e_2 - e_1) = 2$ and, moreover, summed up to $e_2 - e_1$, give $e_1 - e_2$.

As a long root we take $\beta = (e_2 + e_3 - 2e_1)$. Then

$$(**G) \qquad \mathcal{R}^{\mathrm{add}}(e_2 + e_3 - 2e_1) = \{e_1 - e_2,\ e_1 - e_3\} \cup \{e_1 + e_2 - 2e_3,\ e_1 + e_3 - 2e_2\}.$$

One checks that in this case $k(e_2 + e_3 - 2e_1) = 4$, with a maximal admissible sequence

$$e_1 - e_2,\ e_1 - e_2,\ e_1 - e_2,\ e_1 + e_2 - 2e_3$$

which indeed sums up to the opposite root $2e_1 - e_2 - e_3$.

The proof is complete. □

Then we are ready to state the main result:

Theorem 2.17. *Let $\mathsf{M}_{\Phi,\sigma}$ be a real orbit which is fundamental and Levi non degenerate of order $k \geq 1$, then k is less or equal to* 3.

Proof. This is a consequence of Prop. 2.16 and the fact that, if β does not belong to $\bar{\mathsf{Q}}_\Phi$, then $-\beta \in \bar{\mathsf{Q}}_\Phi$ because $\bar{\mathsf{Q}}_\Phi$ is a parabolic set of roots. □

2.5 - *Levi non degenerate CR manifolds with large orders*

In this last section we discuss in detail an example of a homogeneous CR manifold M of type $(q{+}1,1)$, for some integer $q{\geq}1$, that is Levi non degenerate of order q. The compact group $\mathbf{SU}(2)$ acts transitively on the complex projective line $\mathbb{CP}^1$. The homogeneous complex structure of $\mathbb{CP}^1$ can be defined by the totally complex CR algebra $(\mathfrak{su}(2),\mathfrak{b})$, where $\mathfrak{su}(2)$ is the real Lie algebra of anti-Hermitian $2{\times}2$ matrices and $\mathfrak{b}$ the Borel subalgebra of upper triangular matrices of its complexification $\mathfrak{sl}_2(\mathbb{C})$. This CR algebra corresponds to the simple cross-marked Satake diagram

$$\overset{\alpha}{\underset{\times}{\bullet}}$$

The root system of the complexification $\mathfrak{sl}_2(\mathbb{C})$ is $\mathcal{R}=\{\pm(e_1{-}e_2)\}$, and we take $\alpha=(e_1{-}e_2)$, with fundamental weight $\omega=\alpha/2$, i.e. an element of $\mathfrak{h}_{\mathbb{R}}$ equal to the coroot associated to the simple root defined by the condition $2\frac{\langle\omega,\alpha\rangle}{\langle\alpha,\alpha\rangle}{=}1$. With our usual notation $\Phi{=}\{\alpha\}$, so that $\mathfrak{b}=\mathfrak{q}_{\Phi}$, with the linear map defined by $\xi_{\Phi}(e_i){=}(-1)^{i+1}/2$, and $\mathfrak{g}^{\sigma}{=}\mathfrak{su}_2$ with conjugation $\sigma(e_1){=}e_2$, $\sigma(e_2)=e_1$. The irreducible finite dimensional complex linear representations of $\mathfrak{sl}_2(\mathbb{C})$ are indexed by the nonnegative integral multiples $k{\cdot}\omega$ of ω and the corresponding irreducible $\mathfrak{sl}_2(\mathbb{C})$-module $\mathsf{V}_{k\cdot\omega}$ can be identified with the space of complex homogeneous polynomials of degree k in two indeterminates

$$\mathsf{V}_{k\omega}=\left\{\sum\nolimits_{h=0}^{k}a_h z^h w^{k-h}\;\middle|\; a_h\in\mathbb{C}\right\}.$$

We have

$$\mathsf{V}_{k\omega}=\bigoplus\nolimits_{h=0}^{k}\mathsf{V}_{k\omega}^{(k-2h)\omega},$$

where, for a diagonal H in the canonical Cartan subalgebra of $\mathfrak{sl}_2(\mathbb{C})$,

$$\mathsf{V}_{k\omega}^{(k-2h)\omega}=\{v\in\mathsf{V}_{k\omega}\mid H{\cdot}v=(k{-}2h)\omega(H)v\}=\{a\cdot z^h w^{k-h}\mid a\in\mathbb{C}\},\quad 0{\leq}h{\leq}k,$$

are the one-dimensional weight spaces contained in $\mathsf{V}_{k\omega}$. Since $\bar{\omega}=-\omega$, we have $\overline{\mathsf{V}}_{k\omega}=\mathsf{V}_{k\omega}$. The anti-$\mathbb{C}$-linear automorphism $\theta_{k\omega}$ of $\mathsf{V}_{k\omega}$ defined by the conjugation σ comes from $(z,w)\mapsto(-\bar{w},\bar{z})$ and therefore

$$\theta\left(\sum\nolimits_{h=0}^{k}a_h z^h w^{k-h}\right)=\sum\nolimits_{h=0}^{k}(-1)^h\bar{a}_h w^h z^{k-h}.$$

Then $\theta_{k\omega}^2$ equals $\mathrm{id}_{\mathsf{V}_{k\omega}}$ for k even and $-\mathrm{id}_{\mathsf{V}_{k\omega}}$ for k odd. Accordingly, for k even $\mathsf{V}_{k\omega}$ is the complexification of an irreducible $(k{+}1)$-dimensional representation of the real type, that we will denote by $\mathsf{V}^{\mathbb{R}}_{k\omega}$, while for k odd is isomorphic to a $2(k{+}1)$-dimensional irreducible representation of the quaternionic type of $\mathfrak{su}_2$ (see e.g. [**5**, Ch.IX, App.II, Prop.2]).

Remark 2.18. Studying irreducible representation of $\mathfrak{su}_2$ turns out to be of some interest in quantum physics, as they arise when considering rotations on fermionic and bosonic systems (for more details see [**24**, Ch.5 , §5]).

The subspace

$$\mathsf{V}^-_{k\omega} = \bigoplus\nolimits_{k<2h\leq 2k} \mathsf{V}^{(k-2h)\omega}_{k\omega}$$

is a $\mathfrak{b}$-submodule of $\mathsf{V}_{k\omega}$ and we can consider the semidirect sum $\mathfrak{b}\oplus\mathsf{V}^-_{k\omega}$ as a subalgebra of the abelian extension $\mathfrak{sl}_2(\mathbb{C})\oplus\mathsf{V}_{k\omega}$ (cf. e.g. [**23**, Ch.VII,§3] and see [**16**] for more details on gradation of non semisimple Lie algebras). We may consider the map $\mathbf{SL}_2(\mathbb{C})\to\mathbb{CP}^1$ associated to our choice of a Borel subalgebra $\mathfrak{b}$ as a principal bundle with structure group $\mathbf{B}$. Then the Lie pair $(\mathfrak{sl}_2(\mathbb{C})\oplus\mathsf{V}_{k\omega},\mathfrak{b}\oplus\mathsf{V}^-_{k\omega})$ is the CR algebra of a complex holomorphic vector bundle E_k with base $\mathbb{CP}^1$ and typical fiber $\mathsf{V}_{k\omega}/\mathsf{V}^-_{k\omega}\simeq\bigoplus_{2h\leq k}\mathsf{V}^{(k-2h)\omega}_{k\omega}$ (this is an example of Mostow fibration, see [**18**,**19**,**20**] for an introduction to this topics).

Proposition 2.19. *Let q be any positive integer. Then*

$$(\mathfrak{g}_{\mathbb{R}},\mathfrak{q}'_\Phi)=(\mathfrak{su}_2\oplus V^{\mathbb{R}}_{2q\omega},\mathfrak{b}\oplus V^-_{2q\omega})$$

is the CR algebra of a CR manifold E_{2q}, of CR dimension $q{+}1$ and CR codimension 1, which is fundamental and Levi non degenerate of order q.

Proof. We have

$$\bar{\mathsf{V}}^-_{2q\omega}=\mathsf{V}^+_{2q\omega}=\bigoplus\nolimits_{h=1}^{q}\mathsf{V}^{2h\omega}_{2q\omega}\quad\text{and}\quad\mathsf{V}_{2q\omega}=\mathsf{V}^-_{2q\omega}\oplus\mathsf{V}^0_{2q\omega}\oplus\mathsf{V}^+_{2q\omega}.$$

If $Z_\alpha,Z_{-\alpha},H$ is the canonical basis of $\mathfrak{sl}_2(\mathbb{C})$ and w a nonzero vector of $\mathsf{V}^{-2q\omega}_{2q\omega}$, then the images of $X_{-\alpha},w,X_\alpha w,\dots,X^{q-1}_\alpha w$ generate $\mathfrak{q}'_\Phi/(\mathfrak{q}'_\Phi\cap\bar{\mathfrak{q}}'_\Phi)$. Since

$$[\underbrace{X_\alpha,\dots,X_\alpha}_{h\text{ times}},X^{q-h}_\alpha w]=X^q_\alpha w\in\mathsf{V}^0_{2q\omega}\backslash\{0\},\quad [X^{q+1}_\alpha w,X_{-\alpha}]=-2X^q_\alpha w\in\mathsf{V}^0_{2q\omega}\backslash\{0\}$$

we obtain that E_{2q} is fundamental and Levi nondegenerate.

With the notation of the previous section, we have $\mathsf{Q}^c_\Phi\cap\bar{\mathsf{Q}}_\Phi=\{\alpha\}$, with $\xi_\Phi(\alpha)=1$ and $\xi_\Phi(-2j\omega)=-j$. Since $\mathfrak{g}/(\mathfrak{q}'_\Phi+\bar{\mathfrak{q}}'_\Phi)$ is generated by the image of $\mathsf{V}^0_{2q\omega}$, by the above considerations the Levi order of an element of $\mathsf{V}^{-2j\omega}_{2q\omega}$ equals j. This shows that the Levi order of $(\mathfrak{g}_{\mathbb{R}},\mathfrak{q}'_\Phi)$ is q. □

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