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Relative Dolbeault cohomology*In memory of Pierre Dolbeault*

Abstract. This is a partially expository paper, in which the notion of relative Dolbeault cohomology and related topics are reviewed and discussed together with various examples and applications. We deal with this cohomology theory from two viewpoints. One is the Čech theoretical approach, which is convenient to define such operations as the cup product and integration and leads to the study of local duality. Along the way we also establish some notable canonical isomorphisms among various cohomologies. The other is to regard it as the cohomology of a certain complex, which is interpreted as a notion dual to the mapping cone in the theory of derived categories. This approach shows that the cohomology goes well with derived functors. In any case the relative Dolbeault cohomology turns out to be canonically isomorphic with the local (relative) cohomology of A. Grothendieck and M. Sato with coefficients in the sheaf of holomorphic forms so that it provides a handy way of representing the latter. We also give some examples and indicate applications, including simple explicit expressions of Sato hyperfunctions, fundamental operations on them and related local duality theorems.

Keywords. Dolbeault cohomology of an open embedding, Čech-Dolbeault cohomology, relative Dolbeault theorem, complex analytic Alexander morphism, Sato hyperfunctions.

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1 - Introduction

In [29] we discussed the cohomology theory of sheaf complexes for open embeddings of topological spaces and gave some general ways of representing

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the relative cohomology of a sheaf in terms of a soft or fine resolution of the sheaf. In this paper we apply the theory to the case of Dolbeault complex. This naturally leads to the notion of the relative Dolbeault cohomology of a complex manifold. As is explained in [29], there are two ways to approach this cohomology. One is to define it as a special case of Čech-Dolbeault cohomology. This viewpoint goes well with such operations as the cup product and the integration, which enable us to deal with the local duality problem. The integration theory here is a descendent of the one on the Čech-de Rham cohomology, which is defined using honeycomb systems. The other is to see it as the cohomology of a certain complex called co-mapping cone, a notion dual to the mapping cone in the theory of derived categories. From this viewpoint we see that the cohomology goes well with derived functors. It is readily generalized to the cohomology of holomorphic maps between complex manifolds. In any case we have the relative Dolbeault theorem which says that the relative Dolbeault cohomology is canonically isomorphic with the local (relative) cohomology of A. Grothendieck and M. Sato (cf. [9], [23]) with coefficients in the sheaf of holomorphic forms (cf. Theorems 2.3 and 4.8). We also present some canonical isomorphisms that appear along the way (Theorem 3.6) and give some examples and applications.

The paper is organized as follows. In Section 2, we introduce the Dolbeault cohomology for open embeddings of complex manifolds and state the aforementioned relative Dolbeault theorem. We also give generalizations of them to the case of holomorphic maps. We recall, in Section 3, the Čech-Dolbeault cohomology and some related canonical isomorphisms. In Section 4 we review the relative Dolbeault cohomology from the Čech theoretical viewpoint and indicate an alternative proof of the relative Dolbeault theorem.

There are certain cases where there is a significant relation between the de Rham and Dolbeault cohomologies, which are taken up in Section 5. In Section 6, we discuss the cup product and the integration. As mentioned above, the integration theory is a descendant of the one on the Čech-de Rham cohomology, which we briefly recall. In Section 7, we discuss global and local dualities. In the global case where the manifold is compact, we have the Kodaira-Serre duality. In the local case we have the duality morphism, which we call the $\bar{\partial}$ -Alexander morphism. We prove an exact sequence and a commutative diagram giving relation between global and local dualities (Theorems 7.2 and 7.4). We then recall the theory of Fréchet-Schwartz and dual Fréchet-Schwartz spaces and state a theorem where we have the local duality (Theorem 7.7).

Finally we give, in Section 8, some examples and applications. The correspondence of the Bochner-Martinelli form and the higher dimensional Cauchy form in the isomorphism of Dolbeault and Čech cohomologies is rather well-

known (cf. [8], [11]). Here we give the canonical correspondence of them together with integrations in our context (Theorems 8.1 and 8.3). We also present the local duality theorem of A. Martineau in our framework (Theorem 8.4) and, as a special case, describe the local residue pairing. These are closely related to the Sato hyperfunction theory. In fact, it is one of the major topics to which the relative Dolbeault theory can be applied. This application to hyperfunctions is discussed in detail in [14]. Here we take up some of the essence. As another important applications, there is the localization theory of Atiyah classes, including the theory of analytic Thom classes (cf. [1], [2], [28]).

2 - Dolbeault cohomology of open embeddings

In this section we recall the contents of [29, Section 2] specializing them to our setting.

In the sequel, by a sheaf we mean a sheaf with at least the structure of Abelian groups. For a sheaf \mathcal{S} on a topological space X and an open set V in X , we denote by $\mathcal{S}(V)$ the group of sections of \mathcal{S} on V . Also for an open subset V' of V , we denote by $\mathcal{S}(V, V')$ the sections on V that vanish on V' .

2.1 - Cohomology via flabby resolutions

As reference cohomology theory, we adopt the one via flabby resolutions. Recall that a sheaf \mathcal{F} is *flabby* if the restriction $\mathcal{F}(X) \rightarrow \mathcal{F}(V)$ is surjective for every open set V in X . Recall also that every sheaf admits a flabby resolution. Let \mathcal{S} be a sheaf on X and

$$0 \longrightarrow \mathcal{S} \longrightarrow \mathcal{F}^0 \xrightarrow{d} \dots \xrightarrow{d} \mathcal{F}^q \xrightarrow{d} \dots$$

a flabby resolution of \mathcal{S} . For an open set X' in X , the q -th cohomology $H^q(X, X'; \mathcal{S})$ of (X, X') with coefficients in \mathcal{S} is the q -th cohomology of the complex $(\mathcal{F}^\bullet(X, X'), d)$. Note that it is determined uniquely modulo canonical isomorphisms, independently of the flabby resolution. We denote $H^q(X, \emptyset; \mathcal{S})$ by $H^q(X; \mathcal{S})$. We have $H^0(X, X'; \mathcal{S}) = \mathcal{S}(X, X')$. Setting $S = X \setminus X'$, it will also be denoted by $H_S^q(X; \mathcal{S})$. This cohomology in the first expression is referred to as the *relative cohomology* of \mathcal{S} on (X, X') and in the second expression the *local cohomology* of \mathcal{S} on X with support in S (cf. [9], [23]).

2.2 - Dolbeault cohomology

Let X be a complex manifold of dimension n . We always assume that it has a countable basis so that it is paracompact and has only countably many connected components. Without loss of generality, we may assume that the coverings we consider are locally finite. We denote by $\mathcal{E}_X^{(p,q)}$ and $\mathcal{O}_X^{(p)}$, respectively, the sheaves of C^∞ (p, q) -forms and of holomorphic p -forms on X . We denote $\mathcal{O}_X^{(0)}$ simply by \mathcal{O}_X . We also omit the suffix X on the sheaf notation if there is no fear of confusion. By the Dolbeault-Grothendieck lemma, the complex $\mathcal{E}^{(p,\bullet)}$ gives a fine resolution of $\mathcal{O}^{(p)}$:

$$0 \longrightarrow \mathcal{O}^{(p)} \longrightarrow \mathcal{E}^{(p,0)} \xrightarrow{\bar{\partial}} \mathcal{E}^{(p,1)} \xrightarrow{\bar{\partial}} \cdots \xrightarrow{\bar{\partial}} \mathcal{E}^{(p,n)} \longrightarrow 0.$$

The *Dolbeault cohomology* $H_{\bar{\partial}}^{p,q}(X)$ of X of type (p, q) is the q -th cohomology of the complex $(\mathcal{E}^{(p,\bullet)}(X), \bar{\partial})$. The “de Rham type theorem” in [29, Section 2] reads (cf. Remark 3.9 below):

Theorem 2.1 (Canonical Dolbeault theorem). *There is a canonical isomorphism:*

$$H_{\bar{\partial}}^{p,q}(X) \simeq H^q(X; \mathcal{O}^{(p)}).$$

2.3 - Dolbeault cohomology of open embeddings

We recall the contents of [29, Subsection 2.3] in our situation. Let X be a complex manifold of dimension n as above. For an open set X' in X with inclusion $i : X' \hookrightarrow X$, we define a complex $\mathcal{E}^{(p,\bullet)}(i)$ as follows. We set

$$\mathcal{E}^{(p,q)}(i) = \mathcal{E}^{(p,q)}(X) \oplus \mathcal{E}^{(p,q-1)}(X')$$

and define the differential

$$\bar{\partial} : \mathcal{E}^{(p,q)}(i) = \mathcal{E}^{(p,q)}(X) \oplus \mathcal{E}^{(p,q-1)}(X') \rightarrow \mathcal{E}^{(p,q+1)}(i) = \mathcal{E}^{(p,q+1)}(X) \oplus \mathcal{E}^{(p,q)}(X')$$

by

$$\bar{\partial}(\omega, \theta) = (\bar{\partial}\omega, i^*\omega - \bar{\partial}\theta),$$

where $i^* : \mathcal{E}^{(p,q)}(X) \rightarrow \mathcal{E}^{(p,q)}(X')$ denotes the pull-back of differential forms by i , the restriction to X' in this case. Obviously we have $\bar{\partial} \circ \bar{\partial} = 0$.

Definition 2.2. The *Dolbeault cohomology* $H_{\bar{\partial}}^{p,q}(i)$ of $i : X' \rightarrow X$ is the cohomology of $(\mathcal{E}^{(p,\bullet)}(i), \bar{\partial})$.

Denoting by $\mathcal{E}^{(p,\bullet)}[-1]$ the complex with $\mathcal{E}^{(p,q)}[-1] = \mathcal{E}^{(p,q-1)}$ and the differential $-\bar{\partial}$, we define morphisms $\alpha^* : \mathcal{E}^{(p,\bullet)}(i) \rightarrow \mathcal{E}^{(p,\bullet)}(X)$ and $\beta^* : \mathcal{E}^{(p,\bullet)}[-1](X') \rightarrow \mathcal{E}^{(p,\bullet)}(i)$ by

$$\begin{aligned} \alpha^* : \mathcal{E}^{(p,q)}(i) &= \mathcal{E}^{(p,q)}(X) \oplus \mathcal{E}^{(p,q-1)}(X') \longrightarrow \mathcal{E}^{(p,q)}(X), & (\omega, \theta) &\mapsto \omega, \quad \text{and} \\ \beta^* : \mathcal{E}^{(p,q)}[-1](X') &= \mathcal{E}^{(p,q-1)}(X') \longrightarrow \mathcal{E}^{(p,q)}(i) = \mathcal{E}^{(p,q)}(X) \oplus \mathcal{E}^{(p,q-1)}(X'), \\ &\theta &\mapsto (0, \theta). \end{aligned}$$

Then we have the exact sequence of complexes

$$(2.1) \quad 0 \longrightarrow \mathcal{E}^{(p,\bullet)}[-1](X') \xrightarrow{\beta^*} \mathcal{E}^{(p,\bullet)}(i) \xrightarrow{\alpha^*} \mathcal{E}^{(p,\bullet)}(X) \longrightarrow 0,$$

which gives rise to the exact sequence

$$(2.2) \quad \cdots \longrightarrow H_{\bar{\partial}}^{p,q-1}(X') \xrightarrow{\beta^*} H_{\bar{\partial}}^{p,q}(i) \xrightarrow{\alpha^*} H_{\bar{\partial}}^{p,q}(X) \xrightarrow{i^*} H_{\bar{\partial}}^{p,q}(X') \longrightarrow \cdots.$$

The “Relative de Rham type theorem” in [29, Subsection 2.3] reads in our case:

Theorem 2.3 (Relative Dolbeault theorem). *There is a canonical isomorphism:*

$$H_{\bar{\partial}}^{p,q}(i) \simeq H^q(X, X'; \mathcal{O}^{(p)}).$$

Remark 2.4. 1. The above cohomology $H_{\bar{\partial}}^{p,q}(i)$ has already appeared in a number of literatures, e.g., [15] and [16]. For the de Rham complex it is introduced in [3] in a little more general setting (cf. Remark 2.6.1 below).

2. The complex $\mathcal{E}^{(p,\bullet)}(i)$ is nothing but the “co-mapping cone” $M^*(i^*)$ of the morphism $i^* : \mathcal{E}^{(p,\bullet)}(X) \rightarrow \mathcal{E}^{(p,\bullet)}(X')$ (cf. [29, Section 5]). It is also identical with the complex $\mathcal{E}^{(p,\bullet)}(\mathcal{V}^*, \mathcal{V}')$ considered in Section 4 below and the cohomology $H_{\bar{\partial}}^{p,q}(i)$ is identical with $H_{\bar{\partial}}^{p,q}(X, X')$, the *relative Dolbeault cohomology* of (X, X') (cf. (4.3)).

3. If we follow the notation of [29], $\mathcal{E}^{(p,\bullet)}(i)$ should be denoted something like $\mathcal{E}(i)^{(p,\bullet)}$. The same remark applies to the notation such as $\mathcal{E}^{(p,\bullet)}(\mathcal{W}, \mathcal{W}')$ in the subsequent sections.

Dolbeault cohomology of holomorphic maps: Let $f : Y \rightarrow X$ be a holomorphic map of complex manifolds. We may directly generalize the above construction to this situation, replacing X' and i with Y and f . Thus we set

$$\mathcal{E}^{(p,q)}(f) = \mathcal{E}_X^{(p,q)}(X) \oplus \mathcal{E}_Y^{(p,q-1)}(Y)$$

and define $\bar{\partial} : \mathcal{E}^{(p,q)}(f) \rightarrow \mathcal{E}^{(p,q+1)}(f)$ by

$$\bar{\partial}(\omega, \theta) = (\bar{\partial}\omega, f^*\omega - \bar{\partial}\theta).$$

Then $(\mathcal{E}^{(p,\bullet)}(f), \bar{\partial})$ is a complex.

Definition 2.5. The Dolbeault cohomology $H_{\bar{\partial}}^{p,q}(f)$ of f of type (p, q) is defined as the q -th cohomology of $(\mathcal{E}^{(p,\bullet)}(f), \bar{\partial})$.

We denote by $\mathcal{E}_Y^{(p,\bullet)}(Y)[-1]$ the complex such that $\mathcal{E}_Y^{(p,\bullet)}(Y)[-1]^q = \mathcal{E}^{(p,q-1)}(Y)$ with the differential given by $-\bar{\partial}$. Then we have the exact sequence of complexes

$$0 \longrightarrow \mathcal{E}_Y^{(p,\bullet)}(Y)[-1]^q \xrightarrow{\beta^*} \mathcal{E}^{(p,\bullet)}(f) \xrightarrow{\alpha^*} \mathcal{E}_X^{(p,\bullet)}(X) \longrightarrow 0,$$

where $\alpha^*(\omega, \theta) = \omega$ and $\beta^*(\theta) = (0, \theta)$. The above sequence yields a long exact sequence

$$\cdots \longrightarrow H_{\bar{\partial}}^{p,q-1}(Y) \xrightarrow{\beta^*} H_{\bar{\partial}}^{p,q}(f) \xrightarrow{\alpha^*} H_{\bar{\partial}}^{p,q}(X) \xrightarrow{f^*} H_{\bar{\partial}}^{p,q}(Y) \longrightarrow \cdots.$$

In the case $Y = X'$ is an open set in X and $f = i$ is the inclusion, the above cohomology is nothing but $H_{\bar{\partial}}^{p,q}(i)$ defined before.

Remark 2.6. 1. Similar construction is done in [3] for the de Rham case.

2. The complex $\mathcal{E}^{(p,\bullet)}(f)$ is nothing but the co-mapping cone $M^*(f^*)$ of the morphism $f^* : \mathcal{E}_X^{(p,\bullet)}(X) \rightarrow \mathcal{E}_Y^{(p,\bullet)}(Y)$ (cf. [29, Section 5]).

There is the notion of the cohomology of a sheaf morphism (cf. [29, Section 6] and references therein). In our case it is defined as follows. We first consider the space $Z(f) = X \amalg Y$ (disjoint union). For an open set U in X , we set $\tilde{U} = U \amalg f^{-1}U$ and endow $Z(f)$ with the topology whose basis of open sets consists of $\{\tilde{U} \mid U \subset X \text{ open sets}\}$ and $\{V \mid V \subset Y \text{ open sets}\}$. Then we have the closed embedding $X \hookrightarrow Z(f)$ and the open embedding $Y \hookrightarrow Z(f)$. Recall in general that, for a sheaf \mathcal{T} on Y , the direct image $f_*\mathcal{T}$ is the sheaf on X defined by the presheaf $U \mapsto \mathcal{T}(f^{-1}U)$. In our situation, there is the sheaf morphism $f^* : \mathcal{O}_X^{(p)} \rightarrow f_*\mathcal{O}_Y^{(p)}$ given by the pull-back of differential forms. Let $\mathcal{Z}^*(f^*) = \mathcal{Z}^*(\mathcal{O}_Y^{(p)} \xleftarrow{f^*} \mathcal{O}_X^{(p)})$ be the sheaf on $Z(f)$ defined by the presheaf $\tilde{U} \mapsto \mathcal{O}_X^{(p)}(U)$ and $V \mapsto \mathcal{O}_Y^{(p)}(V)$. The presheaf is a sheaf, i.e., $\mathcal{Z}^*(f^*)(\tilde{U}) = \mathcal{O}_X^{(p)}(U)$ and $\mathcal{Z}^*(f^*)(V) = \mathcal{O}_Y^{(p)}(V)$. The restriction $\mathcal{Z}^*(f^*)(\tilde{U}) = \mathcal{O}_X^{(p)}(U) \rightarrow \mathcal{Z}^*(f^*)(f^{-1}U) = \mathcal{O}_Y^{(p)}(f^{-1}U)$ is given by f^* . Then the cohomology $H^q(f; f^*) = H^q(Y \xrightarrow{f} X; \mathcal{O}_Y^{(p)} \xleftarrow{f^*} \mathcal{O}_X^{(p)})$ of the morphism $f^* : \mathcal{O}_X^{(p)} \rightarrow f_*\mathcal{O}_Y^{(p)}$ is defined by (cf. [29, Section 6])

$$H^q(Y \xrightarrow{f} X; \mathcal{O}_Y^{(p)} \xleftarrow{f^*} \mathcal{O}_X^{(p)}) = H^q(Z(f), Z(f) \setminus X; \mathcal{Z}^*(\mathcal{O}_X^{(p)} \xrightarrow{f^*} \mathcal{O}_Y^{(p)})).$$

There is an exact sequence :

$$\cdots \rightarrow H^{q-1}(Y; \mathcal{O}_Y^{(p)}) \rightarrow H^q(f; f^*) \rightarrow H^q(X; \mathcal{O}_X^{(p)}) \rightarrow H^q(Y; \mathcal{O}_Y^{(p)}) \rightarrow \cdots.$$

In the case $f : Y \hookrightarrow X$ is an open embedding, we may identify f^* with the pull-back f^{-1} of sections and we denote $H^q(Y \xrightarrow{f} X; \mathcal{O}_Y^{(p)} \xleftarrow{f^*} \mathcal{O}_X^{(p)})$ by $H^q(f; \mathcal{O}_X^{(p)})$. Thus we have (cf. loc. cit) :

Proposition 2.7. *In the case $f : Y \hookrightarrow X$ is an open embedding, there is a canonical isomorphism*

$$H^q(f; \mathcal{O}_X^{(p)}) \simeq H^q(X, Y; \mathcal{O}_X^{(p)}).$$

In general, since we have the commutative the diagram

$$\begin{array}{ccccc} 0 & \longrightarrow & \mathcal{O}_X^{(p)} & \longrightarrow & \mathcal{O}_X^{(p, \bullet)} \\ & & \downarrow f^* & & \downarrow f^* \\ 0 & \longrightarrow & f_* \mathcal{O}_Y^{(p)} & \longrightarrow & f_* \mathcal{O}_Y^{(p, \bullet)}, \end{array}$$

we have :

Theorem 2.8 (Generalized relative Dolbeault theorem). *For a holomorphic map $f : Y \rightarrow X$ of complex manifolds, there is a canonical isomorphism:*

$$H_{\bar{\partial}}^{p,q}(f) \simeq H^q(Y \xrightarrow{f} X; \mathcal{O}_Y^{(p)} \xleftarrow{f^*} \mathcal{O}_X^{(p)}).$$

In the case $f : Y \hookrightarrow X$ is an open embedding, the above reduces to Theorem 2.3.

3 - Čech-Dolbeault cohomology

We recall the contents of [29, Section 3] specializing them to our setting.

3.1 - Čech cohomology

Let X be a topological space, \mathcal{S} a sheaf on X and $\mathcal{W} = \{W_\alpha\}_{\alpha \in I}$ an open covering of X . We set $W_{\alpha_0 \dots \alpha_q} = W_{\alpha_0} \cap \cdots \cap W_{\alpha_q}$ and consider the direct product

$$C^q(\mathcal{W}; \mathcal{S}) = \prod_{(\alpha_0, \dots, \alpha_q) \in I^{q+1}} \mathcal{S}(W_{\alpha_0 \dots \alpha_q}).$$

The q -th Čech cohomology $H^q(\mathcal{W}; \mathcal{S})$ of \mathcal{S} on \mathcal{W} is the q -th cohomology of the complex $(C^\bullet(\mathcal{W}; \mathcal{S}), \check{\delta})$ with $\check{\delta} : C^q(\mathcal{W}; \mathcal{S}) \rightarrow C^{q+1}(\mathcal{W}; \mathcal{S})$ defined by

$$(\check{\delta}\sigma)_{\alpha_0 \dots \alpha_{q+1}} = \sum_{\nu=0}^{q+1} (-1)^\nu \sigma_{\alpha_0 \dots \widehat{\alpha_\nu} \dots \alpha_{q+1}}.$$

Let X' be an open set in X . Let $\mathcal{W} = \{W_\alpha\}_{\alpha \in I}$ be a covering of X such that $\mathcal{W}' = \{W_\alpha\}_{\alpha \in I'}$ is a covering of X' for some $I' \subset I$. We set

$$C^q(\mathcal{W}, \mathcal{W}'; \mathcal{S}) = \{\sigma \in C^q(\mathcal{W}; \mathcal{S}) \mid \sigma_{\alpha_0 \dots \alpha_q} = 0 \text{ if } \alpha_0, \dots, \alpha_q \in I'\}.$$

The operator $\check{\delta}$ restricts to $C^q(\mathcal{W}, \mathcal{W}'; \mathcal{S}) \rightarrow C^{q+1}(\mathcal{W}, \mathcal{W}'; \mathcal{S})$. The q -th Čech cohomology $H^q(\mathcal{W}, \mathcal{W}'; \mathcal{S})$ of \mathcal{S} on $(\mathcal{W}, \mathcal{W}')$ is the q -th cohomology of $(C^\bullet(\mathcal{W}, \mathcal{W}'; \mathcal{S}), \check{\delta})$.

We have the following:

Theorem 3.1 (Relative Leray theorem). *If $H^{q_2}(W_{\alpha_0 \dots \alpha_{q_1}}, \mathcal{S}) = 0$ for $q_1 \geq 0$ and $q_2 \geq 1$, there is a canonical isomorphism*

$$H^q(\mathcal{W}, \mathcal{W}'; \mathcal{S}) \simeq H^q(X, X'; \mathcal{S}).$$

3.2 - Čech-Dolbeault cohomology

We review the contents of [29, Subsection 3.2] in our case.

Let X , $\mathcal{E}_X^{(p,q)}$ and $\mathcal{O}_X^{(p)}$ be as in Subsection 2.2. Also let X' be an open set in X and let \mathcal{W} and \mathcal{W}' be coverings of X and X' as before. Then we have a double complex $(C^\bullet(\mathcal{W}, \mathcal{W}'; \mathcal{E}^{(p,\bullet)}), \check{\delta}, (-1)^\bullet \bar{\partial})$. We consider the associated single complex $(\mathcal{E}^{(p,\bullet)}(\mathcal{W}, \mathcal{W}'), \bar{\vartheta})$. Thus

$$\mathcal{E}^{(p,q)}(\mathcal{W}, \mathcal{W}') = \bigoplus_{q_1+q_2=q} C^{q_1}(\mathcal{W}, \mathcal{W}'; \mathcal{E}^{(p,q_2)}), \quad \bar{\vartheta} = \check{\delta} + (-1)^{q_1} \bar{\partial}.$$

Definition 3.2. The Čech-Dolbeault cohomology $H_{\bar{\vartheta}}^{p,q}(\mathcal{W}, \mathcal{W}')$ of type (p, q) on $(\mathcal{W}, \mathcal{W}')$ is the q -th cohomology of the complex $(\mathcal{E}^{(p,\bullet)}(\mathcal{W}, \mathcal{W}'), \bar{\vartheta})$.

In the case $X' = \emptyset$, we take \emptyset as I' and denote $\mathcal{E}^{(p,\bullet)}(\mathcal{W}, \mathcal{W}')$ and $H_{\bar{\vartheta}}^{p,q}(\mathcal{W}, \mathcal{W}')$ by $\mathcal{E}^{(p,\bullet)}(\mathcal{W})$ and $H_{\bar{\vartheta}}^{p,q}(\mathcal{W})$.

We recall the description of the differential $\bar{\vartheta}$ as given in [29]. Note that a cochain ξ in $\mathcal{E}^{(p,q)}(\mathcal{W}, \mathcal{W}')$ may be expressed as $\xi = (\xi^{q_1})_{0 \leq q_1 \leq q}$ with ξ^{q_1} in $C^{q_1}(\mathcal{W}, \mathcal{W}'; \mathcal{E}^{(p,q-q_1)})$. In the sequel $\xi_{\alpha_0 \dots \alpha_{q_1}}^{q_1}$ is also written $\xi_{\alpha_0 \dots \alpha_{q_1}}$. Then $\bar{\vartheta} : \mathcal{E}^{(p,q)}(\mathcal{W}, \mathcal{W}') \rightarrow \mathcal{E}^{(p,q+1)}(\mathcal{W}, \mathcal{W}')$ is given by

$$(3.1) \quad (\bar{\vartheta}\xi)^{q_1} = \check{\delta}\xi^{q_1-1} + (-1)^{q_1} \bar{\partial}\xi^{q_1}, \quad 0 \leq q_1 \leq q+1,$$

where we set $\xi^{-1} = 0$ and $\xi^{q+1} = 0$. In particular, for $q_1 = 0, 1$,

$$(3.2) \quad (\bar{\partial}\xi)_{\alpha_0} = \bar{\partial}\xi_{\alpha_0}, \quad (\bar{\partial}\xi)_{\alpha_0\alpha_1} = \xi_{\alpha_1} - \xi_{\alpha_0} - \bar{\partial}\xi_{\alpha_0\alpha_1}.$$

Thus the condition for ξ being a cocycle is given by

$$\begin{cases} \bar{\partial}\xi^0 = 0, \\ \bar{\partial}\xi^{q_1-1} + (-1)^{q_1}\bar{\partial}\xi^{q_1} = 0, & 1 \leq q_1 \leq q, \\ \bar{\partial}\xi^q = 0. \end{cases}$$

We have $H_{\bar{\partial}}^{p,0}(\mathcal{W}, \mathcal{W}') = \mathcal{O}^{(p)}(X, X')$.

For a triple $(\mathcal{W}, \mathcal{W}', \mathcal{W}'')$, we have the exact sequence

$$(3.3) \quad 0 \longrightarrow \mathcal{E}^{(p,\bullet)}(\mathcal{W}, \mathcal{W}') \longrightarrow \mathcal{E}^{(p,\bullet)}(\mathcal{W}, \mathcal{W}'') \longrightarrow \mathcal{E}^{(p,\bullet)}(\mathcal{W}', \mathcal{W}'') \longrightarrow 0$$

yielding an exact sequence

$$(3.4) \quad \begin{aligned} \cdots \longrightarrow H_{\bar{\partial}}^{p,q-1}(\mathcal{W}', \mathcal{W}'') &\xrightarrow{\delta} H_{\bar{\partial}}^{p,q}(\mathcal{W}, \mathcal{W}') \longrightarrow \\ &H_{\bar{\partial}}^{p,q}(\mathcal{W}, \mathcal{W}'') \longrightarrow H_{\bar{\partial}}^{p,q}(\mathcal{W}', \mathcal{W}'') \longrightarrow \cdots \end{aligned}$$

Remark 3.3. We may use only “alternating cochains” in the above construction and the resulting cohomology is canonically isomorphic with the one defined above.

Some special cases: **I.** In the case $\mathcal{W} = \{X\}$, we have $(\mathcal{E}^{(p,\bullet)}(\mathcal{W}), \bar{\partial}) = (\mathcal{E}^{(p,\bullet)}(X), \bar{\partial})$ and $H_{\bar{\partial}}^{p,q}(\mathcal{W}) = H_{\bar{\partial}}^{p,q}(X)$.

II. In the case \mathcal{W} consists of two open sets W_0 and W_1 , we may write (cf. Remark 3.3)

$$\begin{aligned} \mathcal{E}^{(p,q)}(\mathcal{W}) &= C^0(\mathcal{W}, \mathcal{E}^{(p,q)}) \oplus C^1(\mathcal{W}, \mathcal{E}^{(p,q-1)}) \\ &= \mathcal{E}^{(p,q)}(W_0) \oplus \mathcal{E}^{(p,q)}(W_1) \oplus \mathcal{E}^{(p,q-1)}(W_{01}). \end{aligned}$$

Thus a cochain $\xi \in \mathcal{E}^{(p,q)}(\mathcal{W})$ is expressed as a triple $\xi = (\xi_0, \xi_1, \xi_{01})$ and the differential $\bar{\partial} : \mathcal{E}^{(p,q)}(\mathcal{W}) \rightarrow \mathcal{E}^{(p,q+1)}(\mathcal{W})$ is given by

$$\bar{\partial}(\xi_0, \xi_1, \xi_{01}) = (\bar{\partial}\xi_0, \bar{\partial}\xi_1, \xi_1 - \xi_0 - \bar{\partial}\xi_{01}).$$

If we set $Z^{p,q}(\mathcal{W}) = \text{Ker } \bar{\partial}^{p,q}$ and $B^{p,q}(\mathcal{W}) = \text{Im } \bar{\partial}^{p,q-1}$, then by definition, $H_{\bar{\partial}}^{p,q}(\mathcal{W}) = Z^{p,q}(\mathcal{W})/B^{p,q}(\mathcal{W})$. We may somewhat simplify the coboundary group $B^{p,q}(\mathcal{W})$ (cf. [29]):

Proposition 3.4. *The group $B^{p,q}(\mathcal{W})$ is identical with the set*

$$\{\xi \in \mathcal{E}^{(p,q)}(\mathcal{W}) \mid \xi = (\bar{\partial}\eta_0, \bar{\partial}\eta_1, \eta_1 - \eta_0), \text{ for some } \eta_i \in \mathcal{E}^{p,q-1}(W_i), i = 0, 1\}.$$

In the relative case, if we set $\mathcal{W}' = \{W_0\}$, then

$$\mathcal{E}^{(p,q)}(\mathcal{W}, \mathcal{W}') = \{\xi \in \mathcal{E}^{(p,q)}(\mathcal{W}) \mid \xi_0 = 0\} = \mathcal{E}^{(p,q)}(W_1) \oplus \mathcal{E}^{p,q-1}(W_{01}).$$

Thus a cochain $\xi \in \mathcal{E}^{(p,q)}(\mathcal{W}, \mathcal{W}')$ is expressed as a pair $\xi = (\xi_1, \xi_{01})$ and the differential

$$\bar{\partial} : \mathcal{E}^{(p,q)}(\mathcal{W}, \mathcal{W}') \rightarrow \mathcal{E}^{p,q+1}(\mathcal{W}, \mathcal{W}') \quad \text{is given by} \quad \bar{\partial}(\xi_1, \xi_{01}) = (\bar{\partial}\xi_1, \xi_1 - \bar{\partial}\xi_{01}).$$

The q -th cohomology of $(\mathcal{E}^{p,\bullet}(\mathcal{W}, \mathcal{W}'), \bar{\partial})$ is $H_{\bar{\partial}}^{p,q}(\mathcal{W}, \mathcal{W}')$.

If we set $\mathcal{W}'' = \emptyset$, then $H_{\bar{\partial}}^{p,q-1}(\mathcal{W}', \mathcal{W}'') = H_{\bar{\partial}}^{p,q-1}(\mathcal{W}') = H_{\bar{\partial}}^{p,q-1}(W_0)$ and the connecting morphism δ in (3.4) assigns to the class of a $\bar{\partial}$ -closed form ξ_0 on W_0 the class of $(0, -\xi_0)$ (restricted to W_1) in $H_{\bar{\partial}}^{p,q}(\mathcal{W}, \mathcal{W}')$.

We discuss this case more in detail in the subsequent section.

III. Suppose \mathcal{W} consists of three open sets W_0 , W_1 and W_2 and set $\mathcal{W}' = \{W_0, W_1\}$ and $\mathcal{W}'' = \{W_0\}$. Then

$$\begin{aligned} \mathcal{E}^{(p,q)}(\mathcal{W}) &= \bigoplus_{i=0}^2 \mathcal{E}^{(p,q)}(W_i) \oplus \bigoplus_{0 \leq i < j \leq 2} \mathcal{E}^{(p,q-1)}(W_{ij}) \oplus \mathcal{E}^{(p,q-2)}(W_{012}), \\ \mathcal{E}^{(p,q)}(\mathcal{W}, \mathcal{W}'') &= \bigoplus_{i=1}^2 \mathcal{E}^{(p,q)}(W_i) \oplus \bigoplus_{0 \leq i < j \leq 2} \mathcal{E}^{(p,q-1)}(W_{ij}) \oplus \mathcal{E}^{(p,q-2)}(W_{012}), \\ \mathcal{E}^{(p,q)}(\mathcal{W}, \mathcal{W}') &= \mathcal{E}^{(p,q)}(W_2) \oplus \mathcal{E}^{(p,q-1)}(W_{02}) \oplus \mathcal{E}^{(p,q-1)}(W_{12}) \oplus \mathcal{E}^{(p,q-2)}(W_{012}), \\ \mathcal{E}^{(p,q)}(\mathcal{W}', \mathcal{W}'') &= \mathcal{E}^{(p,q)}(W_1) \oplus \mathcal{E}^{(p,q-1)}(W_{01}). \end{aligned}$$

The morphism δ in (3.4) assigns to the class of (θ_1, θ_{01}) in $H_{\bar{\partial}}^{p,q-1}(\mathcal{W}', \mathcal{W}'')$ the class of $(0, 0, -\theta_1, \theta_{01})$ (restricted to W_2) in $H_{\bar{\partial}}^{p,q}(\mathcal{W}, \mathcal{W}')$.

Canonical isomorphisms : We say that a covering $\mathcal{W} = \{W_\alpha\}$ of X is *Stein*, if every non-empty finite intersection $W_{\alpha_0 \dots \alpha_{q_1}}$ is a Stein manifold. In fact, for this it is sufficient if each W_α is Stein (cf. [7], [22]). Note that every complex manifold X admits a Stein covering and that the Stein coverings are cofinal in the set of coverings of X . We quote :

Theorem 3.5 (Oka-Cartan). *For every coherent sheaf \mathcal{S} on a Stein manifold W ,*

$$H^q(W; \mathcal{S}) = 0 \quad \text{for } q \geq 1.$$

By the above and Theorem 2.1, we see that a Stein covering is good for $\mathcal{E}^{(p,\bullet)}$ in the sense of [29, Section 3]. Thus in our case, we have :

Theorem 3.6. *We have the following canonical isomorphisms:*

1. *For every covering \mathcal{W} ,*

$$H_{\bar{\partial}}^{p,q}(X) \xrightarrow{\sim} H_{\bar{\partial}}^{p,q}(\mathcal{W}).$$

2. *For a Stein covering \mathcal{W} ,*

$$H_{\bar{\partial}}^{p,q}(\mathcal{W}, \mathcal{W}') \xleftarrow{\sim} H^q(\mathcal{W}, \mathcal{W}'; \mathcal{O}^{(p)}) \simeq H^q(X, X'; \mathcal{O}^{(p)}).$$

Remark 3.7. 1. From 1 above we see that $H_{\bar{\partial}}^{p,q}(\mathcal{W})$ does not depend on the covering \mathcal{W} . The isomorphism there is induced from the inclusion of complexes:

$$\mathcal{E}^{(p,\bullet)}(X) \hookrightarrow C^0(\mathcal{W}; \mathcal{E}^{(p,\bullet)}) \subset \mathcal{E}^{(p,\bullet)}(\mathcal{W}).$$

In particular, if $W_\alpha = X$ for some $\alpha \in I$, it can be shown that the morphism $\mathcal{E}^{(p,\bullet)}(\mathcal{W}) \rightarrow \mathcal{E}^{(p,\bullet)}(X)$ given by $\xi \mapsto \xi_\alpha$ induces the inverse of the above isomorphism (cf. [29]). See also Propositions 3.10 and 3.11 below.

2. The first isomorphism in 2 above is induced from the inclusion of complexes:

$$C^\bullet(\mathcal{W}, \mathcal{W}'; \mathcal{O}^{(p)}) \hookrightarrow C^\bullet(\mathcal{W}, \mathcal{W}'; \mathcal{E}^{(p,0)}) \subset \mathcal{E}^{(p,\bullet)}(\mathcal{W}, \mathcal{W}').$$

The second isomorphism follows from Theorem 3.1.

From Theorem 3.6 we have:

Corollary 3.8. *If \mathcal{W} is Stein, there is a canonical isomorphism:*

$$H_{\bar{\partial}}^{p,q}(X) \simeq H^q(\mathcal{W}; \mathcal{O}^{(p)}).$$

In the above, we think of a Dolbeault cocycle $\omega \in \mathcal{E}^{(p,q)}(X)$ and a Čech cocycle $c \in C^q(\mathcal{W}; \mathcal{O}^{(p)})$ as being Čech-Dolbeault cocycles, i.e., cocycles in $\mathcal{E}^{(p,q)}(\mathcal{W})$. The classes $[\omega] \in H_{\bar{\partial}}^q(X)$ and $[c] \in H^q(\mathcal{W}; \mathcal{O}^{(p)})$ correspond in the above isomorphism, if and only if ω and c define the same class in $H_{\bar{\partial}}^q(\mathcal{W})$, i.e., there exists a $(q-1)$ -cochain $\chi \in \mathcal{E}^{(p,q-1)}(\mathcal{W})$ such that

$$\omega - c = \bar{\partial}\chi.$$

The above relation is rephrased as, for χ^{q_1} in $C^{q_1}(\mathcal{W}; \mathcal{E}^{(p,q-q_1-1)})$, $0 \leq q_1 \leq q-1$,

$$(3.5) \quad \begin{cases} \omega = \bar{\partial}\chi^0, \\ 0 = \check{\delta}\chi^{q_1-1} + (-1)^{q_1}\bar{\partial}\chi^{q_1}, & 1 \leq q_1 \leq q-1, \\ -c = \check{\delta}\chi^{q-1}. \end{cases}$$

Note that the composition of the isomorphism of Corollary 3.8 and the second isomorphism of Theorem 3.6.2 for $X' = \emptyset$ is equal to the isomorphism in Theorem 2.1.

Remark 3.9. It is possible to establish an isomorphism as in Corollary 3.8 without introducing the Čech-Dolbeault cohomology, using the so-called Weil lemma instead. However this correspondence is different from the one in Corollary 3.8, the difference being a sign of $(-1)^{\frac{q(q+1)}{2}}$, see [29, Section 3] for details.

The seemingly standard proof in the textbooks, e.g., [8], [12], of the isomorphism as in Theorem 2.1 or Corollary 3.8 gives a correspondence same as the one given by the Weil lemma. Thus there is a sign difference as above. For example, in Theorem 8.1 below, the sign $(-1)^{\frac{n(n-1)}{2}}$ does not appear this way (cf. [8], [11]).

We finish this section by discussing the isomorphism of Theorem 3.6.1 in some special cases. Recall that it is induced by the inclusion $\mathcal{E}^{(p,q)}(X) \hookrightarrow C^0(\mathcal{W}; \mathcal{E}^{(p,q)}) \subset \mathcal{E}^{(p,q)}(\mathcal{W})$.

In the case $\mathcal{W} = \{W_0, W_1\}$ (cf. the case II above),

$$\mathcal{E}^{(p,q)}(\mathcal{W}) = \mathcal{E}^{(p,q)}(W_0) \oplus \mathcal{E}^{(p,q)}(W_1) \oplus \mathcal{E}^{(p,q-1)}(W_{01})$$

and the inclusion is given by $\omega \mapsto (\omega|_{W_0}, \omega|_{W_1}, 0)$.

Proposition 3.10. *In the case $\mathcal{W} = \{W_0, W_1\}$, the inverse of the above isomorphism is given by assigning to the class of ξ the class of $\omega = \rho_0\xi_0 + \rho_1\xi_1 - \bar{\partial}\rho_0 \wedge \xi_{01}$, where (ρ_0, ρ_1) is a partition of unity subordinate to \mathcal{W} .*

Proof. Recall that ω is given by $\xi_1 - \bar{\partial}(\rho_0\xi_{01})$ on W_1 (cf. [29, Section 3]). Using the cocycle condition $\xi_1 - \xi_0 - \bar{\partial}\xi_{01} = 0$, it can be written $\rho_0\xi_0 + \rho_1\xi_1 - \bar{\partial}\rho_0 \wedge \xi_{01}$, which is a global expression of ω . \square

Likewise we may prove (cf. the case III above):

Proposition 3.11. *In the case $\mathcal{W} = \{W_0, W_1, W_2\}$, the inverse of the above isomorphism is given by assigning to the class of ξ the class of*

$$\omega = \sum_{i=0}^2 \rho_i \xi_i + \sum_{0 \leq i < j \leq 2} (\rho_i \bar{\partial} \rho_j - \rho_j \bar{\partial} \rho_i) \wedge \xi_{ij} + (\bar{\partial} \rho_0 \wedge \bar{\partial} \rho_1 + \bar{\partial} \rho_1 \wedge \bar{\partial} \rho_2) \wedge \xi_{012},$$

where $\{\rho_0, \rho_1, \rho_2\}$ is a partition of unity subordinate to \mathcal{W} .

4 - Relative Dolbeault cohomology

We specialize the contents of [29, Section 4] to our setting.

Let X be a complex manifold and X' an open set in X . Letting $V_0 = X'$ and V_1 a neighborhood of the closed set $S = X \setminus X'$, consider the coverings

$\mathcal{V} = \{V_0, V_1\}$ and $\mathcal{V}' = \{V_0\}$ of X and X' (cf. the case II in Section 3). We have the cohomology $H_{\bar{\partial}}^{p,q}(\mathcal{V}, \mathcal{V}')$ as the cohomology of the complex $(\mathcal{E}^{(p,\bullet)}(\mathcal{V}, \mathcal{V}'), \bar{\partial})$, where

$$\mathcal{E}^{(p,q)}(\mathcal{V}, \mathcal{V}') = \mathcal{E}^{(p,q)}(V_1) \oplus \mathcal{E}^{(p,q-1)}(V_{01}), \quad V_{01} = V_0 \cap V_1,$$

and $\bar{\partial} : \mathcal{E}^{(p,q)}(\mathcal{V}, \mathcal{V}') \rightarrow \mathcal{E}^{(p,q+1)}(\mathcal{V}, \mathcal{V}')$ is given by $\bar{\partial}(\xi_1, \xi_{01}) = (\bar{\partial}\xi_1, \xi_1 - \bar{\partial}\xi_{01})$. Noting that $\mathcal{E}^{(p,q)}(\{V_0\}) = \mathcal{E}^{(p,q)}(X')$, we have the exact sequence

$$(4.1) \quad 0 \longrightarrow \mathcal{E}^{(p,\bullet)}(\mathcal{V}, \mathcal{V}') \xrightarrow{j^*} \mathcal{E}^{(p,\bullet)}(\mathcal{V}) \xrightarrow{i^*} \mathcal{E}^{(p,\bullet)}(X') \longrightarrow 0,$$

where $j^*(\xi_1, \xi_{01}) = (0, \xi_1, \xi_{01})$ and $i^*(\xi_0, \xi_1, \xi_{01}) = \xi_0$. This gives rise to the exact sequence (cf. (3.4))

$$(4.2) \quad \cdots \longrightarrow H_{\bar{\partial}}^{p,q-1}(X') \xrightarrow{\delta} H_{\bar{\partial}}^{p,q}(\mathcal{V}, \mathcal{V}') \xrightarrow{j^*} H_{\bar{\partial}}^{p,q}(\mathcal{V}) \xrightarrow{i^*} H_{\bar{\partial}}^{p,q}(X') \longrightarrow \cdots,$$

where δ assigns to the class of θ the class of $(0, -\theta)$.

Now we consider the special case where $V_1 = X$. Thus, letting $V_0 = X'$ and $V_1^* = X$, we consider the coverings $\mathcal{V}^* = \{V_0, V_1^*\}$ and $\mathcal{V}' = \{V_0\}$ of X and X' .

Definition 4.1. We denote $H_{\bar{\partial}}^{p,q}(\mathcal{V}^*, \mathcal{V}')$ by $H_{\bar{\partial}}^{p,q}(X, X')$ and call it the *relative Dolbeault cohomology* of (X, X') .

In the case $X' = \emptyset$, it coincides with $H_{\bar{\partial}}^{p,q}(X)$. If we denote by $i : X' \hookrightarrow X$ the inclusion, by construction we see that (cf. Subsection 2.3):

$$(4.3) \quad \mathcal{E}^{(p,\bullet)}(\mathcal{V}^*, \mathcal{V}') = \mathcal{E}^{(p,\bullet)}(i) \quad \text{and} \quad H_{\bar{\partial}}^{p,q}(X, X') = H_{\bar{\partial}}^{p,q}(i).$$

By Theorem 3.6.1, there is a canonical isomorphism $H_{\bar{\partial}}^{p,q}(X) \xrightarrow{\sim} H_{\bar{\partial}}^{p,q}(\mathcal{V}^*)$, which assigns to the class of s the class of $(s|_{X'}, s, 0)$. Its inverse assigns to the class of (ξ_0, ξ_1, ξ_{01}) the class of ξ_1 (cf. Remark 3.7.1). Thus from (4.2) we have the exact sequence

$$(4.4) \quad \cdots \longrightarrow H_{\bar{\partial}}^{p,q-1}(X') \xrightarrow{\delta} H_{\bar{\partial}}^{p,q}(X, X') \xrightarrow{j^*} H_{\bar{\partial}}^{p,q}(X) \xrightarrow{i^*} H_{\bar{\partial}}^{p,q}(X') \longrightarrow \cdots,$$

where j^* assigns to the class of (ξ_1, ξ_{01}) the class of ξ_1 and i^* assigns to the class of s the class of $s|_{X'}$. It coincides with the sequence (2.2), except $\delta = -\beta^*$.

We have the following propositions (cf. [29]):

Proposition 4.2. *For a triple (X, X', X'') , there is an exact sequence*

$$\begin{aligned} \cdots \longrightarrow H_{\bar{\partial}}^{p,q-1}(X', X'') &\xrightarrow{\delta} H_{\bar{\partial}}^{p,q}(X, X') \xrightarrow{j^*} \\ &H_{\bar{\partial}}^{p,q}(X, X'') \xrightarrow{i^*} H_{\bar{\partial}}^{p,q}(X', X'') \longrightarrow \cdots \end{aligned}$$

Let $\mathcal{V} = \{V_0, V_1\}$ be as in the beginning of this section, with V_1 an arbitrary open set containing $X \setminus X'$. By Theorem 3.6.1, there is a canonical isomorphism $H_{\bar{\partial}}^{p,q}(\mathcal{V}) \simeq H_{\bar{\partial}}^{p,q}(X)$ and in (4.2), j^* assigns to the class of (ξ_1, ξ_{01}) the class of $(0, \xi_1, \xi_{01})$ or the class of $\rho_1 \xi_1 - \bar{\partial} \rho_0 \wedge \xi_{01}$ (or the class of ξ_1 if $V_1 = X$) (cf. Proposition 3.10, also Remark 3.7.1).

Proposition 4.3. *The restriction $\mathcal{E}^{(p,\bullet)}(\mathcal{V}^*, \mathcal{V}') \rightarrow \mathcal{E}^{(p,\bullet)}(\mathcal{V}, \mathcal{V}')$ induces an isomorphism*

$$H_{\bar{\partial}}^{p,q}(X, X') \xrightarrow{\sim} H_{\bar{\partial}}^{p,q}(\mathcal{V}, \mathcal{V}').$$

Corollary 4.4. *The cohomology $H_{\bar{\partial}}^{p,q}(\mathcal{V}, \mathcal{V}')$ is uniquely determined modulo canonical isomorphisms, independently of the choice of V_1 .*

Remark 4.5. This freedom of choice of V_1 is one of the advantages of expressing $H_{\bar{\partial}}^{p,q}(i)$ as $H_{\bar{\partial}}^{p,q}(X, X')$.

Proposition 4.6 (Excision). *Let S be a closed set in X . Then, for every open set V in X containing S , there is a canonical isomorphism*

$$H_{\bar{\partial}}^{p,q}(X, X \setminus S) \xrightarrow{\sim} H_{\bar{\partial}}^{p,q}(V, V \setminus S).$$

Now we indicate an alternative proof of Theorem 2.3 and refer to [29] for details. Let $\mathcal{W} = \{W_\alpha\}_{\alpha \in I}$ be a covering of X and $\mathcal{W}' = \{W_\alpha\}_{\alpha \in I'}$ a covering of X' , $I' \subset I$. Letting $V_1^* = X$ as before, we define a morphism

$$\varphi : \mathcal{E}^{(p,q)}(\mathcal{V}^*, \mathcal{V}') \longrightarrow C^0(\mathcal{W}, \mathcal{W}'; \mathcal{E}^{(p,q)}) \oplus C^1(\mathcal{W}, \mathcal{W}'; \mathcal{E}^{(p,q-1)}) \subset \mathcal{E}^{(p,q)}(\mathcal{W}, \mathcal{W}')$$

by setting, for $\xi = (\xi_1, \xi_{01})$,

$$\varphi(\xi)_\alpha = \begin{cases} 0 & \alpha \in I' \\ \xi_1|_{W_\alpha} & \alpha \in I \setminus I', \end{cases} \quad \varphi(\xi)_{\alpha\beta} = \begin{cases} \xi_{01}|_{W_{\alpha\beta}} & \alpha \in I', \beta \in I \setminus I' \\ -\xi_{01}|_{W_{\alpha\beta}} & \alpha \in I \setminus I', \beta \in I' \\ 0 & \text{otherwise.} \end{cases}$$

Theorem 4.7. *The above morphism φ induces an isomorphism*

$$H_{\bar{\partial}}^{p,q}(X, X') \xrightarrow{\sim} H_{\bar{\partial}}^{p,q}(\mathcal{W}, \mathcal{W}').$$

Using the above we have an alternative proof of the relative Dolbeault theorem (Theorem 2.3):

Theorem 4.8. *There is a canonical isomorphism:*

$$H_{\bar{\partial}}^{p,q}(X, X') \simeq H^q(X, X'; \mathcal{O}^{(p)}).$$

The sequence in Proposition 4.2 is compatible with the corresponding sequence for the relative cohomology and the excision of Proposition 4.6 is compatible with that of the relative cohomology, both via the isomorphism of Theorem 4.8.

We finish this section by presenting the following topic:

Differential: Let X be a complex manifold of dimension n and X' an open set in X . We consider coverings \mathcal{W} and \mathcal{W}' of X and X' as before.

First note that the second isomorphism of Theorem 3.6.2 is compatible with the differential $d : \mathcal{O}^{(p)} \rightarrow \mathcal{O}^{(p+1)}$, in fact $d = \partial$ in this case. We define

$$\partial : \mathcal{E}^{(p,q)}(\mathcal{W}, \mathcal{W}') \longrightarrow \mathcal{E}^{(p+1,q)}(\mathcal{W}, \mathcal{W}') \quad \text{by } (\partial\xi)^{q_1} = (-1)^{q-q_1} \partial\xi^{q_1}$$

for $0 \leq q_1 \leq q$. A straightforward computation shows that it is compatible with the operator $\bar{\partial}$, i.e., the following diagram is commutative:

$$\begin{array}{ccc} \mathcal{E}^{(p,q)}(\mathcal{W}, \mathcal{W}') & \xrightarrow{\partial} & \mathcal{E}^{(p+1,q)}(\mathcal{W}, \mathcal{W}') \\ \downarrow \bar{\partial} & & \downarrow \bar{\partial} \\ \mathcal{E}^{(p,q+1)}(\mathcal{W}, \mathcal{W}') & \xrightarrow{\partial} & \mathcal{E}^{(p+1,q+1)}(\mathcal{W}, \mathcal{W}'). \end{array}$$

Thus we have

$$\partial : H_{\bar{\partial}}^{p,q}(\mathcal{W}, \mathcal{W}') \longrightarrow H_{\bar{\partial}}^{p+1,q}(\mathcal{W}, \mathcal{W}').$$

Proposition 4.9. *If \mathcal{W} is Stein, we have the following commutative diagram:*

$$\begin{array}{ccc} H_{\bar{\partial}}^{p,q}(\mathcal{W}, \mathcal{W}') & \xrightarrow{\partial} & H_{\bar{\partial}}^{p+1,q}(\mathcal{W}, \mathcal{W}') \\ \uparrow \wr & & \uparrow \wr \\ H^q(\mathcal{W}, \mathcal{W}'; \mathcal{O}^{(p)}) & \xrightarrow{d} & H^q(\mathcal{W}, \mathcal{W}'; \mathcal{O}^{(p+1)}), \end{array}$$

where the vertical isomorphisms are the ones in Theorem 3.6.2.

Example 4.10. In the case $\mathcal{W} = \{X\}$, we have $H_{\bar{\partial}}^{p,q}(\mathcal{W}) = H_{\bar{\partial}}^{p,q}(X)$ (cf. the case I in Subsection 3.2) and $\partial : H_{\bar{\partial}}^{p,q}(\mathcal{W}) \rightarrow H_{\bar{\partial}}^{p+1,q}(\mathcal{W})$ is induced by $\theta \mapsto (-1)^q \partial\theta$.

In the case $\mathcal{W} = \{W_0, W_1\}$ (cf. the case II in Subsection 3.2), $\partial : H_{\bar{\partial}}^{p,q}(\mathcal{W}) \rightarrow H_{\bar{\partial}}^{p+1,q}(\mathcal{W})$ is induced by

$$(\xi_0, \xi_1, \xi_{01}) \mapsto (-1)^q (\partial\xi_0, \partial\xi_1, -\partial\xi_{01}).$$

From the above we have the differential

$$\partial : H_{\bar{\partial}}^{p,q}(X, X') \longrightarrow H_{\bar{\partial}}^{p+1,q}(X, X') \quad \text{induced by } (\xi_1, \xi_{01}) \mapsto (-1)^q (\partial\xi_1, -\partial\xi_{01}).$$

From Proposition 4.9, we have

Proposition 4.11. *We have the following commutative diagram:*

$$\begin{array}{ccc} H_{\bar{\partial}}^{p,q}(X, X') & \xrightarrow{\partial} & H_{\bar{\partial}}^{p+1,q}(X, X') \\ \wr \downarrow & & \downarrow \wr \\ H^q(X, X'; \mathcal{O}^{(p)}) & \xrightarrow{d} & H^q(X, X'; \mathcal{O}^{(p+1)}), \end{array}$$

where the vertical isomorphisms are the ones in Theorem 4.8.

5 - Relation with the case of de Rham complex

5.1 - Relative de Rham cohomology

We refer to [3] and [26] for details on the Čech-de Rham cohomology. For the relative de Rham cohomology and the Thom class in this context, see [26].

In this subsection, we let X denote a C^∞ manifold of dimension m with a countable basis. We assume that the coverings we consider are locally finite. We denote by $\mathcal{E}_X^{(q)}$ the sheaf of C^∞ q -forms on X . Recall that, by the Poincaré lemma, $\mathcal{E}_X^{(\bullet)}$ gives a fine resolution of the constant sheaf \mathbb{C}_X :

$$0 \longrightarrow \mathbb{C} \longrightarrow \mathcal{E}^{(0)} \xrightarrow{d} \mathcal{E}^{(1)} \xrightarrow{d} \cdots \xrightarrow{d} \mathcal{E}^{(m)} \longrightarrow 0.$$

de Rham cohomology: The de Rham cohomology $H_d^q(X)$ is the cohomology of the complex $(\mathcal{E}^{(\bullet)}(X), d)$. The de Rham theorem says that there is an isomorphism

$$H_d^q(X) \simeq H^q(X; \mathbb{C}_X).$$

Note that among the isomorphisms, there is a canonical one (cf. [29]).

Čech-de Rham cohomology: Let X' be an open set in X and $(\mathcal{W}, \mathcal{W}')$ a pair of coverings of (X, X') . The Čech-de Rham cohomology $H_D^q(\mathcal{W}, \mathcal{W}')$ on $(\mathcal{W}, \mathcal{W}')$ is the cohomology of the single complex $(\mathcal{E}^{(\bullet)}(\mathcal{W}, \mathcal{W}'), D)$ associated with the double complex $(C^\bullet(\mathcal{W}, \mathcal{W}'; \mathcal{E}^{(\bullet)}), \check{\delta}, (-1)^{\bullet} d)$, i.e.,

$$\mathcal{E}^{(q)}(\mathcal{W}, \mathcal{W}') = \bigoplus_{q_1+q_2=q} C^{q_1}(\mathcal{W}, \mathcal{W}'; \mathcal{E}^{(q_2)}), \quad D = \check{\delta} + (-1)^{q_1} d.$$

We say that \mathcal{W} is *good* if every non-empty finite intersection $W_{\alpha_0 \dots \alpha_q}$ is diffeomorphic with \mathbb{R}^m . Note that every C^∞ manifold X admits a good covering and that the good coverings are cofinal in the set of coverings of X . By the Poincaré lemma, we see that a good covering is good for $\mathcal{E}^{(\bullet)}$ in the sense of [29, Section 3]. Thus we have the following canonical isomorphisms:

$$(1) \text{ For every covering } \mathcal{W}, H_d^q(X) \xrightarrow{\sim} H_D^q(\mathcal{W}).$$

$$(2) \text{ For a good covering } \mathcal{W},$$

$$H_D^q(\mathcal{W}, \mathcal{W}') \xleftarrow{\sim} H^q(\mathcal{W}, \mathcal{W}'; \mathbb{C}) \simeq H^q(X, X'; \mathbb{C}).$$

Relative de Rham cohomology : We may also define the relative de Rham cohomology as in the case of relative Dolbeault cohomology. Thus let S be a closed set in X . Letting $V_0 = X \setminus S$ and V_1 a neighborhood of S in X , we consider the coverings $\mathcal{V} = \{V_0, V_1\}$ and $\mathcal{V}' = \{V_0\}$ of X and $X \setminus S$, as before. We set

$$\mathcal{E}^{(q)}(\mathcal{V}, \mathcal{V}') = \mathcal{E}^{(q)}(V_1) \oplus \mathcal{E}^{(q-1)}(V_{01})$$

and define

$$D : \mathcal{E}^{(q)}(\mathcal{V}, \mathcal{V}') \longrightarrow \mathcal{E}^{(q+1)}(\mathcal{V}, \mathcal{V}') \quad \text{by} \quad D(\sigma_1, \sigma_{01}) = (d\sigma_1, \sigma_1 - d\sigma_{01}).$$

Definition 5.1. The q -th *relative de Rham cohomology* $H_D^q(\mathcal{V}, \mathcal{V}')$ is the q -th cohomology of the complex $(\mathcal{E}^{(\bullet)}(\mathcal{V}, \mathcal{V}'), D)$.

As in the case of Dolbeault complex, we may show that it does not depend on the choice of V_1 and we denote it by $H_D^q(X, X \setminus S)$. We have the relative de Rham theorem which says that there is a canonical isomorphism (cf. [27], [29]):

$$(5.1) \quad H_D^q(X, X \setminus S) \simeq H^q(X, X \setminus S; \mathbb{C}_X).$$

Remark 5.2. 1. The sheaf cohomology $H^q(X; \mathbb{Z}_X)$ is canonically isomorphic with the singular cohomology $H^q(X; \mathbb{Z})$ of X with \mathbb{Z} -coefficients on finite chains and the relative sheaf cohomology $H^q(X, X \setminus S; \mathbb{Z}_X)$ is isomorphic with the relative singular cohomology $H^q(X, X \setminus S; \mathbb{Z})$.

2. In [3], the relative de Rham cohomology is introduced somewhat in a different way (cf. Remark 2.4.1).

Thom class : Let $\pi : E \rightarrow M$ be a C^∞ real vector bundle of rank l on a C^∞ manifold M . We identify M with the image of the zero section. Suppose it is orientable as a bundle and is specified with an orientation, i.e., oriented. Then we have the Thom isomorphism

$$T : H^{q-l}(M; \mathbb{Z}) \xrightarrow{\sim} H^q(E, E \setminus M; \mathbb{Z}).$$

The *Thom class* $\Psi_E \in H^l(E, E \setminus M; \mathbb{Z})$ of E is the image of $[1] \in H^0(M; \mathbb{Z})$ by T .

The Thom isomorphism with \mathbb{C} -coefficients is expressed in terms of the de Rham and relative de Rham cohomologies :

$$T : H_d^{q-l}(M) \xrightarrow{\sim} H_D^q(E, E \setminus M).$$

Its inverse is given by the integration along the fibers of π (cf. [26, Ch.II, Theorem 5.3]). Let $W_0 = E \setminus M$ and $W_1 = E$ and consider the coverings $\mathcal{W} = \{W_0, W_1\}$ and $\mathcal{W}' = \{W_0\}$ of E and $E \setminus M$. Then, $H_D^q(E, E \setminus M) = H_D^q(\mathcal{W}, \mathcal{W}')$ and we have :

Proposition 5.3. *For the trivial bundle $E = \mathbb{R}^l \times M$, Ψ_E is represented by the cocycle*

$$(0, -\psi_l) \quad \text{in } \mathcal{E}^{(l)}(\mathcal{W}, \mathcal{W}'),$$

where ψ_l is the angular form on \mathbb{R}^l .

Recall that ψ_l is given by

$$(5.2) \quad \psi_l = C_l \frac{\sum_{i=1}^l \Phi_i(x)}{\|x\|^l}, \quad \Phi_i(x) = (-1)^{i-1} x_i dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_l,$$

where C_l is a constant equal to $\frac{(k-1)!}{2\pi^k}$, if $l = 2k$, and $\frac{(2k)!}{2^l \pi^k k!}$, if $l = 2k + 1$. The important fact is that it is a closed $(l-1)$ -form and $\int_{S^{l-1}} \psi_l = 1$ for a usually oriented $(l-1)$ -sphere in $\mathbb{R}^l \setminus \{0\}$.

In the above situation, if M is orientable, the total space E is orientable. We endow them with orientations so that the orientation of the fiber of π followed by the orientation of M gives the orientation of E .

Let X be a C^∞ manifold of dimension m and $M \subset X$ a closed submanifold of dimension n . Set $l = m - n$. If we denote by $T_M X$ the normal bundle of M in X , by the tubular neighborhood theorem and excision, we have a canonical isomorphism

$$H^q(X, X \setminus M; \mathbb{Z}) \simeq H^q(T_M X, T_M X \setminus M; \mathbb{Z}).$$

Note that if X and M are orientable, the bundle $T_M X$ is orientable and thus the total space is also orientable. We endow them with orientations according to the above rule. In this case the Thom class $\Psi_M \in H^l(X, X \setminus M; \mathbb{Z})$ of M in X is defined to be the class corresponding to the Thom class of $T_M X$ under the above isomorphism for $q = l$. We also have the Thom isomorphism

$$(5.3) \quad T : H^{q-l}(M; \mathbb{Z}) \xrightarrow{\sim} H^q(X, X \setminus M; \mathbb{Z}).$$

5.2 - Relative de Rham and relative Dolbeault cohomologies

Let X be a complex manifold of dimension n . We consider the following two cases where there is a natural relation between the two cohomology theories.

(I) Noting that, for every (n, q) -form ω , $\bar{\partial}\omega = d\omega$, there is a natural morphism

$$(5.4) \quad H_{\bar{\partial}}^{n,q}(\mathcal{W}, \mathcal{W}') \longrightarrow H_D^{n+q}(\mathcal{W}, \mathcal{W}').$$

In particular, this is used to define the integration on the Čech-Dolbeault cohomology in the subsequent section.

(II) We define $\rho^q : \mathcal{E}^{(q)} \longrightarrow \mathcal{E}^{(0,q)}$ by assigning to a q -form ω its $(0, q)$ -component $\omega^{(0,q)}$. Then $\rho^{q+1}(d\omega) = \bar{\partial}(\rho^q \omega)$ and we have:

Proposition 5.4. *There is a natural morphism of complexes*

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & \mathbb{C} & \longrightarrow & \mathcal{E}^{(0)} & \xrightarrow{d} & \mathcal{E}^{(1)} & \xrightarrow{d} & \dots & \xrightarrow{d} & \mathcal{E}^{(q)} & \xrightarrow{d} & \dots \\ & & \downarrow \iota & & \downarrow \rho^0 & & \downarrow \rho^1 & & & & \downarrow \rho^q & & \\ 0 & \longrightarrow & \mathcal{O} & \longrightarrow & \mathcal{E}^{(0,0)} & \xrightarrow{\bar{\partial}} & \mathcal{E}^{(0,1)} & \xrightarrow{\bar{\partial}} & \dots & \xrightarrow{\bar{\partial}} & \mathcal{E}^{(0,q)} & \xrightarrow{\bar{\partial}} & \dots \end{array}$$

Corollary 5.5. *There is a natural morphism ρ^q from $H_D^q(X, X')$ to $H_{\bar{\partial}}^{0,q}(X, X')$ that makes the following diagram commutative:*

$$\begin{array}{ccc} H_D^q(X, X') & \xrightarrow{\rho^q} & H_{\bar{\partial}}^{0,q}(X, X') \\ \downarrow \wr & & \downarrow \wr \\ H^q(X, X'; \mathbb{C}) & \xrightarrow{\iota} & H^q(X, X'; \mathcal{O}). \end{array}$$

Recall that we have the analytic de Rham complex

$$0 \longrightarrow \mathbb{C} \xrightarrow{\iota} \mathcal{O} \xrightarrow{d} \mathcal{O}^{(1)} \xrightarrow{d} \dots \xrightarrow{d} \mathcal{O}^{(n)} \longrightarrow 0$$

and we have an isomorphism of complexes (cf. Proposition 4.11):

$$(5.5) \quad \begin{array}{ccccccc} 0 \longrightarrow & H_D^q(X, X') & \xrightarrow{\rho^q} & H_{\bar{\partial}}^{0,q}(X, X') & \xrightarrow{\partial} & \dots & \xrightarrow{\partial} & H_{\bar{\partial}}^{n,q}(X, X') & \longrightarrow & 0 \\ & \downarrow \wr & & \downarrow \wr & & & & \downarrow \wr & & \\ 0 \longrightarrow & H^q(X, X'; \mathbb{C}) & \xrightarrow{\iota} & H^q(X, X'; \mathcal{O}) & \xrightarrow{d} & \dots & \xrightarrow{d} & H^q(X, X'; \mathcal{O}^{(n)}) & \longrightarrow & 0. \end{array}$$

Although the following appears to be well-known, we give a proof for the sake of completeness.

Theorem 5.6. *If $H^q(X, X'; \mathbb{C}) = 0$ and $H^q(X, X'; \mathcal{O}^{(p)}) = 0$ for $p \geq 0$ and $q \neq q_0$, then*

$$0 \longrightarrow H^{q_0}(X, X'; \mathbb{C}) \xrightarrow{\iota} H^{q_0}(X, X'; \mathcal{O}) \xrightarrow{d} \dots \xrightarrow{d} H^{q_0}(X, X'; \mathcal{O}^{(n)}) \longrightarrow 0$$

is exact.

Proof. Let $(\mathcal{F}^{\bullet, \bullet}, d_1, d_2)$ be a double complex of flabby sheaves such that, in the following diagram, each row is exact, each column is a flabby resolution and the diagram consisting of the first and second rows is commutative (note that $d_1 \circ d_2 + d_2 \circ d_1 = 0$):

$$\begin{array}{ccccccc}
& 0 & & 0 & & 0 & & 0 \\
& \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \mathbb{C} & \xrightarrow{\iota} & \mathcal{O} & \xrightarrow{d} & \mathcal{O}^{(1)} & \xrightarrow{d} \dots \xrightarrow{d} \mathcal{O}^{(n)} \longrightarrow 0 \\
& \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \mathcal{F}^{-1,0} & \xrightarrow{d_1} & \mathcal{F}^{0,0} & \xrightarrow{d_1} & \mathcal{F}^{1,0} & \xrightarrow{d_1} \dots \xrightarrow{d_1} \mathcal{F}^{n,0} \longrightarrow 0 \\
& \downarrow d_2 & & \downarrow d_2 & & \downarrow d_2 & & \downarrow d_2 \\
0 & \longrightarrow & \mathcal{F}^{-1,1} & \xrightarrow{d_1} & \mathcal{F}^{0,1} & \xrightarrow{d_1} & \mathcal{F}^{1,1} & \xrightarrow{d_1} \dots \xrightarrow{d_1} \mathcal{F}^{n,1} \longrightarrow 0 \\
& \downarrow d_2 & & \downarrow d_2 & & \downarrow d_2 & & \downarrow d_2 \\
& \vdots & & \vdots & & \vdots & & \vdots
\end{array}$$

We have the associated double complex $(F^{\bullet,\bullet}, d_1, d_2)$, $F^{p,q} = \mathcal{F}^{p,q}(X, X')$. Denoting by F^\bullet the single complex associated with $F^{\bullet,\bullet}$, consider the first spectral sequence

$$'E_2^{p,q} = H_{d_1}^p H_{d_2}^q(F^{\bullet,\bullet}) \implies H^{p+q}(F^\bullet).$$

By assumption $H_{d_2}^q(F^{p,\bullet}) = 0$ for $p \geq -1$ and $q \neq q_0$ and $H_{d_2}^{q_0}(F^{p,\bullet}) = H^{q_0}(X, X'; \mathcal{O}^{(p)})$. Thus $H_{d_1}^p H^{q_0}(X, X'; \mathcal{O}^{(\bullet)}) \simeq H^{p+q_0}(F^\bullet)$. On the other hand, in the second spectral sequence

$$''E_2^{q,p} = H_{d_2}^q H_{d_1}^p(F^{\bullet,\bullet}) \implies H^{p+q}(F^\bullet),$$

$H_{d_1}^p(F^{\bullet,q}) = 0$, for $p \geq -1$ and $q \geq 0$, so that $H^r(F^\bullet) = 0$ for all r . \square

As an application, combining with (5.5), we have the de Rham complex for “hyperforms” (cf. (8.5) below and [14]).

6 - Cup product and integration

Let X be a complex manifold of dimension n and $\mathcal{W} = \{W_\alpha\}_{\alpha \in I}$ a covering of X .

6.1 - Cup product

We have the complex $\mathcal{E}^{(p,\bullet)}(\mathcal{W})$ as considered in Subsection 3.2. We define the “cup product”

$$(6.1) \quad \mathcal{E}^{(p,q)}(\mathcal{W}) \times \mathcal{E}^{(p',q')}(\mathcal{W}) \longrightarrow \mathcal{E}^{(p+p',q+q')}(\mathcal{W})$$

by assigning to ξ in $\mathcal{E}^{(p,q)}(\mathcal{W})$ and η in $\mathcal{E}^{(p',q')}(\mathcal{W})$ the cochain $\xi \smile \eta$ in $\mathcal{E}^{(p+p',q+q')}(\mathcal{W})$ given by

$$(\xi \smile \eta)_{\alpha_0 \dots \alpha_r} = \sum_{\nu=0}^r (-1)^{(p+q-\nu)(r-\nu)} \xi_{\alpha_0 \dots \alpha_\nu} \wedge \eta_{\alpha_{\nu+1} \dots \alpha_r}.$$

Then $\xi \smile \eta$ is bilinear in (ξ, η) and we have

$$(6.2) \quad \bar{\partial}(\xi \smile \eta) = \bar{\partial}\xi \smile \eta + (-1)^{p+q} \xi \smile \bar{\partial}\eta.$$

Thus it induces the cup product

$$(6.3) \quad H_{\bar{\partial}}^{p,q}(\mathcal{W}) \times H_{\bar{\partial}}^{p',q'}(\mathcal{W}) \longrightarrow H_{\bar{\partial}}^{p+p',q+q'}(\mathcal{W})$$

compatible, via the isomorphism of Theorem 3.6.1, with the product in the Dolbeault cohomology induced by the exterior product of forms.

If \mathcal{W}' is a subcovering of \mathcal{W} , the cup product (6.1) induces

$$\mathcal{E}^{(p,q)}(\mathcal{W}, \mathcal{W}') \times \mathcal{E}^{(p',q')}(\mathcal{W}) \longrightarrow \mathcal{E}^{(p+p',q+q')}(\mathcal{W}, \mathcal{W}'),$$

which in turn induces the cup product

$$H_{\bar{\partial}}^{p,q}(\mathcal{W}, \mathcal{W}') \times H_{\bar{\partial}}^{p',q'}(\mathcal{W}) \longrightarrow H_{\bar{\partial}}^{p+p',q+q'}(\mathcal{W}, \mathcal{W}').$$

In the case of a covering $\mathcal{V} = \{V_0, V_1\}$ with two open sets, the cup product

$$(6.4) \quad \mathcal{E}^{(p,q)}(\mathcal{V}) \times \mathcal{E}^{(p',q')}(\mathcal{V}) \longrightarrow \mathcal{E}^{(p+p',q+q')}(\mathcal{V})$$

assigns to ξ in $\mathcal{E}^{(p,q)}(\mathcal{V})$ and η in $\mathcal{E}^{(p',q')}(\mathcal{V})$ the cochain $\xi \smile \eta$ in $\mathcal{E}^{(p+p',q+q')}(\mathcal{V})$ given by

$$\begin{aligned} (\xi \smile \eta)_0 &= \xi_0 \wedge \eta_0, & (\xi \smile \eta)_1 &= \xi_1 \wedge \eta_1 \quad \text{and} \\ (\xi \smile \eta)_{01} &= (-1)^{p+q} \xi_0 \wedge \eta_{01} + \xi_{01} \wedge \eta_1. \end{aligned}$$

Suppose S is a closed set in X . Let $V_0 = X \setminus S$ and V_1 a neighborhood of S and consider the covering $\mathcal{V} = \{V_0, V_1\}$. Then we see that (6.4) induces a pairing

$$(6.5) \quad \mathcal{E}^{(p,q)}(\mathcal{V}, V_0) \times \mathcal{E}^{(p',q')}(V_1) \longrightarrow \mathcal{E}^{(p+p',q+q')}(\mathcal{V}, V_0)$$

assigning to $\xi = (\xi_1, \xi_{01})$ and η_1 the cochain $(\xi_1 \wedge \eta_1, \xi_{01} \wedge \eta_1)$. It induces the pairing

$$(6.6) \quad H_{\bar{\partial}}^{p,q}(X, X \setminus S) \times H_{\bar{\partial}}^{p',q'}(V_1) \longrightarrow H_{\bar{\partial}}^{p+p',q+q'}(X, X \setminus S).$$

More generally, let S_i be a closed set in X , $i = 1, 2$. Let $V_0^{(i)} = X \setminus S_i$ and $V_1^{(i)}$ a neighborhood of S_i and consider the covering $\mathcal{V}^{(i)} = \{V_0^{(i)}, V_1^{(i)}\}$ of X . We set $S = S_1 \cap S_2$, $V_0 = X \setminus S$ and V_1 an open neighborhood of S contained in $V_1^{(1)} \cap V_1^{(2)}$ and consider the covering $\mathcal{V} = \{V_0, V_1\}$ of X . The set V_0 is covered by two open sets $V_0^{(1)}$ and $V_0^{(2)}$. Let $\{\rho^{(1)}, \rho^{(2)}\}$ be a partition of unity subordinate to the covering. We define a paring

$$(6.7) \quad \mathcal{E}^{(p,q)}(\mathcal{V}^{(1)}, V_0^{(1)}) \times \mathcal{E}^{(p',q')}(\mathcal{V}^{(2)}, V_0^{(2)}) \longrightarrow \mathcal{E}^{(p+p',q+q')}(\mathcal{V}, V_0)$$

by

$$(\xi_1, \xi_{01}) \smile (\eta_1, \eta_{01}) = (\xi_1 \wedge \eta_1, \rho^{(1)} \xi_{01} \wedge \eta_1 + (-1)^{p+q} (\rho^{(2)} \xi_1 \wedge \eta_{01} - \bar{\partial} \rho^{(1)} \wedge \xi_{01} \wedge \eta_{01})).$$

Then we see that the equality (6.2) also holds and we have the product

$$(6.8) \quad H_{\bar{\partial}}^{p,q}(X, X \setminus S_1) \times H_{\bar{\partial}}^{p',q'}(X, X \setminus S_2) \longrightarrow H_{\bar{\partial}}^{p+p',q+q'}(X, X \setminus S).$$

It is not difficult to see that (6.8) does not depend on the choice of the partition of unity $\{\rho^{(1)}, \rho^{(2)}\}$.

In particular, if $S_2 = X$, we may set $\rho^{(1)} \equiv 1$ and $\rho^{(2)} \equiv 0$ and (6.7) reduces to (6.5).

The above may be used to define, for two pairs (X, S) and (Y, T) the product

$$H_{\bar{\partial}}^{p,q}(X, X \setminus S) \times H_{\bar{\partial}}^{p',q'}(Y, Y \setminus T) \longrightarrow H_{\bar{\partial}}^{p+p',q+q'}(X \times Y, X \times Y \setminus S \times T).$$

6.2 - Integration

Recall that there is a natural morphism $H_{\bar{\partial}}^{n,q}(\mathcal{W}) \rightarrow H_D^{n+q}(\mathcal{W})$ (cf. (5.4)). Thus the integration on $H_D^{2n}(\mathcal{W})$ carries directly on to $H_{\bar{\partial}}^{n,n}(\mathcal{W})$. We briefly recall the integration theory on the Čech-de Rham cohomology and refer to [20] and [26] for details.

Let X be a C^∞ manifold of dimension m and $\mathcal{W} = \{W_\alpha\}_{\alpha \in I}$ a covering of X . We assume that I is an ordered set such that if $W_{\alpha_0 \dots \alpha_q} \neq \emptyset$, the induced order on the subset $\{\alpha_0, \dots, \alpha_q\}$ is total. We set

$$I^{(q)} = \{(\alpha_0, \dots, \alpha_q) \in I^{q+1} \mid \alpha_0 < \dots < \alpha_q\}.$$

A *honeycomb system* adapted to \mathcal{W} is a collection $\{R_\alpha\}_{\alpha \in I}$ such that

- (1) each R_α is an m -dimensional manifold with piecewise C^∞ boundary in W_α and $X = \bigcup_\alpha R_\alpha$,

- (2) if $\alpha \neq \beta$, then $\text{Int } R_\alpha \cap \text{Int } R_\beta = \emptyset$,
- (3) if $W_{\alpha_0 \dots \alpha_q} \neq \emptyset$ and if the α_ν 's are distinct, then $R_{\alpha_0 \dots \alpha_q}$ is an $(m - q)$ -dimensional manifold with piecewise C^∞ boundary,
- (4) if the set $\{\alpha_0, \dots, \alpha_q\}$ is maximal, then $R_{\alpha_0 \dots \alpha_q}$ has no boundary.

In the above, we denote by $\text{Int } R$ the interior of a subset R in X and we set $R_{\alpha_0 \dots \alpha_q} = \bigcap_{\nu=0}^q R_{\alpha_\nu}$, which is equal to $\bigcap_{\nu=0}^q \partial R_{\alpha_\nu}$ by (2) above. Also, the set $\{\alpha_0, \dots, \alpha_q\}$ being maximal means that if $W_{\alpha, \alpha_0, \dots, \alpha_q} \neq \emptyset$, then α is in $\{\alpha_0, \dots, \alpha_q\}$.

Suppose X is oriented. Let R be an m -dimensional manifold with C^∞ boundary ∂R in X . Then R is oriented so that it has the same orientation as X . In this case, the boundary ∂R is orientable and is oriented as follows. Let p be a point in ∂R . There exist a neighborhood U of p and a C^∞ coordinate system (x_1, \dots, x_m) on U such that $R \cap U = \{x \in U \mid x_1 \leq 0\}$. We orient ∂R so that if (x_1, \dots, x_m) is a positive coordinate system on X , (x_2, \dots, x_m) is a positive coordinate system on ∂R . A manifold with piecewise C^∞ boundary is oriented similarly.

If $\{R_\alpha\}$ is a honeycomb system, we orient $R_{\alpha_0 \dots \alpha_q}$ by the following rules :

- (1) each R_α and its boundary are oriented as above,
- (2) for $(\alpha_0, \dots, \alpha_q)$ in $I^{(q)}$, $q \geq 1$, $R_{\alpha_0 \dots \alpha_q}$ is oriented as a part of $\partial R_{\alpha_0 \dots \alpha_{q-1}}$,
- (3) for $(\alpha_0, \dots, \alpha_q)$ in I^{q+1} , we set

$$R_{\alpha_0 \dots \alpha_q} = \begin{cases} \text{sgn } \rho \cdot R_{\alpha_{\rho(0)} \dots \alpha_{\rho(q)}} & \text{if } W_{\alpha_0 \dots \alpha_q} \neq \emptyset \text{ and the } \alpha_i \text{'s are distinct,} \\ \emptyset & \text{otherwise,} \end{cases}$$

where ρ is the permutation such that $\alpha_{\rho(0)} < \dots < \alpha_{\rho(q)}$.

With the above convention, we may write :

$$(6.9) \quad \partial R_{\alpha_0 \dots \alpha_q} = \sum_{\alpha \in I} R_{\alpha_0 \dots \alpha_q \alpha}.$$

Suppose moreover that X is compact, then each R_α is compact and we may define the integration

$$\mathcal{E}^{(m)}(\mathcal{W}) \xrightarrow{\int_X} \mathbb{C}$$

by setting

$$\int_X \xi = \sum_{q=0}^m \left(\sum_{(\alpha_0, \dots, \alpha_q) \in I^{(q)}} \int_{R_{\alpha_0 \dots \alpha_q}} \xi_{\alpha_0 \dots \alpha_q} \right) \quad \text{for } \xi \in \mathcal{E}^{(m)}(\mathcal{W}).$$

Then we see that it induces the integration on the cohomology

$$H_D^m(\mathcal{W}) \xrightarrow{\int_X} \mathbb{C},$$

which is compatible with the usual integration on the de Rham cohomology $H_d^m(X)$.

Now let X be a complex manifold of dimension n and \mathcal{W} a covering of X . As a real manifold, X is always orientable and we specify an orientation in the sequel. However we note that the orientation we consider is not necessarily the “usual one”. Here we say an orientation of X is usual if $(x_1, y_1, \dots, x_n, y_n)$ is a positive coordinate system when (z_1, \dots, z_n) , $z_i = x_i + \sqrt{-1}y_i$, is a coordinate system on X .

Using the natural morphism $H_{\bar{\partial}}^{n,q}(\mathcal{W}) \rightarrow H_D^{n+q}(\mathcal{W})$, if X is compact, we may define the integration on $H_{\bar{\partial}}^{n,n}(\mathcal{W})$ as the composition

$$(6.10) \quad H_{\bar{\partial}}^{n,n}(\mathcal{W}) \longrightarrow H_D^{2n}(\mathcal{W}) \xrightarrow{\int_X} \mathbb{C}.$$

Let K be a compact set in X (X may not be compact). Letting $V_0 = X \setminus K$ and V_1 a neighborhood of K , we consider the coverings $\mathcal{V} = \{V_0, V_1\}$ and $\mathcal{V}' = \{V_0\}$. Let $\{R_0, R_1\}$ be a honeycomb system adapted to \mathcal{V} . In this case we may take as R_1 a compact $2n$ -dimensional manifold with C^∞ boundary in V_1 containing K in its interior and set $R_0 = X \setminus \text{Int } R_1$. Then $R_{01} = -\partial R_1$ (cf. (6.9)) and we have the integration on $\mathcal{E}^{(n,n)}(\mathcal{V}, \mathcal{V}')$ given by, for $\xi = (\xi_1, \xi_{01})$,

$$\int_X \xi = \int_{R_1} \xi_1 + \int_{R_{01}} \xi_{01}.$$

This again induces the integration on the cohomology

$$(6.11) \quad H_{\bar{\partial}}^{n,n}(X, X \setminus K) \xrightarrow{\int_X} \mathbb{C}.$$

7 - Local duality morphism

Let X be a complex manifold of dimension n .

First, if X is compact, the bilinear pairing

$$H_{\bar{\partial}}^{p,q}(\mathcal{W}) \times H_{\bar{\partial}}^{n-p,n-q}(\mathcal{W}) \xrightarrow{\smile} H_{\bar{\partial}}^{n,n}(\mathcal{W}) \xrightarrow{\int_X} \mathbb{C}$$

given as the composition of the cup product (6.3) and the integration (6.10) induces the Kodaira-Serre duality

$$(7.1) \quad KS_X : H_{\bar{\partial}}^{p,q}(X) \simeq H_{\bar{\partial}}^{p,q}(\mathcal{W}) \xrightarrow{\smile} H_{\bar{\partial}}^{n-p,n-q}(\mathcal{W})^* \simeq H_{\bar{\partial}}^{n-p,n-q}(X)^*,$$

where, for a complex vector space V , V^* denotes its algebraic dual.

Now we consider the case where X may not be compact. Let $\mathcal{E}_c^{(p,q)}(X)$ denote the space of (p, q) -forms with compact support on X . The q -th cohomology of the complex $(\mathcal{E}_c^{(p,\bullet)}(X), \bar{\partial})$ will be denoted by $H_{\bar{\partial},c}^{p,q}(X)$. The bilinear pairing

$$\mathcal{E}_c^{(p,q)}(X) \times \mathcal{E}_c^{(n-p,n-q)}(X) \xrightarrow{\wedge} \mathcal{E}_c^{(n,n)}(X) \xrightarrow{\int_X} \mathbb{C}$$

induces the Serre morphism

$$S_X : H_{\bar{\partial}}^{p,q}(X) \longrightarrow H_{\bar{\partial},c}^{n-p,n-q}(X)^*.$$

Let K be a compact set in X . The cup product (6.6) followed by the integration (6.11) gives a bilinear pairing

$$H_{\bar{\partial}}^{p,q}(X, X \setminus K) \times H_{\bar{\partial}}^{n-p,n-q}(V_1) \xrightarrow{\smile} H_{\bar{\partial}}^{p,q}(X, X \setminus K) \xrightarrow{\int_X} \mathbb{C}.$$

Setting

$$H_{\bar{\partial}}^{n-p,n-q}[K] = \varinjlim_{V_1 \supset K} H_{\bar{\partial}}^{n-p,n-q}(V_1),$$

where V_1 runs through open neighborhoods of K , this induces a morphism

$$(7.2) \quad \bar{A}_{X,K} : H_{\bar{\partial}}^{p,q}(X, X \setminus K) \longrightarrow H_{\bar{\partial}}^{n-p,n-q}[K]^*$$

which we call the *complex analytic Alexander morphism*, or the $\bar{\partial}$ -*Alexander morphism* for short. We have the following commutative diagram :

$$\begin{array}{ccc} H_{\bar{\partial}}^{p,q}(X, X \setminus K) & \xrightarrow{j^*} & H_{\bar{\partial}}^{p,q}(X) \\ \downarrow \bar{A}_{X,K} & & \downarrow S_X \\ H_{\bar{\partial}}^{n-p,n-q}[K]^* & \xrightarrow{j_*} & H_{\bar{\partial},c}^{n-p,n-q}(X)^*, \end{array}$$

which will be extended to a commutative diagram of long exact sequences (cf. Theorem 7.4 below).

An exact sequence : Let S be a closed set in X . Since a differential form on $X \setminus S$ with compact support may naturally be thought of as a form on X with compact support, there is a natural morphism

$$i^* : H_{\bar{\partial},c}^{p,q}(X \setminus S) \longrightarrow H_{\bar{\partial},c}^{p,q}(X).$$

We also have a natural morphism

$$j^* : H_{\bar{\partial},c}^{p,q}(X) \longrightarrow H_{\bar{\partial}}^{p,q}[S] = \lim_{V_1 \supset S} H_{\bar{\partial}}^{p,q}(V_1)$$

as the composition

$$H_{\bar{\partial},c}^{p,q}(X) \longrightarrow H_{\bar{\partial}}^{p,q}(X) \longrightarrow H_{\bar{\partial}}^{p,q}[S].$$

Let K be a compact set in X . We define a morphism

$$(7.3) \quad \gamma^* : H_{\bar{\partial}}^{p,q}[K] \longrightarrow H_{\bar{\partial},c}^{p,q+1}(X \setminus K)$$

as follows. Take an element a in $H_{\bar{\partial}}^{p,q}[K]$. Then it is represented by $[\eta]$ in $H_{\bar{\partial}}^{p,q}(V_1)$ for some neighborhood V_1 of K , which may be assumed to be relatively compact. Let $V_0 = X \setminus K$ and consider the covering $\mathcal{V} = \{V_0, V_1\}$ of X . Let $\{\rho_0, \rho_1\}$ be a C^∞ partition of unity subordinate to \mathcal{V} . Then, noting that the support of $\bar{\partial}\rho_1$ is in $V_{01} = V_1 \setminus K$, we see that $\eta \wedge \bar{\partial}\rho_1$ is a $\bar{\partial}$ -closed $(p, q+1)$ -form with compact support in $X \setminus K$.

Lemma 7.1. *The class of $\eta \wedge \bar{\partial}\rho_1$ in $H_{\bar{\partial},c}^{p,q+1}(X \setminus K)$ is uniquely determined by a .*

Proof. Suppose a is also represented by $[\eta']$ in $H_{\bar{\partial}}^{p,q}(V'_1)$, $V'_1 \subset V_1$. Then there exists a $(p, q-1)$ -form ξ on V'_1 such that $\eta' - \eta = \bar{\partial}\xi$ on V'_1 . We compute

$$\eta' \wedge \bar{\partial}\rho'_1 - \eta \wedge \bar{\partial}\rho_1 = (\eta' - \eta) \wedge \bar{\partial}\rho'_1 + \eta \wedge (\bar{\partial}\rho'_1 - \bar{\partial}\rho_1).$$

The form $\xi \wedge \bar{\partial}\rho'_1$ is a (p, q) -form with compact support in $X \setminus K$ and $(\eta' - \eta) \wedge \bar{\partial}\rho'_1 = \bar{\partial}(\xi \wedge \bar{\partial}\rho'_1)$. Also the support of $\rho'_1 - \rho_1$ is in V_{01} and $\eta \wedge (\bar{\partial}\rho'_1 - \bar{\partial}\rho_1) = (-1)^{p+q}\bar{\partial}((\rho'_1 - \rho_1)\eta)$. \square

Thus we define the morphism (7.3) by $\gamma^*(a) = [\eta \wedge \bar{\partial}\rho_1]$.

Theorem 7.2. *The following sequence is exact:*

$$\cdots \longrightarrow H_{\bar{\partial},c}^{p,q}(X \setminus K) \xrightarrow{i^*} H_{\bar{\partial},c}^{p,q}(X) \xrightarrow{j^*} H_{\bar{\partial}}^{p,q}[K] \xrightarrow{\gamma^*} H_{\bar{\partial},c}^{p,q+1}(X \setminus K) \xrightarrow{i^*} \cdots$$

Proof. To show $\text{Im } i^* \subset \text{Ker } j^*$, let $[\omega_0]$ be a class in $H_{\bar{\partial},c}^{p,q}(X \setminus K)$. Take V_1 so that it avoids the support of ω_0 . Then the class $i^*[\omega_0]$ is mapped to zero by $H_{\bar{\partial},c}^{p,q}(X) \rightarrow H_{\bar{\partial}}^{p,q}(X) \rightarrow H_{\bar{\partial}}^{p,q}(V_1)$. To show $\text{Ker } j^* \subset \text{Im } i^*$, let $[\omega]$ be a class in $H_{\bar{\partial},c}^{p,q}(X)$ such that $j^*([\omega]) = 0$. Then there exist a neighborhood V_1 of K and a $(p, q-1)$ -form θ on V_1 such that $\omega = \bar{\partial}\theta$ on V_1 . The form $\rho_1\theta$ is a $(p, q-1)$ -form

on X and $\omega' = \omega - \bar{\partial}(\rho_1\theta)$ is a $\bar{\partial}$ -closed (p, q) -form with compact support in $X \setminus K$. Thus $[\omega] = i^*[\omega']$.

To show $\text{Im } j^* \subset \text{Ker } \gamma^*$, let $[\omega]$ be a class in $H_{\bar{\partial},c}^{p,q}(X)$. Then $\gamma^*j^*[\omega]$ is by definition, the class of $\omega \wedge \bar{\partial}\rho_1$. The form $\rho_0\omega$ is a (p, q) -form with compact support in $X \setminus K$ and $\omega \wedge \bar{\partial}\rho_1 = (-1)^{p+q+1}\bar{\partial}(\rho_0\omega)$. To show $\text{Ker } \gamma^* \subset \text{Im } j^*$, let a be a class in $H_{\bar{\partial}}^{p,q}[K]$ represented by (V_1, η) . If $\gamma^*a = 0$, there exists a (p, q) -form ξ on $X \setminus K$ with compact support such that $\eta \wedge \bar{\partial}\rho_1 = \bar{\partial}\xi$ on $X \setminus K$. We may think of ξ as a (p, q) -form on X and the equality holds on X , the both sides being 0 near K . The form $\rho_1\eta$ is a (p, q) -form on X and $\omega = \xi + (-1)^{p+q+1}\rho_1\eta$ is a (p, q) -form on X extending η (restricted to a neighborhood of K) and we have $\bar{\partial}\omega = \bar{\partial}\xi - \eta \wedge \bar{\partial}\rho_1 = 0$.

To show $\text{Im } \gamma^* \subset \text{Ker } i^*$, let a be a class in $H_{\bar{\partial}}^{p,q}[K]$ represented by (V_1, η) . By definition, γ^*a is the class of $\eta \wedge \bar{\partial}\rho_1$, which may be written $(-1)^{p+q}\bar{\partial}(\rho_1\eta)$ and $\rho_1\eta$ is a (p, q) -form on X with compact support. To show $\text{Ker } i^* \subset \text{Im } \gamma^*$, let $[\alpha]$ be a class in $H_{\bar{\partial},c}^{p,q+1}(X \setminus K)$. If $i^*[\alpha] = 0$, there exists a (p, q) -form β with compact support in X such that $\alpha = \bar{\partial}\beta$. Take ρ_0 so that $\rho_0 \equiv 1$ on the support of α . Then $\bar{\partial}(\rho_0\beta) = \bar{\partial}\rho_0 \wedge \beta + \rho_0\bar{\partial}\beta$, $\rho_0\beta$ is a (p, q) -form with compact support in $X \setminus K$ and $\rho_0\bar{\partial}\beta = \alpha$. Thus $\alpha = (-1)^{p+q}\beta \wedge \bar{\partial}\rho_1 + \bar{\partial}(\rho_0\beta)$. \square

Remark 7.3. The above is an expression of the following exact sequence (cf. [5], [19]) in our framework:

$$\begin{aligned} \cdots \longrightarrow H_c^q(X \setminus K; \mathcal{O}^{(p)}) &\xrightarrow{i^*} H_c^q(X; \mathcal{O}^{(p)}) \xrightarrow{j^*} \\ &H^q(K; \mathcal{O}^{(p)}) \xrightarrow{\gamma^*} H_c^{q+1}(X \setminus K; \mathcal{O}^{(p)}) \xrightarrow{i^*} \cdots \end{aligned}$$

Theorem 7.4. *In the above situation, we have a commutative diagram with exact rows:*

$$\begin{array}{ccccccc} \cdots \longrightarrow & H_{\bar{\partial}}^{p,q-1}(X \setminus K) & \xrightarrow{\delta} & H_{\bar{\partial}}^{p,q}(X, X \setminus K) & \xrightarrow{j^*} & H_{\bar{\partial}}^{p,q}(X) & \xrightarrow{i^*} H_{\bar{\partial}}^{p,q}(X \setminus K) \longrightarrow \cdots \\ & \downarrow S_{X \setminus K} & & \downarrow \bar{A} & & \downarrow S_X & \downarrow S_{X \setminus K} \\ \cdots \longrightarrow & H_{\bar{\partial},c}^{n-p,n-q+1}(X \setminus K)^* & \xrightarrow{\gamma^*} & H_{\bar{\partial}}^{n-p,n-q}[K]^* & \xrightarrow{j_*} & H_{\bar{\partial},c}^{n-p,n-q}(X)^* & \xrightarrow{i_*} H_{\bar{\partial},c}^{n-p,n-q}(X \setminus K)^* \longrightarrow \cdots \end{array}$$

where the second row is the sequence dual to the one in Theorem 7.2.

Proof. For the first rectangle, it amounts to showing that

$$\int_{X \setminus K} \theta \wedge \eta \wedge \bar{\partial}\rho_1 = - \int_{R_{01}} \theta \wedge \eta,$$

where θ is a $\bar{\partial}$ -closed $(p, q-1)$ -form on $X \setminus K$, η is a $\bar{\partial}$ -closed $(n-p, n-q)$ -form on V_1 , $\{R_0, R_1\}$ is a honeycomb system adapted to \mathcal{V} and $\{\rho_0, \rho_1\}$ is a

partition of unity subordinate to \mathcal{V} . We may assume that $\rho_1 \equiv 1$ on R_1 , thus in particular the support of $\bar{\partial}\rho_1$ is in R_0 . Since $\rho_1\eta$ is a $(n-p, n-q)$ -form on X , $\theta \wedge \rho_1\eta$ is an $(n, n-1)$ -form on $X \setminus K$. Since it is ∂ -closed, we have

$$d(\theta \wedge \rho_1\eta) = \bar{\partial}(\theta \wedge \rho_1\eta) = -\theta \wedge \eta \wedge \bar{\partial}\rho_1$$

and by the Stokes theorem

$$\int_{R_0} \theta \wedge \eta \wedge \bar{\partial}\rho_1 = - \int_{R_{01}} \theta \wedge \rho_1\eta = - \int_{R_{01}} \theta \wedge \eta.$$

For the second rectangle, it amounts to showing that

$$(7.4) \quad \int_X (\rho_1\sigma_1 - \bar{\partial}\rho_0 \wedge \sigma_{01}) \wedge \omega = \int_{R_1} \sigma_1 \wedge \omega + \int_{R_{01}} \sigma_{01} \wedge \omega,$$

where (σ_1, σ_{01}) is a cocycle representing a class in $H_{\bar{\partial}}^{p,q}(X, X \setminus K)$ and ω a $\bar{\partial}$ -closed $(n-p, n-q)$ -form on X with compact support. We take $\{\rho_0, \rho_1\}$ so that the support of ρ_1 is contained in R_1 . Then the left hand side is the integral on R_1 and is written

$$\int_{R_1} \sigma_1 \wedge \omega - \int_{R_1} (\rho_0\sigma_1 + \bar{\partial}\rho_0 \wedge \sigma_{01}) \wedge \omega.$$

The form $\rho_0\sigma_{01}$ is defined on V_1 and $\rho_0\sigma_{01} \wedge \omega$ is an $(n, n-1)$ -form defined on V_1 . We have

$$d(\rho_0\sigma_{01} \wedge \omega) = \bar{\partial}(\rho_0\sigma_{01} \wedge \omega) = (\rho_0\sigma_1 + \bar{\partial}\rho_0 \wedge \sigma_{01}) \wedge \omega$$

and we have (7.4) by the Stokes theorem.

The commutativity of the third rectangle follows directly from the definition. \square

An interesting problem would be to see when \bar{A} is an isomorphism. For this, we need to consider topological duals instead of algebraic duals and we briefly recall the theory of topological vector spaces and the Serre duality (cf. [19], [25], [32]). In the sequel, for a locally convex topological vector space \mathbf{V} , we denote by \mathbf{V}' its strong topological dual.

A Fréchet-Schwartz space, an FS space for short, is a locally convex space \mathbf{V} that can be expressed as the inverse limit $\mathbf{V} = \varprojlim \mathbf{V}_i$ of a descending sequence of Banach spaces $(\mathbf{V}_i, u_{i,i+1})$ with each $u_{i,i+1} : \mathbf{V}_{i+1} \rightarrow \mathbf{V}_i$ a compact linear map. A closed subspace \mathbf{W} of an FS space \mathbf{V} is FS. The quotient \mathbf{V}/\mathbf{W} is also FS. A dual Fréchet-Schwartz space, a DFS space for short, is a locally convex space

V that can be expressed as the direct limit $V = \varinjlim V_i$ of an ascending sequence of Banach spaces $(V_i, u_{i+1,i})$ with each $u_{i+1,i} : V_i \rightarrow V_{i+1}$ an injective compact linear map. A closed subspace W of a DFS space V is DFS. The quotient V/W is also DFS.

If $V = \varinjlim V_i$ is FS, V' is a DFS space, which may be written $V' = \varinjlim V'_i$. Also, if $V = \varinjlim V_i$ is DFS, V' is an FS space, which may be written $V' = \varinjlim V'_i$. In either case we have $(V')' = V$. Let $T : V_1 \rightarrow V_2$ be a continuous linear map of FS spaces and ${}^tT : V'_2 \rightarrow V'_1$ its transpose, which is continuous. In this situation, $\text{Im } T$ is closed if and only if $\text{Im } {}^tT$ is closed (the closed range theorem). Note that $\text{Ker } T$ and $\text{Ker } {}^tT$ are always closed.

Let

$$V_1 \xrightarrow{T} V_2 \xrightarrow{S} V_3$$

be a sequence of continuous linear maps of FS spaces such that $S \circ T = 0$. We set $H = \text{Ker } S / \text{Im } T$ and $H^D = \text{Ker } {}^tT / \text{Im } {}^tS$.

Lemma 7.5 (Serre). *In the above situation, suppose $\text{Im } T$ and $\text{Im } S$ are closed so that H is FS and H^D is DFS. In this case, H^D is isomorphic with H' .*

The space $\mathcal{E}^{(p,q)}(X)$ has a natural structure of FS space. In the sequence

$$(7.5) \quad \mathcal{E}^{(p,q-1)}(X) \xrightarrow{\bar{\partial}^{p,q-1}} \mathcal{E}^{(p,q)}(X) \xrightarrow{\bar{\partial}^{p,q}} \mathcal{E}^{(p,q+1)}(X),$$

$\text{Ker } \bar{\partial}^{p,q}$ is always closed. Thus if $\text{Im } \bar{\partial}^{p,q-1}$ is closed, $H_{\bar{\partial}}^{p,q}(X)$ has a natural structure of FS space. The strong dual $\mathcal{E}^{(p,q)}(X)'$ of $\mathcal{E}^{(p,q)}(X)$ is isomorphic with the space $\mathcal{D}_c^{(n-p,n-q)}(X)$ of the DFS space of $(n-p, n-q)$ -currents with compact support in X , the isomorphism is given by assigning to T in $\mathcal{D}_c^{(n-p,n-q)}(X)$ the linear functional

$$\omega \mapsto \int_X \omega \wedge T.$$

on $\mathcal{E}^{(p,q)}(X)$ (cf. [19], [25]). The transpose of (7.5) is isomorphic with

$$\mathcal{D}_c^{(n-p,n-q+1)}(X) \xleftarrow{(-1)^{p+q}\bar{\partial}} \mathcal{D}_c^{(n-p,n-q)}(X) \xleftarrow{(-1)^{p+q+1}\bar{\partial}} \mathcal{D}_c^{(n-p,n-q-1)}(X).$$

Thus if $\text{Im } \bar{\partial}^{p,q}$ in (7.5) is closed, $H^{n-q}(\mathcal{D}_c^{(n-p,\bullet)}(X))$ has a natural structure of DFS space. By Lemma 7.5, we have:

Theorem 7.6 (Serre). *If both $\text{Im } \bar{\partial}^{p,q-1}$ and $\text{Im } \bar{\partial}^{p,q}$ in (7.5) are closed, there is a natural isomorphism.*

$$H^{n-q}(\mathcal{D}_c^{(n-p,\bullet)}(X)) \simeq H_{\bar{\partial}}^{p,q}(X)'.$$

Note that there is a natural isomorphism $H^q(\mathcal{D}_c^{(p,\bullet)}(X)) \simeq H_{\bar{\partial},c}^{p,q}(X)$ and we may endow the latter with the DFS structure so that the isomorphism is an isomorphism as topological vector space. With this, under the assumption of the above theorem, we have the Serre duality

$$(7.6) \quad H_{\bar{\partial}}^{p,q}(X) \simeq H_{\bar{\partial},c}^{n-p,n-q}(X)'.$$

By a lemma of L. Schwartz, if $\dim H_{\bar{\partial}}^{p,q}(X)$ is finite, $\text{Im } \bar{\partial}^{p,q-1}$ is closed (cf. [19], [25]). Thus, if X is compact, (7.6) reduces to (7.1) for all p and q . In the case X is Stein, we have $H_{\bar{\partial}}^{p,q}(X) = 0$ for $p \geq 0$ and $q \geq 1$. Since $\bar{\partial}^{p,-1} = 0$, (7.6) holds for all p and q . In particular, $H_{\bar{\partial},c}^{p,q}(X) = 0$ for $p \geq 0$ and $0 \leq q \leq n-1$ and $H_{\bar{\partial},c}^{p,n}(X) \simeq H_{\bar{\partial}}^{n-p,0}(X)'$.

Theorem 7.7. *Suppose X is Stein. Let q be an integer with $q \geq 2$. Suppose that in the sequence*

$$\mathcal{E}^{(p,q-2)}(X \setminus K) \xrightarrow{\bar{\partial}^{p,q-2}} \mathcal{E}^{(p,q-1)}(X \setminus K) \xrightarrow{\bar{\partial}^{p,q-1}} \mathcal{E}^{(p,q)}(X \setminus K),$$

$\text{Im } \bar{\partial}^{p,q-2}$ and $\text{Im } \bar{\partial}^{p,q-1}$ are closed. Then $H_{\bar{\partial}}^{p,q}(X, X \setminus K)$ and $H_{\bar{\partial}}^{n-p,n-q}[K]$ admit natural structures of FS and DFS spaces, respectively, and

$$\bar{A} : H_{\bar{\partial}}^{p,q}(X, X \setminus K) \xrightarrow{\sim} H_{\bar{\partial}}^{n-p,n-q}[K]'.$$

Proof. By assumption, we have the Serre duality for X and, for $q \geq 2$,

$$\begin{aligned} \delta : H_{\bar{\partial}}^{p,q-1}(X \setminus K) &\xrightarrow{\sim} H_{\bar{\partial}}^{p,q}(X, X \setminus K) \quad \text{and} \\ \gamma^* : H_{\bar{\partial}}^{n-p,n-q}[K] &\xrightarrow{\sim} H_{\bar{\partial},c}^{n-p,n-q+1}(X \setminus K). \end{aligned}$$

Also by assumption, $H_{\bar{\partial}}^{p,q-1}(X \setminus K)$ is FS and $H_{\bar{\partial},c}^{n-p,n-q+1}(X \setminus K)$ is DFS. Thus if we endow $H_{\bar{\partial}}^{p,q}(X, X \setminus K)$ and $H_{\bar{\partial}}^{n-p,n-q}[K]$ with FS and DFS structures so that δ and γ^* become isomorphisms, we have the duality. \square

8 - Examples, applications and related topics

8.1 - A canonical Dolbeault-Čech correspondence

We consider the covering $\mathcal{W}' = \{W_i\}_{i=1}^n$ of $\mathbb{C}^n \setminus \{0\}$ given by $W_i = \{z_i \neq 0\}$. Here we put “'” as we later consider the covering $\mathcal{W} = \{W_i\}_{i=1}^{n+1}$ of \mathbb{C}^n with

$W_{n+1} = \mathbb{C}^n$ (cf. Remark 8.12.2 below). In the sequel we denote $\mathbb{C}^n \setminus \{0\}$ by $\mathbb{C}^n \setminus 0$. We set

$$\Phi(z) = dz_1 \wedge \cdots \wedge dz_n \quad \text{and} \quad \Phi_i(z) = (-1)^{i-1} z_i dz_1 \wedge \cdots \wedge \widehat{dz_i} \wedge \cdots \wedge dz_n.$$

Then on the one hand we have the Bochner-Martinelli form

$$\beta_n = C_n \frac{\sum_{i=1}^n \overline{\Phi_i(z)} \wedge \Phi(z)}{\|z\|^{2n}}, \quad C_n = (-1)^{\frac{n(n-1)}{2}} \frac{(n-1)!}{(2\pi\sqrt{-1})^n},$$

which is a $\bar{\partial}$ -closed $(n, n-1)$ -form on $\mathbb{C}^n \setminus 0$. On the other hand we have the Cauchy form in n -variables

$$\kappa_n = \left(\frac{1}{2\pi\sqrt{-1}} \right)^n \frac{\Phi(z)}{z_1 \cdots z_n},$$

which may be thought of as a cocycle c in $C^{n-1}(\mathcal{W}'; \mathcal{O}^{(n)})$ given by $c_{1\dots n} = \kappa_n$.

Theorem 8.1. *Under the isomorphism*

$$H_{\bar{\partial}}^{n,n-1}(\mathbb{C}^n \setminus 0) \simeq H^{n-1}(\mathcal{W}'; \mathcal{O}^{(n)})$$

of Corollary 3.8, the class of β_n corresponds to the class of $(-1)^{\frac{n(n-1)}{2}} \kappa_n$.

Proof. If $n = 1$, the cohomologies are the same and $\beta_1 = \kappa_1$. Thus we assume $n \geq 2$. We may think of β_n as being in $C^0(\mathcal{W}'; \mathcal{E}^{(n,n-1)}) \subset \mathcal{E}^{(n,n-1)}(\mathcal{W}')$ and κ_n in $C^{n-1}(\mathcal{W}'; \mathcal{E}^{(n,0)}) \subset \mathcal{E}^{(n,n-1)}(\mathcal{W}')$. We construct a cochain χ in $\mathcal{E}^{(n,n-2)}(\mathcal{W}') = \bigoplus_{p=0}^{n-2} C^p(\mathcal{W}'; \mathcal{E}^{(n,q)})$, $q = n - p - 2$, so that

$$\beta_n - (-1)^{\frac{n(n-1)}{2}} \kappa_n = \bar{\partial}\chi \quad \text{in } \mathcal{E}^{(n,n-1)}(\mathcal{W}').$$

Writing $\chi = \sum_{p=0}^{n-2} \chi^p$, $\chi^p \in C^p(\mathcal{W}'; \mathcal{E}^{(n,q)})$, this is expressed as (cf. (3.5))

$$(8.1) \quad \begin{cases} \beta_n = \bar{\partial}\chi^0, \\ 0 = \check{\delta}\chi^{p-1} + (-1)^p \bar{\partial}\chi^p, & 1 \leq p \leq n-2, \\ -(-1)^{\frac{n(n-1)}{2}} \kappa_n = \check{\delta}\chi^{n-2}. \end{cases}$$

Note that the condition in the middle is vacuous if $n = 2$.

Let $0 \leq p \leq n-2$ so that $0 \leq q \leq n-2$. For a $(p+1)$ -tuple of integers $I = (i_0, \dots, i_p)$ with $1 \leq i_0 < \cdots < i_p \leq n$, let $I^* = (j_0, \dots, j_q)$ denote the

complement of $\{i_0, \dots, i_p\}$ in $\{1, \dots, n\}$ with $1 \leq j_0 < \dots < j_q \leq n$. Setting $z_I = z_{i_0} \cdots z_{i_p}$, $|I^*| = j_0 + \dots + j_q$ and

$$\bar{\Phi}_{I^*}(z) = \sum_{\mu=0}^q (-1)^\mu \bar{z}_{j_\mu} d\bar{z}_{j_0} \wedge \cdots \wedge \widehat{d\bar{z}_{j_\mu}} \wedge \cdots \wedge d\bar{z}_{j_q},$$

we define a cochain χ^p by

$$\chi_I^p = (-1)^{\varepsilon_I} \frac{q! C_n}{(n-1)!} \frac{\bar{\Phi}_{I^*}(z) \wedge \Phi(z)}{z_I \|z\|^{2(q+1)}}, \quad \varepsilon_I = |I^*| + \frac{q(n+p-1)}{2}$$

and prove that it satisfies (8.1).

If we set $d\bar{z}_{I^*} = d\bar{z}_{j_0} \wedge \cdots \wedge d\bar{z}_{j_q}$, we have

$$\bar{\partial} \bar{\Phi}_{I^*}(z) = (q+1) d\bar{z}_{I^*}.$$

We also have

$$\bar{\partial} \left(\frac{1}{\|z\|^{2(q+1)}} \right) = -\frac{q+1}{\|z\|^{2(q+2)}} \sum_{i=1}^n z_i d\bar{z}_i.$$

Using these, we compute

$$\bar{\partial} \chi_I^p = (-1)^{\varepsilon_I} C_n \frac{(q+1)!}{(n-1)!} \frac{(\|z\|^2 d\bar{z}_{I^*} - \sum_{i=1}^n z_i d\bar{z}_i \wedge \bar{\Phi}_{I^*}(z)) \wedge \Phi(z)}{z_I \|z\|^{2(q+2)}}.$$

To see the numerator, note that

$$\sum_{i=1}^n z_i d\bar{z}_i \wedge \bar{\Phi}_{I^*}(z) = \sum_{\nu=0}^p z_{i_\nu} d\bar{z}_{i_\nu} \wedge \bar{\Phi}_{I^*}(z) + \sum_{\mu=0}^q |z_{j_\mu}|^2 d\bar{z}_{I^*}$$

so that

$$\|z\|^2 d\bar{z}_{I^*} - \sum_{i=1}^n z_i d\bar{z}_i \wedge \bar{\Phi}_{I^*}(z) = \sum_{\nu=0}^p z_{i_\nu} (\bar{z}_{i_\nu} d\bar{z}_{I^*} - d\bar{z}_{i_\nu} \wedge \bar{\Phi}_{I^*}(z)).$$

Thus, setting $I_\nu = (i_0, \dots, \widehat{i_\nu}, \dots, i_p)$, we have

$$(8.2) \quad \bar{\partial} \chi_I^p = (-1)^{\varepsilon_I} C_n \frac{(q+1)!}{(n-1)!} \sum_{\nu=0}^p \frac{(\bar{z}_{i_\nu} d\bar{z}_{I^*} - d\bar{z}_{i_\nu} \wedge \bar{\Phi}_{I^*}(z)) \wedge \Phi(z)}{z_{I_\nu} \|z\|^{2(q+2)}}.$$

Now we verify the three identities in (8.1) successively.

(I) First identity. If $p = 0$, then $q = n - 2$. In this case I is of the form (r) , $r = 1, \dots, n$, and $(r)^* = (1, \dots, \hat{r}, \dots, n)$. Denoting (r) by r , from (8.2), we have

$$\bar{\partial}\chi_r^0 = (-1)^{\varepsilon_r} C_n \frac{(\bar{z}_r d\bar{z}_{(r)^*} - d\bar{z}_r \wedge \bar{\Phi}_{(r)^*}(z)) \wedge \Phi(z)}{\|z\|^{2n}}.$$

On the other hand, we compute

$$\sum_{i=1}^n \overline{\Phi_i(z)} = \overline{\Phi_r(z)} + \sum_{i \neq r} \overline{\Phi_i(z)} = (-1)^{r-1} (\bar{z}_r d\bar{z}_{(r)^*} - d\bar{z}_r \wedge \bar{\Phi}_{(r)^*}(z)).$$

Noting that $\varepsilon_r - r + 1 = n(n-1) - 2(r-1)$, which is always even, we have the first identity.

(II) The second identity. This applies for $n \geq 3$. Suppose $1 \leq p \leq n-2$ so that $0 \leq q \leq n-3$. By definition

$$(\delta\chi^{p-1})_I = \sum_{\nu=0}^p (-1)^\nu \chi_{I_\nu}^{p-1}.$$

We have

$$\chi_{I_\nu}^{p-1} = (-1)^{\varepsilon_{I_\nu}} \frac{(q+1)! C_n}{(n-1)!} \frac{\bar{\Phi}_{I_\nu^*}(z) \wedge \Phi(z)}{z_{I_\nu} \|z\|^{2(q+2)}}.$$

To compute $\bar{\Phi}_{I_\nu^*}(z)$, let r_ν denote the integer with $-1 \leq r_\nu \leq q$ such that $j_{r_\nu} < i_\nu < j_{r_\nu+1}$, where we set $j_{-1} = 0$ and $j_{q+1} = n+1$. Then we have

$$\bar{\Phi}_{I_\nu^*}(z) = (-1)^{r_\nu+1} (\bar{z}_{i_\nu} d\bar{z}_{I^*} - d\bar{z}_{i_\nu} \wedge \bar{\Phi}_{I^*}(z)).$$

Thus, comparing with (8.2), it suffices to show that the parity of $\nu + \varepsilon_{I_\nu} + r_\nu + 1$ is different from that of $p + \varepsilon_I$. We have

$$\varepsilon_{I_\nu} = |I_\nu^*| + \frac{(q+1)(n+p-2)}{2} = |I^*| + i_\nu + \frac{(q+1)(n+p-2)}{2}.$$

Therefore

$$\nu + \varepsilon_{I_\nu} + r_\nu + 1 - (p + \varepsilon_I) = \nu + i_\nu + r_\nu + 1.$$

We show, by induction on ν , that $\nu + i_\nu + r_\nu$ is even. Suppose $\nu = 0$. If $i_0 < j_0$, then $i_0 = 1$ and $r_0 = -1$ so that it is even. If $i_0 > j_0$, then $i_0 = r_0 + 2$ and it is even. Suppose it is even for ν . If $i_{\nu+1} < j_{r_\nu+1}$, then $i_{\nu+1} = i_\nu + 1$ and $r_{\nu+1} = r_\nu$ so that it is even for $\nu + 1$. If $i_{\nu+1} > j_{r_\nu+1}$, then $i_{\nu+1} = i_\nu + r_{\nu+1} - r_\nu + 1$ and it is even.

(III) The third identity. If $p = n - 2$, then $q = 0$ and, for $r = 1, \dots, n$, we set $I^{(r)} = (1, \dots, \widehat{r}, \dots, n)$. Then $I^{(r)*} = (r)$ and we have

$$\begin{aligned}\chi_{I^{(r)}}^{n-2} &= (-1)^{r+\frac{n(n-1)}{2}} \left(\frac{1}{2\pi\sqrt{-1}} \right)^n \frac{\bar{z}_r \Phi(z)}{z_{I^{(r)}} \|z\|^2} \\ &= (-1)^{r+\frac{n(n-1)}{2}} \left(\frac{1}{2\pi\sqrt{-1}} \right)^n \frac{|z_r|^2 \Phi(z)}{\|z\|^2 z_1 \cdots z_n}.\end{aligned}$$

By definition

$$\check{\delta}\chi_{1\dots n}^{n-2} = \sum_{r=1}^n (-1)^{r-1} \chi_{I^{(r)}}^{n-2} = -(-1)^{\frac{n(n-1)}{2}} \kappa_n.$$

□

Remark 8.2. 1. If we set

$$\beta_n^0 = C_n \frac{\sum_{i=1}^n \overline{\Phi_i(z)}}{\|z\|^2}, \quad \kappa_n^0 = \left(\frac{1}{2\pi\sqrt{-1}} \right)^n \frac{1}{z_1 \cdots z_n},$$

under the isomorphism

$$H_{\bar{\partial}}^{0,n-1}(\mathbb{C}^n \setminus 0) \simeq H^{n-1}(\mathcal{W}'; \mathcal{O}),$$

the class of β_n^0 corresponds to the class of $(-1)^{\frac{n(n-1)}{2}} \kappa_n^0$.

2. Let V be a Stein neighborhood of 0 in \mathbb{C}^n and $\mathcal{V}' = \{V_i\}_{i=1}^n$ the covering of $V \setminus 0$ given by $V_i = V \cap W_i$. In this situation, we have a canonical isomorphism $H_{\bar{\partial}}^{n,n-1}(V \setminus 0) \simeq H^{n-1}(\mathcal{V}'; \mathcal{O}^{(n)})$, under which the the class of β_n (restricted to $V \setminus 0$) corresponds to the class of $(-1)^{\frac{n(n-1)}{2}} \kappa_n$ (restricted to \mathcal{V}'). Suppose the class of θ corresponds to the class of γ under the above isomorphism. If h is a holomorphic function on V , since $\bar{\partial}h = 0$, we see that the class of $h\theta$ corresponds to the class of $h\gamma$ (cf. (3.5), (8.1)).

In the sequel we endow $\mathbb{C}^n = \{(z_1, \dots, z_n)\}$, $z_i = x_i + \sqrt{-1}y_i$, with the usual orientation, i.e., the one where $(x_1, y_1, \dots, x_n, y_n)$ is a positive coordinate system. In the above situation set

$$R_1 = \{z \in \mathbb{C}^n \mid \|z\|^2 \leq n\varepsilon^2\}.$$

The boundary ∂R_1 is a usually oriented $(2n-1)$ -sphere S^{2n-1} . We also set

$$\Gamma = \{z \in \mathbb{C}^n \mid |z_i| = \varepsilon, i = 1, \dots, n\},$$

which is an n -cycle oriented so that $\arg z_1 \wedge \cdots \wedge \arg z_n$ is positive.

Theorem 8.3. *Let θ be a $\bar{\partial}$ -closed $(n, n-1)$ -form on $\mathbb{C}^n \setminus 0$ and γ a cocycle in $C^{n-1}(\mathcal{W}'; \mathcal{O}^{(n)})$. If the class of θ corresponds to the class of γ by the canonical isomorphism*

$$H_{\bar{\partial}}^{n, n-1}(\mathbb{C}^n \setminus 0) \simeq H^{n-1}(\mathcal{W}'; \mathcal{O}^{(n)}),$$

then

$$\int_{S^{2n-1}} \theta = (-1)^{\frac{n(n-1)}{2}} \int_{\Gamma} \gamma.$$

Proof. Recall that we have canonical isomorphisms

$$H_{\bar{\partial}}^{n, n-1}(\mathbb{C}^n \setminus 0) \xrightarrow{\sim} H_{\bar{\partial}}^{n, n-1}(\mathcal{W}') \xleftarrow{\sim} H^{n-1}(\mathcal{W}'; \mathcal{O}^{(n)}).$$

The assumption implies that there exists a cochain χ in $\mathcal{E}^{(n, n-2)}(\mathcal{W}')$ such that $\theta - \gamma = \bar{\partial}\chi$. Consider the commutative diagram

$$\begin{array}{ccccc} \mathcal{E}^{(n, n-2)}(\mathcal{W}') & \longrightarrow & \mathcal{E}^{(2n-2)}(\mathcal{W}') & \longrightarrow & \mathcal{E}^{(2n-2)}(\mathcal{W}' \cap S^{2n-1}) \\ \downarrow \bar{\partial} & & \downarrow D & & \downarrow D \\ \mathcal{E}^{(n, n-1)}(\mathcal{W}') & \longrightarrow & \mathcal{E}^{(2n-1)}(\mathcal{W}') & \longrightarrow & \mathcal{E}^{(2n-1)}(\mathcal{W}' \cap S^{2n-1}) \xrightarrow{\int_{S^{2n-1}}} \mathbb{C}, \end{array}$$

where $\mathcal{W}' \cap S^{2n-1}$ denotes the covering of S^{2n-1} consisting of the $W_i \cap S^{2n-1}$'s. For each $i = 1, \dots, n$, we set

$$Q_i = \{z \in S^{2n-1} \mid |z_i| \geq |z_j| \text{ for all } j \neq i\}.$$

Then $\{Q_i\}$ is a honeycomb system adapted to $\mathcal{W}' \cap S^{2n-1}$ and, by the Stokes formula for Čech-de Rham cochains,

$$0 = \int_{S^{2n-1}} D\chi = \sum_{p=0}^{n-1} \sum_{i_0 < \dots < i_p} \int_{Q_{i_0 \dots i_p}} (D\chi)_{i_0 \dots i_p} = \sum_{i=1}^n \int_{Q_i} \theta - \int_{Q_{1 \dots n}} \gamma.$$

Noting that $Q_{1 \dots n} = (-1)^{\frac{n(n-1)}{2}} \Gamma$, we have the theorem. \square

Note that the above is consistent with Theorem 8.1:

$$\int_{S^{2n-1}} \beta_n = 1 = \int_{\Gamma} \kappa_n.$$

8.2 - Local duality

A theorem of Martineau : The following theorem of A. Martineau [21] (see also [10], [19]) may naturally be interpreted in our framework as one of the cases where the $\bar{\partial}$ -Alexander morphism is an isomorphism with topological duals so that the duality pairing is given by the cup product followed by integration as described in Section 7.

In the below we assume that \mathbb{C}^n is oriented, but the orientation may not be the usual one.

Theorem 8.4. *Let K be a compact set in \mathbb{C}^n such that $H_{\bar{\partial}}^{p,q}[K] = 0$ for $q \geq 1$. Then for every open set $V \supset K$, $H_{\bar{\partial}}^{p,q}(V, V \setminus K)$ and $H_{\bar{\partial}}^{n-p, n-q}[K]$ admits natural structures of FS and DFS spaces, respectively, and we have :*

$$\bar{A} : H_{\bar{\partial}}^{p,q}(V, V \setminus K) \xrightarrow{\sim} H_{\bar{\partial}}^{n-p, n-q}[K]' = \begin{cases} 0 & q \neq n \\ \mathcal{O}^{(n-p)}[K]' & q = n. \end{cases}$$

The theorem is originally stated for $p = 0$ in terms of local cohomology. This is proved by applying Theorem 7.7. First, by excision we may assume that V is Stein. Thus the essential point is to prove that the hypothesis of Theorem 7.7 holds, which is done using a theorem of Malgrange (cf. [19]). Incidentally, the hypothesis $H_{\bar{\partial}}^{p,q}[K] = 0$, for $q \geq 1$, is satisfied if K is a subset of \mathbb{R}^n by the following theorem (cf. [6]) :

Theorem 8.5 (Grauert). *Any subset of \mathbb{R}^n admits a fundamental system of neighborhoods consisting of Stein open sets in \mathbb{C}^n .*

In our framework, the duality is described as follows (cf. Section 7). Let $V_0 = V \setminus K$ and V_1 a neighborhood of K in V and consider the coverings $\mathcal{V}_K = \{V_0, V_1\}$ and $\mathcal{V}'_K = \{V_0\}$ of V and V_0 . The duality pairing is given, for a cocycle (ξ_1, ξ_{01}) in $\mathcal{E}^{(p,n)}(\mathcal{V}_K, \mathcal{V}'_K)$ and a holomorphic $(n-p)$ -form η near K , by

$$(8.3) \quad \int_{R_1} \xi_1 \wedge \eta + \int_{R_{01}} \xi_{01} \wedge \eta,$$

where R_1 is a compact real $2n$ -dimensional manifold with C^∞ boundary in V_1 containing K in its interior and $R_{01} = -\partial R_1$. We may always choose a cocycle with $\xi_1 = 0$, if V is Stein.

Local residue pairing : Now we consider Theorem 8.4 in the case $K = \{0\}$ and $(p, q) = (n, n)$. We also let $V = \mathbb{C}^n$. In this paragraph we consider the usual orientation on \mathbb{C}^n . We have the exact sequence

$$\cdots \longrightarrow H_{\bar{\partial}}^{n, n-1}(\mathbb{C}^n \setminus 0) \xrightarrow{\delta} H_{\bar{\partial}}^{n, n}(\mathbb{C}^n, \mathbb{C}^n \setminus 0) \longrightarrow 0.$$

Thus every element in $H_{\bar{\partial}}^{n,n}(\mathbb{C}^n, \mathbb{C}^n \setminus 0)$ is represented by a cocycle of the form $(0, -\theta)$. Since $\mathcal{O}[K] = \mathcal{O}_{\mathbb{C}^n,0} = \mathcal{O}_n$ in this case, the duality in Theorem 8.4 is induced by the pairing

$$H_{\bar{\partial}}^{n,n}(\mathbb{C}^n, \mathbb{C}^n \setminus 0) \times \mathcal{O}_n \xrightarrow{\int} \mathbb{C}$$

given by

$$((0, -\theta), h) \mapsto - \int_{R_{01}} h\theta = \int_{S^{2n-1}} h\theta.$$

In the above, h is a holomorphic function in a neighborhood V of 0 in \mathbb{C}^n . We may take as R_1 a $2n$ -ball around 0 in V so that $R_{01} = -\partial R_1 = -S^{2n-1}$ with S^{2n-1} a usually oriented $(2n-1)$ -sphere. Thus if θ corresponds to γ , the above integral is equal to

$$(-1)^{\frac{n(n-1)}{2}} \int_{\Gamma} h\gamma$$

(cf. Remark 8.2.2 and Theorem 8.3). In particular, if $\theta = \beta_n$ the pairing is given by

$$(8.4) \quad \int_{S^{2n-1}} h\beta_n = \int_{\Gamma} h\kappa_n = \left(\frac{1}{2\pi\sqrt{-1}} \right)^n \int_{\Gamma} \frac{hdz_1 \wedge \cdots \wedge dz_n}{z_1 \cdots z_n} = h(0).$$

Likewise in the case $(p, q) = (0, n)$, the duality in Theorem 8.4 is induced by the pairing

$$H_{\bar{\partial}}^{0,n}(\mathbb{C}^n, \mathbb{C}^n \setminus 0) \times \mathcal{O}_{\mathbb{C}^n,0}^{(n)} \xrightarrow{\int} \mathbb{C}$$

given by

$$((0, -\theta), \eta) \mapsto - \int_{R_{01}} \theta \wedge \eta = \int_{S^{2n-1}} \theta \wedge \eta.$$

8.3 - Sato hyperfunctions

Sato hyperfunctions are defined in terms of relative cohomology with coefficients in the sheaf of holomorphic functions and the theory is developed in the language of derived functors (cf. [23], [24]). The use of relative Dolbeault cohomology via the relative Dolbeault theorem (Theorems 2.3, 4.8) provides us with another way of treating the theory. This approach gives simple and explicit expressions of hyperfunctions and some fundamental operations on them and leads to a number of new results. These are discussed in detail in [14], see also [13] and [30]. Here we pick up some of the essentials of the contents therein. In general the theory of hyperfunctions may be developed on an arbitrary real analytic manifold and it involves various orientation sheaves. However for simplicity, here we consider hyperfunctions on open sets in \mathbb{R}^n fixing various orientations.

Hyperfunctions and hyperforms: We consider the standard inclusion $\mathbb{R}^n \subset \mathbb{C}^n$, i.e., if (z_1, \dots, z_n) , $z_i = x_i + \sqrt{-1}y_i$, is a coordinate system on \mathbb{C}^n , then \mathbb{R}^n is given by $y_i = 0$, $i = 1, \dots, n$. We orient \mathbb{R}^n and \mathbb{C}^n so that (x_1, \dots, x_n) and $(y_1, \dots, y_n, x_1, \dots, x_n)$ are positive coordinate systems. Thus (y_1, \dots, y_n) is a positive coordinate system in the normal direction. Note that the difference between this orientation of \mathbb{C}^n and the usual one, where $(x_1, y_1, \dots, x_n, y_n)$ is positive, is a sign of $(-1)^{\frac{n(n+1)}{2}}$.

With these, for an open set U in \mathbb{R}^n , the space of hyperfunctions on U is given by

$$\mathcal{B}(U) = H_U^n(V; \mathcal{O}),$$

where V is an open set in \mathbb{C}^n containing U as a closed set and \mathcal{O} the sheaf of holomorphic functions on \mathbb{C}^n . We call such a V a complex neighborhood of U . Note that, by excision, the definition does not depend on the choice of the complex neighborhood V . By the relative Dolbeault theorem (cf. Theorems 2.3 and 4.8), there is a canonical isomorphism:

$$\mathcal{B}(U) \simeq H_{\bar{\partial}}^{0,n}(V, V \setminus U).$$

More generally we introduce the following:

Definition 8.6. The space of p -hyperforms on U is defined by

$$\mathcal{B}^{(p)}(U) = H_{\bar{\partial}}^{p,n}(V, V \setminus U).$$

Note that the definition does not depend on the choice of V by excision (cf. Proposition 4.6). Denoting by $\mathcal{O}^{(p)}$ the sheaf of holomorphic p -forms on \mathbb{C}^n , we have a canonical isomorphism (cf. Theorems 2.3 and 4.8):

$$H_{\bar{\partial}}^{p,n}(V, V \setminus U) \simeq H_U^n(V; \mathcal{O}^{(p)})$$

so that $\mathcal{B}^{(0)}(U)$ is canonically isomorphic with $\mathcal{B}(U)$. Hyperforms are essentially equal to what have conventionally been referred to as differential forms with coefficients in hyperfunctions.

Remark 8.7. In the above we implicitly used the fact that \mathbb{R}^n is “purely n -codimensional” in \mathbb{C}^n with respect to $\mathcal{O}_{\mathbb{C}^n}^{(p)}$ and $\mathbb{Z}_{\mathbb{C}^n}$ (cf. [17]). For the latter, this can be seen from the Alexander isomorphism or the Thom isomorphism (cf. [26], [27], see also Subsection 5.1).

Expression of hyperforms: Let U and V be as above. Letting $V_0 = V \setminus U$ and V_1 a neighborhood of U in V , we consider the open coverings $\mathcal{V} = \{V_0, V_1\}$ and $\mathcal{V}' = \{V_0\}$ of V and $V \setminus U$. Then $\mathcal{B}^{(p)}(U) = H_{\bar{\partial}}^{p,n}(V, V \setminus U) = H_{\bar{\partial}}^{p,n}(\mathcal{V}, \mathcal{V}')$

and a p -hyperform is represented by a pair (ξ_1, ξ_{01}) with ξ_1 a (p, n) -form on V_1 , which is automatically $\bar{\partial}$ -closed, and ξ_{01} a $(p, n-1)$ -form on V_{01} such that $\xi_1 = \bar{\partial}\xi_{01}$ on V_{01} . We have the exact sequence (cf. (4.2), (4.4)):

$$H_{\bar{\partial}}^{p, n-1}(V) \longrightarrow H_{\bar{\partial}}^{p, n-1}(V \setminus U) \xrightarrow{\delta} \mathcal{B}^{(p)}(U) \xrightarrow{j^*} H_{\bar{\partial}}^{p, n}(V).$$

By Theorem 8.5, we may take as V a Stein open set and, if we do this, we have $H_{\bar{\partial}}^{p, n}(V) \simeq H^n(V; \mathcal{O}^{(p)}) = 0$. Thus δ is surjective and every p -hyperform is represented by a cocycle of the form $(0, -\theta)$ with θ a $\bar{\partial}$ -closed $(p, n-1)$ -form on $V \setminus U$.

In the case $n > 1$, $H_{\bar{\partial}}^{p, n-1}(V) \simeq H^{n-1}(V; \mathcal{O}^{(p)}) = 0$ and δ is an isomorphism:

$$H_{\bar{\partial}}^{p, n-1}(V \setminus U) \simeq \mathcal{B}^{(p)}(U), \quad [\theta] \leftrightarrow [(0, -\theta)].$$

In the case $n = 1$, as $H_{\bar{\partial}}^{p, 0}(V \setminus U) = H^0(V \setminus U; \mathcal{O}^{(p)})$ and $H_{\bar{\partial}}^{p, 0}(V) = H^0(V; \mathcal{O}^{(p)})$, $p = 0, 1$, we have the isomorphism

$$H^0(V \setminus U; \mathcal{O}^{(p)})/H^0(V; \mathcal{O}^{(p)}) \simeq \mathcal{B}^{(p)}(U), \quad [\omega] \leftrightarrow [(0, -\omega)].$$

In particular, for $p = 0$, the left hand side is the original expression of hyperfunctions by Sato in the one-dimensional case and the right hand side is the expression in terms of relative Dolbeault cohomology.

Remark 8.8. Although a hyperform may be represented by a single differential form in most of the cases, it is important to keep in mind that it is represented by a pair (ξ_1, ξ_{01}) in general.

Now we describe some of the operations on hyperforms.

Multiplication by real analytic functions: Let $\mathcal{A}(U)$ denote the space of real analytic functions on U . We define the multiplication

$$\mathcal{A}(U) \times \mathcal{B}^{(p)}(U) \longrightarrow \mathcal{B}^{(p)}(U)$$

by assigning to $(f, [\xi])$ the class of $(\tilde{f}\xi_1, \tilde{f}\xi_{01})$ with \tilde{f} a holomorphic extension of f .

Partial derivatives: We define the partial derivative

$$\frac{\partial}{\partial x_i} : \mathcal{B}(U) \longrightarrow \mathcal{B}(U)$$

as follows. Let (ξ_1, ξ_{01}) represent a hyperfunction on U . We write $\xi_1 = f d\bar{z}_1 \wedge \cdots \wedge d\bar{z}_n$ and $\xi_{01} = \sum_{j=1}^n g_j d\bar{z}_1 \wedge \cdots \wedge \widehat{d\bar{z}_j} \wedge \cdots \wedge d\bar{z}_n$. Then $\frac{\partial}{\partial x_i}[\xi]$ is represented by the cocycle

$$\left(\frac{\partial f}{\partial z_i} d\bar{z}_1 \wedge \cdots \wedge d\bar{z}_n, \sum_{j=1}^n \frac{\partial g_j}{\partial z_i} d\bar{z}_1 \wedge \cdots \wedge \widehat{d\bar{z}_j} \wedge \cdots \wedge d\bar{z}_n \right).$$

Thus for a differential operator $P(x, D)$, $P(x, D) : \mathcal{B}(U) \rightarrow \mathcal{B}(U)$ is well-defined.

Restriction: Let U' be an open subset of U . Take a complex neighborhood V' of U' and a neighborhood V'_1 of U' in V' so that $V' \subset V$ and $V'_1 \subset V_1$. Then the restriction $\mathcal{B}^{(p)}(U) \rightarrow \mathcal{B}^{(p)}(U')$ is defined by assigning to the class of (ξ_1, ξ_{01}) the class of $(\xi_1|_{V'_1}, \xi_{01}|_{V'_{01}})$, $V'_{01} = V'_1 \setminus U'$.

Differential: We define the differential

$$d : \mathcal{B}^{(p)}(U) \longrightarrow \mathcal{B}^{(p+1)}(U).$$

by assigning to the class of (ξ_1, ξ_{01}) the class of $(-1)^n(\partial\xi_1, -\partial\xi_{01})$. From Theorem 5.6, we have the exact sequence (de Rham complex for hyperforms, cf. Remark 8.7):

$$(8.5) \quad 0 \longrightarrow \mathbb{C}(U) \longrightarrow \mathcal{B}(U) \xrightarrow{d} \mathcal{B}^{(1)}(U) \xrightarrow{d} \cdots \xrightarrow{d} \mathcal{B}^{(n)}(U) \longrightarrow 0.$$

We come back to the first part below.

Integration: Let K be a compact set in U . We define the space $\mathcal{B}_K^{(p)}(U)$ of p -hyperforms on U with support in K as the kernel of the restriction $\mathcal{B}^{(p)}(U) \rightarrow \mathcal{B}^{(p)}(U \setminus K)$. Then we have:

Proposition 8.9. *For any complex neighborhood V of U , there is a canonical isomorphism*

$$\mathcal{B}_K^{(p)}(U) \simeq H_{\bar{\partial}}^{p,n}(V, V \setminus K).$$

Proof. Applying Proposition 4.2 for the triple $(V, V \setminus K, V \setminus U)$, we have the exact sequence

$$H_{\bar{\partial}}^{p,n-1}(V \setminus K, V \setminus U) \xrightarrow{\delta} H_{\bar{\partial}}^{p,n}(V, V \setminus K) \xrightarrow{j^*} H_{\bar{\partial}}^{p,n}(V, V \setminus U) \xrightarrow{i^*} H_{\bar{\partial}}^{p,n}(V \setminus K, V \setminus U).$$

By definition, $H_{\bar{\partial}}^{p,n}(V, V \setminus U) = \mathcal{B}^{(p)}(U)$. Since $V \setminus K$ is a complex neighborhood of $U \setminus K$ and $(V \setminus K) \setminus (U \setminus K) = V \setminus U$, $H_{\bar{\partial}}^{p,n}(V \setminus K, V \setminus U) = \mathcal{B}^{(p)}(U \setminus K)$.

On the other hand, $H_{\bar{\partial}}^{p,n-1}(V \setminus K, V \setminus U) = 0$ (cf. Remark 8.7). \square

By the above proposition, we may define the integration on $\mathcal{B}_K^{(n)}(U)$ by directly applying (6.11), which we recall for the sake of completeness. Let V be a complex neighborhood of U and consider the coverings $\mathcal{V}_K = \{V_0, V_1\}$ and $\mathcal{V}'_K = \{V_0\}$, with $V_0 = V \setminus K$ and V_1 a neighborhood of K in V . Then we have a canonical identification $\mathcal{B}_K^{(p)}(U) = H_{\bar{\partial}}^{p,n}(\mathcal{V}_K, \mathcal{V}'_K)$. Let R_1 be a compact real $2n$ -dimensional manifold with C^∞ boundary in V_1 containing K in its interior and set $R_{01} = -\partial R_1$. Then the integration

$$\mathcal{B}_K^{(n)}(U) \xrightarrow{\int_U} \mathbb{C}$$

is given as follows. Noting that $u \in \mathcal{B}_K^{(n)}(U) = H_{\bar{\partial}}(\mathcal{V}_K, \mathcal{V}'_K)$ is represented by

$$\xi = (\xi_1, \xi_{01}) \in \mathcal{E}^{(n,n)}(\mathcal{V}_K, \mathcal{V}'_K) = \mathcal{E}^{(n,n)}(V_1) \oplus \mathcal{E}^{(n,n-1)}(V_{01}),$$

we have

$$\int_U u = \int_{R_1} \xi_1 + \int_{R_{01}} \xi_{01}.$$

Duality: By Theorem 8.4 we have

$$(8.6) \quad \mathcal{B}_K^{(p)}(U) = H_{\bar{\partial}}^{p,n}(V, V \setminus K) \simeq \mathcal{O}^{(n-p)}[K]' = \mathcal{A}^{(n-p)}[K]',$$

where $\mathcal{A}^{(n-p)}$ denotes the sheaf of germs of real analytic $(n-p)$ -forms on \mathbb{R}^n and

$$\mathcal{A}^{(n-p)}[K] = \varinjlim \mathcal{A}^{(n-p)}(U_1),$$

the direct limit over the set of neighborhoods U_1 of K in U . Recall that the pairing is induced by (8.3).

δ -function and δ -form: We consider the case $K = \{0\} \subset \mathbb{R}^n$. Recall the expressions $\mathcal{B}_{\{0\}}(\mathbb{R}^n) = H_{\bar{\partial}}^{0,n}(\mathbb{C}^n, \mathbb{C}^n \setminus \{0\})$ and $\mathcal{B}_{\{0\}}^{(n)}(\mathbb{R}^n) = H_{\bar{\partial}}^{n,n}(\mathbb{C}^n, \mathbb{C}^n \setminus \{0\})$.

Definition 8.10. The δ -function is a hyperfunction in $\mathcal{B}_{\{0\}}(\mathbb{R}^n)$ represented by

$$(0, -(-1)^{\frac{n(n+1)}{2}} \beta_n^0),$$

where β_n^0 is as defined in Remark 8.2.1.

The isomorphism (8.6) reads in this case :

$$\mathcal{B}_0(\mathbb{R}^n) \simeq (\mathcal{A}_0^{(n)})',$$

where $\mathcal{A}_0^{(n)}$ denotes the stalk of $\mathcal{A}^{(n)}$ at 0. For a representative $\omega = h(x)\Phi(x)$ of an element in $\mathcal{A}_0^{(n)}$, $h(z)\Phi(z)$ is its complex representative. Let R_1 be a small $2n$ -ball around 0 in \mathbb{C}^n so that $R_{01} = -\partial R_1 = -(-1)^{\frac{n(n+1)}{2}} S^{2n-1}$ with S^{2n-1} a usually oriented $(2n-1)$ -sphere. Then the δ -function is the hyperfunction that assigns to a representative $\omega = h(x)\Phi(x)$ the value (cf. (8.4))

$$-(-1)^{\frac{n(n+1)}{2}} \int_{R_{01}} h(z)\beta_n = \int_{S^{2n-1}} h(z)\beta_n = h(0).$$

Definition 8.11. The δ -form is a hyperform in $\mathcal{B}_{\{0\}}^{(n)}(\mathbb{R}^n)$ represented by

$$(0, -(-1)^{\frac{n(n+1)}{2}} \beta_n).$$

Recall the isomorphism (8.6), which reads in this case :

$$\mathcal{B}_0^{(n)}(\mathbb{R}^n) \simeq (\mathcal{A}_0)'. \quad (8.6)$$

For a representative $h(x)$ of an element in \mathcal{A}_0 , $h(z)$ is its complex representative. Let R_1 be as above. Then the δ -form is the hyperform that assigns to a representative $h(x)$ the value

$$-(-1)^{\frac{n(n+1)}{2}} \int_{R_{01}} h(z)\beta_n = \int_{S^{2n-1}} h(z)\beta_n = h(0).$$

Remark 8.12. 1. If we orient \mathbb{C}^n so that the usual coordinate system $(x_1, y_1, \dots, x_n, y_n)$ is positive, the delta function $\delta(x)$ is represented by $(0, -\beta_n^0)$. Also, the delta form is represented by $(0, -\beta_n)$. Incidentally, it has the same expression as the Thom class of the trivial complex vector bundle of rank n (cf. [26, Ch.III, Remark 4.6]).

2. Set $W_i = \{z_i \neq 0\}$, $i = 1, \dots, n$, and $W_{n+1} = \mathbb{C}^n$ and consider the coverings $\mathcal{W} = \{W_i\}_{i=1}^{n+1}$ and $\mathcal{W}' = \{W_i\}_{i=1}^n$ of \mathbb{C}^n and $\mathbb{C}^n \setminus 0$. We have the natural isomorphisms

$$\mathcal{B}_{\{0\}}(\mathbb{R}^n) \simeq H_{\bar{\partial}}^{0, n-1}(\mathbb{C}^n \setminus 0) \simeq H^{n-1}(\mathcal{W}'; \mathcal{O}) \simeq H^n(\mathcal{W}, \mathcal{W}'; \mathcal{O}).$$

As noted in Remark 8.2.1, under the middle isomorphism above, the class of β_n^0 corresponds to the class of $(-1)^{\frac{n(n-1)}{2}} \kappa_n^0$. If we choose the usual orientation on \mathbb{C}^n , the class corresponding to $[\kappa_n^0]$ in $H^n(\mathcal{W}, \mathcal{W}'; \mathcal{O})$ is the traditional representation of the δ -function (cf. (8.4)).

1 as a hyperfunction: We examine the map $\mathbb{C}(U) \rightarrow \mathcal{B}(U)$ in (8.5). Let \mathcal{V} and \mathcal{V}' be as before. Then it is given by $\rho^n : H_D^n(\mathcal{V}, \mathcal{V}') \rightarrow H_{\bar{\partial}}^{0,n}(\mathcal{V}, \mathcal{V}')$, which is induced by $(\omega_1, \omega_{01}) \mapsto (\omega_1^{(0,n)}, \omega_{01}^{(0,n-1)})$ (cf. Corollary 5.5). For simplicity we assume that U is connected. Then we have the commutative diagram:

$$\begin{array}{ccc} \mathbb{C} = H^0(U; \mathbb{C}) & \xrightarrow{\sim} & H^n(V, V \setminus U; \mathbb{C}) \xrightarrow{\iota} H^n(V, V \setminus U; \mathcal{O}) = \mathcal{B}(U) \\ & & \downarrow \wr \qquad \qquad \qquad \downarrow \wr \\ & & H_D^n(\mathcal{V}, \mathcal{V}') \xrightarrow{\rho^n} H_{\bar{\partial}}^{0,n}(\mathcal{V}, \mathcal{V}'), \end{array}$$

where T denotes the Thom isomorphism, which sends $1 \in \mathbb{C}$ to the Thom class $\Psi_U \in H^n(V, V \setminus U; \mathbb{C})$ (cf. (5.3)). If Ψ_U is represented by (ν_1, ν_{01}) in $H_D^n(\mathcal{V}, \mathcal{V}')$, as a hyperfunction, 1 is represented by $\rho^n(\nu_1, \nu_{01}) = (\nu_1^{(0,n)}, \nu_{01}^{(0,n-1)})$. In particular, we may set $(\nu_1, \nu_{01}) = (0, -\psi_n(y))$, where $\psi_n(y)$ is the angular form on \mathbb{R}_y^n (cf. Proposition 5.3). Thus as a hyperfunction, 1 is represented by $(0, -\psi_n^{(0,n-1)})$. Noting that $y_i = 1/(2\sqrt{-1})(z_i - \bar{z}_i)$, we compute

$$\psi_n^{(0,n-1)} = (\sqrt{-1})^n C_n \frac{\sum_{i=1}^n (-1)^i (z_i - \bar{z}_i) d\bar{z}_1 \wedge \cdots \wedge \widehat{d\bar{z}_i} \wedge \cdots \wedge d\bar{z}_n}{\|z - \bar{z}\|^n}.$$

In particular, if $n = 1$,

$$\psi_1^{(0,0)} = \frac{1}{2} \frac{y}{|y|}.$$

Embedding of real analytic forms: Let U and V be as above. We define a morphism

$$\mathcal{A}^{(p)}(U) \longrightarrow \mathcal{B}^{(p)}(U) = H_{\bar{\partial}}^{p,n}(V, V \setminus U)$$

by assigning to an element $\omega(x)$ in $\mathcal{A}^{(p)}(U)$ the class of

$$(\nu_1^{(0,n)} \wedge \omega(z), \nu_{01}^{(0,n-1)} \wedge \omega(z)),$$

where (ν_1, ν_{01}) is a representative of the Thom class as above and $\omega(z)$ the complexification of $\omega(x)$. Note that $(\nu^{(0,n)} \wedge \omega, \nu_{01}^{(0,n-1)} \wedge \omega)$ is a cocycle as ω is holomorphic. This induces an embedding $\iota^{(p)} : \mathcal{A}^{(p)}(U) \hookrightarrow \mathcal{B}^{(p)}(U)$ compatible with the differentials d of $\mathcal{A}^{(\bullet)}(U)$ and $\mathcal{B}^{(\bullet)}(U)$. In particular, if $p = 0$, we have the embedding $\mathcal{A}(U) \hookrightarrow \mathcal{B}(U)$, which is given by $f \mapsto (\tilde{f}\nu^{(0,n)}, \tilde{f}\nu_{01}^{(0,n-1)})$ with \tilde{f} the complexification of f .

8.4 - Some others

We may develop the theory of Atiyah classes in the context of Čech-Dolbeault cohomology, which is conveniently used to define their localizations in the

relative Dolbeault cohomology. In particular we have the $\bar{\partial}$ -Thom class of a holomorphic vector bundle, see [1] and [28] for details.

The above idea and techniques may further be pushed forward to develop the theory of Čech-Bott-Chern cohomology. In particular the relative Bott-Chern cohomology which arise naturally in this context is used for the localization of Bott-Chern classes of vector bundles admitting a Hermitian connection compatible with an action of a distribution. For details and the relation with the relative Dolbeault cohomology theory, we refer to [4].

We refer to [2] for another application of the relative Dolbeault cohomology, namely to the study of Hodge structures under blowing-up. We may equally use the complex of currents, instead of that of differential forms, to define the relative Dolbeault cohomology. One of the advantages of this is that the push-forward morphism is available, see [31] for details and applications in the context of [2].

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