Riv. Mat. Univ. Parma, Vol. **13** (2022), 297-305

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Cup and Massey products on the cohomology of compact almost complex manifolds

Abstract. The cohomology of any compact almost complex manifold carries bidegree decompositions induced by a Frölicher-type spectral sequence. In this note we give some restrictions on the possible decompositions on a given manifold and study how cup and Massey products behave with respect to such decompositions.

Keywords. Almost complex manifolds, Frölicher spectral sequence, Massey products.

Mathematics Subject Classification: 32Q60, 53C15, 32Q55.

1 - Introduction

The complex de Rham algebra $\mathcal{A}^*(M) := \mathcal{A}^*_{\mathrm{dR}}(M) \otimes \mathbb{C}$ of any almost complex manifold M decomposes into bidegrees

$$\mathcal{A}^n(M) = \bigoplus_{p+q=n} \mathcal{A}^{p,q}$$

and its exterior differential decomposes as $d = \bar{\mu} + \bar{\partial} + \partial + \mu$ where $\bar{\mu}$ has bidegree $(-1, 2)$ and $\bar{\partial}$ has bidegree $(0, 1)$. The components ∂ and μ are complex conjugate to $\bar{\partial}$ and $\bar{\mu}$ respectively. This decomposition is induced by the eigenspace decomposition defined by the almost complex structure acting on

Received: November 19, 2021; accepted: January 25, 2022.

This work was partially funded by the Spanish State Research Agency (María de Maeztu Program CEX2020-001084-M and I+D+i project PID2020-117971GB-C22/MCIN/AEI/10.13039/501100011033) as well as by the French National Research Agency (ANR-20-CE40-0016). I also thank the Serra Húnter Program.

the complexified tangent bundle of the manifold, and so strongly depends on the almost complex structure. The Hodge-type filtration

$$F^p\mathcal{A}^n(M) := \mathrm{Ker}(\bar{\mu}) \cap \mathcal{A}^{p,n-p} \oplus \bigoplus_{q>p} \mathcal{A}^{q,n-q}$$

introduced in [**CW21**] is compatible with the exterior differential and so it makes sense to consider its associated spectral sequence $E_r^{*,*}(M)$. For a compact complex manifold, for which $\bar{\mu} \equiv 0$, one recovers the usual Frölicher spectral sequence and the vector spaces $E_r^{p,q}(M)$ are finite-dimensional for all $r \geq 1$ and all p, q. As shown in [**CPS21**], some of the vector spaces $E_1^{p,q}(M)$ are infinite-dimensional in the maximally non-integrable case. Still, for an arbitrary compact almost complex manifold, convergence ensures that there is a last page $E_\infty(M) = E_s(M)$ for some finite $s \geq 1$ which is finite-dimensional and there are isomorphisms

$$E_\infty^{p,q}(M) \cong Gr_F^p H_{\mathrm{dR}}^{p+q}(M; \mathbb{C}),$$

where $Gr_F^p = F^p/F^{p+1}$ denotes the graded-p piece of the Hodge filtration F induced on complex de Rham cohomology. In particular, one may choose filtered isomorphisms

$$H_{\mathrm{dR}}^n(M; \mathbb{C}) \cong \bigoplus_{p+q=n} E_\infty^{p,q}(M)$$

giving a refinement of de Rham cohomology. Note that, while such refinement is not functorial, the numbers

$$h_r^{p,q} := \dim E_r^{p,q}(M)$$

are almost complex invariants. We deduce some inequalities concerning the numbers $h_r^{p,q}$, for $r \geq 1$, and analyze the behavior of the multiplicative structure on $H_{\mathrm{dR}}^*(M; \mathbb{C})$ with respect to the above bidegree decompositions. Stronger and more concrete results appear in [**CW22**] in the case of compact almost complex 4-manifolds, for which the E_2-page gives "Hodge-de Rham numbers" with very special properties.

2 - A Hodge-de Rham inequality

In [**PU18**] it is shown that for any compact complex manifold M there is an injective map

$$H_{\mathrm{dR}}^1(M; \mathbb{C}) \longrightarrow E_1^{0,1}(M) \oplus \overline{E_1^{0,1}(M)}$$

and in particular one has the inequality $b^1 \leq 2h^{0,1}_{\bar{\partial}}$, where $b^1 := H^1(M;\mathbb{C})$ is the first Betti number and $h^{0,1}_{\bar{\partial}} = \dim E_1^{0,1}(M) = \dim H^{0,1}_{\bar{\partial}}(M)$ is the dimension of the Dolbeault cohomology vector space in bidegree $(0,1)$. In this section we prove a generalization of this result which is valid for compact almost complex manifolds and carries through any page of the Frölicher-type spectral sequence.

We will be interested in the vector spaces $E_r^{0,1}(M)$ for any $r \geq 1$. General formulae for the various stages of the spectral sequence are given in the Appendix of [**CW21**]. In bidegree $(0,1)$, such formulae are quite simple. Let

$$Z_1^{0,1}(M) := \{\alpha_0 \in \mathcal{A}^{0,1}; \bar{\partial}\alpha_0 = \bar{\mu}\alpha_1 \text{ for some } \alpha_1 \in \mathcal{A}^{1,0}\}.$$
$$Z_2^{0,1}(M) := \{\alpha_0 \in \mathcal{A}^{0,1}; \bar{\partial}\alpha_0 = \bar{\mu}\alpha_1, \partial\alpha_0 = \bar{\partial}\alpha_1 \text{ for some } \alpha_1 \in \mathcal{A}^{1,0}\}.$$
$$Z_3^{0,1}(M) := \{\alpha_0 \in \mathcal{A}^{0,1}; \bar{\partial}\alpha_0 = \bar{\mu}\alpha_1, \partial\alpha_0 = \bar{\partial}\alpha_1, \mu\alpha_0 = \partial\alpha_1 \text{ for } \alpha_1 \in \mathcal{A}^{1,0}\}.$$

We have obvious inclusions $Z_3^{0,1}(M) \subseteq Z_2^{0,1}(M) \subseteq Z_1^{0,1}(M)$. The relations

$$\begin{aligned} \mu^2 &= 0 \\ \mu\partial + \partial\mu &= 0 \\ \mu\bar{\partial} + \bar{\partial}\mu + \partial^2 &= 0 \\ \mu\bar{\mu} + \partial\bar{\partial} + \bar{\partial}\partial + \bar{\mu}\mu &= 0 \\ \bar{\mu}\partial + \partial\bar{\mu} + \bar{\partial}^2 &= 0 \\ \bar{\mu}\bar{\partial} + \bar{\partial}\bar{\mu} &= 0 \\ \bar{\mu}^2 &= 0 \end{aligned}$$

arising from $d^2 = 0$ ensure that $\bar{\partial}(\mathcal{A}^{0,0}) \subseteq Z_3^{0,1}(M)$. For all $1 \leq r \leq 3$ we have

$$E_r^{0,1}(M) \cong Z_r^{0,1}(M)/\bar{\partial}(\mathcal{A}^{0,0}).$$

Also, we have $E_3^{0,1}(M) = E_\infty^{0,1}(M)$ and there are injections

$$E_3^{0,1}(M) \hookrightarrow E_2^{0,1}(M) \hookrightarrow E_1^{0,1}(M).$$

Proposition 2.1. *For any compact almost complex manifold M and any $r \geq 1$ there is a well-defined and injective map*

$$\mathcal{F} : H^1_{\mathrm{dR}}(M;\mathbb{C}) \longrightarrow E_r^{0,1}(M) \oplus \overline{E_r^{0,1}(M)}.$$

Proof. Let $\alpha \in \mathcal{A}^1(M)$ be such that $d\alpha = 0$. We may write $\alpha = \alpha_0 + \alpha_1$ where $\alpha_0 \in \mathcal{A}^{0,1}$ and $\alpha_1 \in \mathcal{A}^{1,0}$. The condition $d\alpha = 0$ gives

$$\begin{cases} \bar{\partial}\alpha_0 + \bar{\mu}\alpha_1 = 0 \\ \partial\alpha_0 + \bar{\partial}\alpha_1 = 0 \\ \mu\alpha_0 + \partial\alpha_1 = 0. \end{cases}$$

These conditions imply that $\alpha_0, \overline{\alpha_1} \in Z_r^{0,1}(M)$ for any $1 \leq r \leq 3$. Since $E_3^{0,1}(M) = E_\infty^{0,1}(M)$, we obtain classes $[\alpha_0]_r, [\overline{\alpha_1}]_r \in E_r^{0,1}(M)$ for all $r \geq 1$. We now show that the map

$$\mathcal{F}([\alpha]) := ([\alpha_0]_r, \overline{[\overline{\alpha_1}]_r})$$

is well-defined. Assume that $\alpha = df$. Then we have $\alpha_0 = \bar{\partial} f$ and $\overline{\alpha_1} = \bar{\partial}\overline{f}$. These conditions readily imply that $[\alpha_0]_r = [\overline{\alpha_1}]_r = 0$. Lastly, we prove that $\mathcal{F}$ is injective. Assume that $\alpha_0 = \bar{\partial} f$ and $\overline{\alpha_1} = \bar{\partial} g$. Then we have

$$0 = \partial\alpha_0 + \bar{\partial}\alpha_1 = \partial\bar{\partial} f + \bar{\partial}\partial\overline{g} = \partial\bar{\partial}(f - \overline{g}).$$

Since M is compact, any 2-form in the image of $\partial\bar{\partial}$ is constant (see Corollary 1 of [**CW20**] for a proof in the possibly non-integrable case). Therefore $f - \overline{g}$ must be constant. This implies that $\partial\overline{g} = \partial f$ and so $\alpha = \alpha_0 + \alpha_1 = \bar{\partial} f + \partial\overline{g} = df$. $\square$

For a compact almost complex manifold M, and any $r \geq 1$, denote

$$h_r^{p,q}(M) := \dim E_r^{p,q}(M),$$

noting that these numbers are always finite in the integrable case, but may be infinite for non-integrable structures. We also consider the finite numbers

$$h_{\mathrm{dR}}^{p,q}(M) := \dim E_\infty^{p,q}(M).$$

Corollary 2.2. *For any compact almost complex manifold we have inequalities* $b^1(M) \leq 2h_r^{0,1}(M)$ *for all* $r \geq 1$ *and* $h_{\mathrm{dR}}^{1,0}(M) \leq h_{\mathrm{dR}}^{0,1}(M)$.

Remark 2.3. Since there are always inequalities

$$h_1^{p,q}(M) \geq h_2^{p,q}(M) \geq \cdots \geq h_{\mathrm{dR}}^{p,q}(M),$$

the above result strengthens the inequality $b^1(M) \leq 2h_{\bar{\partial}}^{0,1}(M) = 2h_1^{0,1}(M)$ proven in [**PU18**] in the integrable case.

3 - Cup and Massey products on the E_∞-page

Let M be a compact almost complex manifold. A natural question is to ask how cup and Massey products in cohomology behave with respect to the choice of a filtered isomorphism

$$H_{\mathrm{dR}}^n(M; \mathbb{C}) \cong \bigoplus_{p+q=n} E_\infty^{p,q}(M).$$

For compact Kähler manifolds, Hodge theory ensures that $E_1 = E_\infty$ and that cup products preserve bidegrees while higher Massey products vanish. For an arbitrary compact complex or almost complex manifold this is not true in general, even when the spectral sequence degenerates at the first stage.

Example 3.1. The twistor space $Z = Tw(\mathbb{T}^4)$ of the 4-torus $\mathbb{T}^4$ is a complex non-Kähler manifold homeomorphic to $\mathbb{T}^4 \times S^2$ which has no holomorphic forms: $E_1^{p,0}(Z) = 0$ for all $p > 0$. A complete description of the Frölicher spectral sequence of Z appears in Section 4 of [**ES93**]. There are four classes in $E_\infty^{0,1}(Z)$ corresponding with the four generators of $\mathbb{T}^4$. These classes multiply to the top class of $\mathbb{T}^4$, which sits in bidegree $(2,2)$ in the spectral sequence for Z. This gives a cup product of bidegree $(2,-2)$. This space is formal and so higher Massey products are all trivial.

Let us briefly recall how Massey products are defined on the cohomology $H^*(\mathcal{A})$ of a dg-algebra $\mathcal{A}$ defined over a field.

Consider cohomology classes $[x],[y],[z] \in H^*(\mathcal{A})$ such that $[x]\cdot[y] = 0$ and $[y]\cdot[z] = 0$. Then there are elements $a, b \in \mathcal{A}$ such that $da = x \cdot y$ and $db = y \cdot z$ and the element

$$a \cdot z - (-1)^{|x|} x \cdot b$$

is a cocycle whose cohomology class depends on the choice of a and b. The set $\langle [x],[y],[z] \rangle$ formed by all the cohomology classes constructed this way is the *triple Massey product* of $[x]$,$[y]$ and $[z]$. More generally, consider cohomology classes $x_1, \cdots, x_k \in H^*(\mathcal{A})$, with $k \geq 3$. A *defining system* for $\{x_1, x_2, \cdots, x_k\}$ is a collection of elements $\{x_{i,j}\}$, for $1 \leq i \leq j \leq k$ with $(i,j) \neq (1,k)$ where $x_i = [x_{i,i}]$ and

$$d(x_{i,j}) = \sum_{q=i}^{j-1} (-1)^{|x_{i,q}|} x_{i,q} x_{q+1,j}.$$

Consider the cocycle

$$\gamma(x_{i,j}) := \sum_{q=1}^{k-1} (-1)^{|x_{1,q}|} x_{1,q} x_{q+1,k}.$$

The *k-tuple Massey product* $\langle x_1, \cdots, x_k \rangle$ is defined to be the set of all cohomology classes $[\gamma(x_{i,j})]$, for all possible defining systems. A Massey product is *trivial* if the trivial cohomology class belongs to its defining set.

Note that the triple Massey product $\langle x_1, x_2, x_3 \rangle$ is defined whenever $x_1 x_2 = 0$ and $x_2 x_3 = 0$. For $k > 3$ one similarly asks that some q-tuple Massey products, with $q < k$, are trivial in a certain compatible way, so that at least one defining system exists.

Remark 3.2. Massey products are extremely related to A_∞-structures. Indeed, given any dg-algebra defined over a field, there is a transferred structure of A_∞-algebra on its cohomology $H^*(\mathcal{A})$, which is unique up to A_∞-isomorphism. The higher operations μ_k of this A_∞-structure give elements in the corresponding Massey sets. Conversely, given $x \in \langle x_1, \cdots, x_k\rangle$ there is always an A_∞-structure on $H^*(\mathcal{A})$ such that $\mu_k(x_1, \cdots, x_k) = \pm x$ [**BMFM20**]. Massey products are also linked to formality: if a dg-algebra is formal then Massey products vanish.

The following result describes the behaviour of cup and Massey products with respect to a fixed bidegree decomposition on the cohomology of almost complex manifolds.

Theorem 3.3. *Let M be a compact almost complex manifold. Let $r \geq 0$ be such that $E_r(M) \neq E_\infty(M)$ and $E_{r+1}(M) = E_\infty(M)$. Then:*

1. *If $\alpha \in E_\infty^{p,*}(M)$ and $\beta \in E_\infty^{p',*}(M)$, then the cup product $\alpha\cdot\beta$ has bidegrees $(p + p' + i, *)$, with $i \geq 0$.*
2. *Let $k \geq 3$ and $\alpha_i \in E_\infty^{p_i,*}(M)$ for $1 \leq i \leq k$. Then the k-tuple Massey product $\langle\alpha_1, \cdots, \alpha_k\rangle$ has bidegrees $(p_1 + \cdots + p_k + (2 - k)r + i, *)$ with $i \geq 0$.*

Proof. The theory of filtered A_∞-structures developed in [**CS**] gives the same bidegrees for the structure maps of the induced A_∞-structure on the spectral sequence of a filtered dg-Algebra (see Remark 3.18 of [**CS**]). It suffices to apply this theory for the Hodge-type filtration

$$F^p\mathcal{A}^n(M) := \mathrm{Ker}(\bar{\mu}) \cap \mathcal{A}^{p,n-p} \oplus \bigoplus_{q>p} \mathcal{A}^{q,n-q}$$

and to interpret the bidegrees obtained in terms of Massey products, as explained in Remark 3.2. □

While the above formulae for bidegrees may not be very enlightening at first, they allow us to understand the behaviour of the topological multiplicative structures on any almost complex manifold, as we will see in the following section.

4 - Applications

We explain some direct consequences of Theorem 3.3.

4.1 - *Cup-length of almost complex manifolds*

The *cup-length* $cl(M)$ of a topological space M is the largest possible integer k such that there are k-cohomology classes in $H^{>0}(M)$ whose cup product is non-trivial. If M is a manifold of dimension $2n$, then the cup-length satisfies $cl(M) \leq 2n$. If M is simply connected then we have $cl(M) \leq n$.

By Theorem 3.3, cup products can have components in bidegree $(i, -i)$ for any $i \geq 0$, but can never have bidegrees with negative first component. This gives:

Corollary 4.1. *If M is a compact almost complex manifold of dimension $2n$ and $h^{0,1}_{\mathrm{dR}}(M) = 0$ then $cl(M) \leq n$.*

Note however that this Corollary is superseded by Corollary 2.2, since the condition $h^{0,1}_{\mathrm{dR}} = 0$ implies $h^{1,0}_{\mathrm{dR}} = 0$ and so the manifold is simply connected. However, the same idea can be applied (a priori non-trivially) to higher bidegrees. For instance, looking at products in degree 2 on an almost complex 6-manifold, we obtain:

Corollary 4.2. *Let M be a compact almost complex 6-manifold and assume that $h^{0,2}_{\mathrm{dR}}(M) = 0$, $h^{1,1}_{\mathrm{dR}}(M) = 0$ and $b^1(M) = 0$. Then, we have $cl(M) \leq 2$.*

4.2 - *Massey products and degeneration*

We now turn our attention to Massey products. According to Theorem 3.3, triple Massey products, which have total degree -1, can only have components in bidegrees $(-r+i, r-1-i)$ for any $i \geq 0$, whenever $E_r(M) \neq E_\infty(M)$. Likewise, k-tuple Massey products, which have total degree $2-k$, can only have components in bidegrees $((2-k)r+i, (2-k)(1-r)-i)$ for any $i \geq 0$, whenever $E_r(M) \neq E_\infty(M)$. Note that Massey products may have bidegrees with negative first component depending on the stage where the spectral sequence degenerates.

As explained in the introduction of [**DGMS75**], the initial motivation of Deligne, Griffiths, Morgan and Sullivan for proving the formality of compact Kähler manifolds was that there were no higher products starting from $H^{*,0}$. We see that, in fact, this property is satisfied for any compact almost complex manifold whose spectral sequence degenerates at the first stage.

Corollary 4.3. *If the spectral sequence of a compact almost complex manifold M satisfies $E_1(M) = E_\infty(M)$ then there are no Massey products starting from $E^{*,0}_\infty(M)$.*

Using the fact that the Frölicher spectral sequence of complex surfaces always degenerates at E_1, we have:

Corollary 4.4. *If on a compact almost complex 4-manifold M there is a non-trivial Massey product $\langle\alpha_1,\alpha_2,\alpha_3\rangle$ with $\alpha_i \in E_\infty^{p_i,q_i}$, and if $p_1+p_2+p_3 \geq 3$, then the structure is non-integrable.*

For instance, the above applies when the bidegrees of α_i are given by

$$|\alpha_1| = |\alpha_2| = (1,0) \text{ and } |\alpha_3| = (1,1).$$

The above results indicate how products and Massey products on the cohomology of an almost complex manifold may be used to discard possible "Hodge-de Rham bidegree decompositions" for compact complex manifolds. In a lucky scenario, given a fixed manifold admitting an almost complex structure, one might use the above techniques to discard all possible integrable E_∞-pages, thus proving that such manifold does not admit integrable structures.

Acknowledgments. I would like to thank Dennis Sullivan for the idea of looking at the cup-length of complex manifolds and Scott Wilson for our constant discussions on almost complex geometry. Thanks also to Anna Sopena and Jonas Stelzig for useful exchanges.

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