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Statistical structures, α-connections and Generalized Geometry

Abstract. The main purpose of this paper is to describe how statistical structures fit perfectly into Generalized Geometry. Firstly, we will briefly present the properties of generalized pseudo-calibrated almost complex structures induced by statistical structures. Then we will characterize the integrability of generalized almost complex structures with respect to the bracket defined by the α-connection, finding conditions under which the concept of integrability is α-invariant. Finally, we consider a pair of generalized dual quasi-statistical connections $(\hat{\nabla}, \hat{\nabla}^*)$ on the generalized tangent bundle $TM \oplus T^*M$ and we provide conditions for $TM \oplus T^*M$ with the α-connections $(\hat{\nabla}^{(\alpha)}, \hat{\nabla}^{(-\alpha)})$ induced by $(\hat{\nabla}, \hat{\nabla}^*)$ to be conjugate Ricci-symmetric.

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1 - Introduction

Statistical manifolds were introduced by S. Amari in 1985 ([**3**], [**2**]). They are manifolds of probability distributions and they constitute a bridge between Differential Geometry, Information Geometry and Theoretical Physics. Moreover, in the framework of Machine Learning, statistical manifolds turned out to be useful for classifying patients with Alzheimer's disease ([**7**]).

Basically, a *statistical structure* on a smooth manifold M consists of a pseudo-Riemannian metric g and a torsion-free affine connection ∇ such that ∇g is a Codazzi tensor field. To every statistical structure (g, ∇) one can naturally associate a dual statistical structure (g, ∇^*), and, (g, ∇, ∇^*) defines a family of connections called α-connections.

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The main purpose of this paper is to describe how statistical structures fit perfectly into Generalized Geometry.

Generalized Geometry consists in to translate problems from the tangent bundle TM of M to the *generalized tangent bundle* $TM \oplus T^*M$ of M, in order to study different geometrical objects from the same point of view.

Examples are given by Dirac structures introduced by A. Weinstein and T. Courant in 1990 ([**6**]), in order to unify Poisson and pre-symplectic structures. Other examples are given by generalized complex structures introduced by N. Hitchin in 2003 ([**10**]), and further investigated by M. Gualtieri ([**9**]), in order to unify complex and symplectic structures.

In this framework, we will firstly describe generalized pseudo-calibrated almost complex structures induced by statistical structures. Then we will characterize the integrability of generalized almost complex structures with respect to the bracket defined by the α-connection, finding conditions under which the concept of integrability is α-invariant. Finally, we will define a pair of generalized dual quasi-statistical connections $(\hat{\nabla}, \hat{\nabla}^*)$ on the generalized tangent bundle $TM \oplus T^*M$ and we provide conditions for $TM \oplus T^*M$ with the α-connections $(\hat{\nabla}^{(\alpha)}, \hat{\nabla}^{(-\alpha)})$ induced by $(\hat{\nabla}, \hat{\nabla}^*)$ to be conjugate Ricci-symmetric.

2 - Statistical manifolds. Examples

Let M be a smooth manifold, let TM be the tangent bundle and T^*M the cotangent bundle of M and denote by $C^\infty(TM)$ (respectively, by $C^\infty(T^*M)$) the smooth sections of TM (respectively, of T^*M).

Definition 2.1. Let g be a pseudo-Riemannian metric on M and let ∇ be a torsion-free affine connection on M. Then (g, ∇) is called a *statistical structure* on M (and (M, g, ∇) a *statistical manifold*) if the Codazzi equation

$$(\nabla_X g)(Y, Z) = (\nabla_Y g)(X, Z)$$

is satisfied, for any $X, Y, Z \in C^\infty(TM)$.

Remark 2.2. If we denote by C the cubic form defined as $C(X, Y, Z) := (\nabla_X g)(Y, Z)$, for $X, Y, Z \in C^\infty(TM)$, we remark that (M, g, ∇) is a statistical manifold if and only if C is totally symmetric.

2.1 - *A trivial example*

Let (M, g) be a pseudo-Riemannian manifold and let ∇ be the Levi-Civita connection of g. Then $C = 0$ and (M, g, ∇) is a statistical manifold.

Remark 2.3. A statistical manifold is a generalization of a pseudo-Riemannian manifold.

2.2 - *Hypersurfaces in* $\mathbb{R}^{n+1}$

Let M be a locally convex hypersurface in $\mathbb{R}^{n+1}$, let g, h, be respectively the first and the second fundamental forms of M and let ∇ be the Levi-Civita connection of g. Then h is a Riemannian metric on M and the Codazzi equation

$$(\nabla_X h)(Y,Z) = (\nabla_Y h)(X,Z)$$

holds, for any $X, Y, Z \in C^\infty(TM)$. Thus, (M, h, ∇) is a statistical manifold.

2.3 - *Hessian manifolds*

Definition 2.4. An *affine manifold* is a smooth manifold provided with a flat, torsion-free, affine connection.

Definition 2.5. Let (M, ∇) be an affine manifold. A Riemannian metric g on M is said to be a *Hessian metric* if g is locally expressed by the Hessian of a locally smooth function f, i.e., $g = \nabla^2 f = \nabla df$. In this case, (g, ∇) is called a *Hessian structure* and (M, g, ∇) is called a *Hessian manifold*.

Proposition 2.6. *A Hessian manifold is a statistical manifold.*

Proof. For $g := \nabla df$, we have

$$g(Y,Z) := (\nabla_Y df)(Z) := Y(df(Z)) - df(\nabla_Y Z)$$

and

$$\begin{aligned}(\nabla_X g)(Y,Z) &= X(g(Y,Z)) - g(\nabla_X Y, Z) - g(Y, \nabla_X Z)\\ &= X(Y(df(Z))) - X(df(\nabla_Y Z)) - (\nabla_X Y)(df(Z)) + df(\nabla_{\nabla_X Y} Z) -\\ &\quad - Y(df(\nabla_X Z)) + df(\nabla_Y \nabla_X Z),\end{aligned}$$

for any $X, Y, Z \in C^\infty(TM)$.

As ∇ is torsion-free and flat, we get

$$\begin{aligned}&(\nabla_X g)(Y,Z) - (\nabla_Y g)(X,Z)\\ &= ([X,Y] - \nabla_X Y + \nabla_Y X)Z(f) + (([\nabla_Y, \nabla_X] - \nabla_{[Y,X]})Z)(f) = 0,\end{aligned}$$

for any $X, Y, Z \in C^\infty(TM)$. $\square$

2.4 - *Holomorphic statistical manifolds*

Definition 2.7. Let (M, g, J) be a Kähler manifold and let ∇ be a torsion-free affine connection on M. Then (M, g, J, ∇) is called a *holomorphic statistical manifold* if

1. (M, g, ∇) is a statistical manifold, and
2. $\omega = g(\cdot, J\cdot)$ is a ∇-parallel 2-form on M.

Remark 2.8. A holomorphic statistical manifold is a generalization of a Kähler manifold and, if ∇ is flat, then (M, g, J, ∇) is a *special Kähler manifold* ([**8**]).

2.5 - *Norden manifolds*

Norden manifolds, also called anti-Kählerian manifolds, were introduced by Norden in 1960 ([**14**]). They have applications both in Mathematics and in Theoretical Physics.

Definition 2.9. (M, g, J) is called a *Norden manifold* if g is a pseudo-Riemannian metric and J is a g-symmetric almost complex structure on M, i.e., $J : TM \to TM$, $J^2 = -I$ and $g(JX, Y) = g(X, JY)$, for any $X, Y \in C^\infty(TM)$.

Denote by $\tilde{g}$ the metric defined by $\tilde{g}(X, Y) := g(X, JY)$, for $X, Y \in C^\infty(TM)$, and called the *twin metric* defined by (g, J).

Let ∇ be a torsion-free affine connection on M and let d^∇ be the *exterior differential* operator associated to ∇, defined for any (tangent bundle valued p-form) $T \in C^\infty(\bigwedge^p T^*M \otimes TM)$ by:

$$(d^\nabla T)(X_1, ..., X_{p+1}) := -\sum_{i=1}^{p+1} (-1)^i (\nabla_{X_i} T)(X_1, ..., \hat{X}_i, ..., X_{p+1}).$$

In particular, for $T = J$, we have:

$$(d^\nabla J)(X, Y) := (\nabla_X J)Y - (\nabla_Y J)X,$$

for $X, Y \in C^\infty(TM)$.

Examples of statistical structures can be obtained by certain almost complex structures, namely a direct computation gives the following.

Proposition 2.10. *Let (M, g, J) be a Norden manifold, let ∇ be the Levi-Civita connection of g and let $\tilde{g}$ be the twin metric defined by (g, J). Then $(M, \tilde{g}, \nabla)$ is a statistical manifold if and only if*

$$d^\nabla J = 0.$$

Remark 2.11. If J is integrable and ∇ is flat, then (M, J, ∇) is a *special complex manifold* ([**1**]).

3 - Quasi-statistical manifolds. Examples

Definition 3.1. Let (M, g) be a pseudo-Riemannian manifold and let ∇ be an affine connection on M with torsion tensor T^∇. Then (g, ∇) is called a *quasi-statistical structure* on M (and (M, g, ∇) a *quasi-statistical manifold*, or *statistical manifold admitting torsion*) if $d^\nabla g = 0$, where

$$(d^\nabla g)(X, Y, Z) := (\nabla_X g)(Y, Z) - (\nabla_Y g)(X, Z) + g(T^\nabla(X, Y), Z),$$

for $X, Y, Z \in C^\infty(TM)$.

Examples of quasi-statistical manifolds can be constructed by means of a pseudo-Riemannian manifold (M, g) and a positive smooth function f on M. Indeed we have the following.

Proposition 3.2. *Let M be a smooth manifold, let g, ∇ and f be respectively a pseudo-Riemannian metric, an affine connection and a positive smooth function on M. Define $\bar\nabla := \nabla - \frac{1}{f} df \otimes I$ and $\bar g := \frac{1}{f} g$. Then:*

$$T^{\bar\nabla} = T^\nabla + \frac{1}{f}(I \otimes df - df \otimes I), \quad \bar\nabla \bar g = \frac{1}{f}\nabla g + \frac{1}{f} df \otimes \bar g.$$

Moreover:

$$d^{\bar\nabla}\bar g = \frac{1}{f} d^\nabla g.$$

Proof. For $X, Y, Z \in C^\infty(TM)$ we have:

$$T^{\bar\nabla}(X, Y) = T^\nabla(X, Y) + \frac{1}{f}(Y(f)X - X(f)Y)$$

$$(\bar\nabla_X \bar g)(Y, Z) = \frac{1}{f}(\nabla_X g)(Y, Z) + \frac{1}{f^2} X(f) g(Y, Z).$$

Then:

$$\begin{aligned}(d^{\bar\nabla}\bar g)(X, Y, Z) &= \frac{1}{f}(d^\nabla g)(X, Y, Z) + \frac{1}{f}(X(f)\bar g(Y, Z) - Y(f)\bar g(X, Z) \\ &\quad + Y(f)\bar g(X, Z) - X(f)\bar g(Y, Z)) = \frac{1}{f}(d^\nabla g)(X, Y, Z). \qquad \square\end{aligned}$$

Corollary 3.3. *(M, g, ∇) is a quasi-statistical manifold if and only if $(M, \bar g, \bar\nabla)$ is a quasi-statistical manifold.*

Corollary 3.4. *Let (M, g) be a pseudo-Riemannian manifold and let ∇ be the Levi-Civita connection of g. Then $(\bar g, \bar\nabla)$ is a quasi-statistical structure on M.*

4 - Dualistic structures and α-connections

Definition 4.1. Let M be a smooth manifold and let g be a pseudo-Riemannian metric on M. Two affine connections ∇ and ∇^* on M are said to be *dual connections* with respect to g if

$$X(g(Y,Z)) = g(\nabla_X Y, Z) + g(Y, \nabla^*_X Z),$$

for any $X, Y, Z \in C^\infty(TM)$ and we call (g, ∇, ∇^*) a *dualistic structure.*

A direct computation gives the following.

Lemma 4.2. *If g is a pseudo-Riemannian metric on M and ∇ is an affine connection on M, then the dual connection ∇^* satisfies:*

$$\nabla^*_X Y = \nabla_X Y + g^{-1}((\nabla_X g)(Y)), \quad \nabla^*_X \beta = \nabla_X \beta - (\nabla_X g)(g^{-1}(\beta)),$$

for any $X, Y \in C^\infty(TM)$ and $\beta \in C^\infty(T^*M)$, where $g : TM \to T^*M$ is identified to the flat musical isomorphism of g.

Moreover, if (g, ∇) is a quasi-statistical structure, then the dual connection ∇^* satisfies:

$$T^{\nabla^*} = 0, \quad \nabla^* g = -\nabla g.$$

In particular, if (M, g, ∇) is a statistical manifold, then (M, g, ∇^*) is a statistical manifold, too.

Lemma 4.3. *If (M, g, J, ∇) is a holomorphic statistical manifold, then*

$$J(\nabla^*_X Y) = \nabla_X JY,$$

for any $X, Y \in C^\infty(TM)$.

Proposition 4.4. *Let (M, g, J) be a Norden manifold, let ∇ be the Levi-Civita connection of g and let $\tilde{g}$ be the twin metric defined by (g, J). Then $(\tilde{g}, \nabla, \nabla^*)$ defines a dualistic structure with the dual connection ∇^* given by:*

$$\nabla^*_X Y = \nabla_X Y - J((\nabla_X J)Y),$$

for any $X, Y \in C^\infty(TM)$. In particular, we get:

$$\nabla^* J = -\nabla J.$$

For the quasi-statistical structure (g, ∇) and ∇^* the dual connection of ∇, we consider the family of α-connections on M, for $\alpha \in \mathbb{R}$:

$$\nabla^{(\alpha)} := \frac{1+\alpha}{2}\nabla + \frac{1-\alpha}{2}\nabla^*.$$

Remark that the dual connection of $\nabla^{(\alpha)}$ is $\nabla^{(-\alpha)}$.

5 - Generalized Geometry

5.1 - *Some geometrical structures on $TM \oplus T^*M$*

Let $E := TM \oplus T^*M$ be the generalized tangent bundle of M. On E we consider the natural indefinite metric:

$$< X + \eta, Y + \beta > := -\frac{1}{2}(\eta(Y) + \beta(X))$$

and the natural symplectic structure:

$$(X + \eta, Y + \beta) := -\frac{1}{2}(\eta(Y) - \beta(X)),$$

for $X, Y \in C^\infty(TM)$ and $\eta, \beta \in C^\infty(T^*M)$.

Furthermore, given an affine connection ∇ on M, we define the ∇-bracket, $[\cdot, \cdot]_\nabla$, as:

$$[X + \eta, Y + \beta]_\nabla := [X, Y] + \nabla_X \beta - \nabla_Y \eta,$$

for $X, Y \in C^\infty(TM)$ and $\eta, \beta \in C^\infty(T^*M)$.

5.2 - *Generalized almost complex structures*

Definition 5.1. A *generalized almost complex structure* on M is an endomorphism $\widehat{J} : E \to E$ such that $\widehat{J}^2 = -I$.

Definition 5.2. ([**12**]) A generalized almost complex structure $\widehat{J}$ is called *pseudo-calibrated* if it is $(\cdot, \cdot)$-invariant and if the bilinear symmetric form defined by $(\cdot, \widehat{J}\cdot)$ is non degenerate. Moreover, $\widehat{J}$ is called *calibrated* if it is pseudo-calibrated and $(\cdot, \widehat{J}\cdot)$ is positive definite.

From the definition, we get the following block matrix form of a generalized pseudo-calibrated almost complex structure:

$$\widehat{J} = \begin{pmatrix} H & -(I + H^2)g^{-1} \\ g & -H^* \end{pmatrix},$$

where g is a pseudo-Riemannian metric on M, $H : TM \to TM$ is a g-symmetric operator and $H^* : T^*M \to T^*M$ is the dual operator of H defined by $H^*(\eta)(X) := \eta(H(X))$, for $X \in C^\infty(TM)$ and $\eta \in C^\infty(T^*M)$.

If $\widehat{J}$ is calibrated, then g is a Riemannian metric, namely:

$$g(X)(Y) = g(X, Y) = 2(X, \widehat{J}Y).$$

5.3 - ∇-*integrability of generalized almost complex structures*

Lemma 5.3. *Let $\widehat{J} : E \to E$ be a generalized almost complex structure on M and let*

$$N^{\nabla}(\widehat{J}) : C^{\infty}(E) \times C^{\infty}(E) \to C^{\infty}(E),$$
$$N^{\nabla}(\widehat{J})(\sigma, \tau) := [\widehat{J}\sigma, \widehat{J}\tau]_{\nabla} - \widehat{J}[\widehat{J}\sigma, \tau]_{\nabla} - \widehat{J}[\sigma, \widehat{J}\tau]_{\nabla} - [\sigma, \tau]_{\nabla},$$

for $\sigma, \tau \in C^{\infty}(E)$. Then $N^{\nabla}(\widehat{J})$ is a skew-symmetric tensor field called the Nijenhuis tensor *of $\widehat{J}$ with respect to ∇.*

Let $E^{\mathbb{C}} := (TM \oplus T^*M) \otimes \mathbb{C}$ be the complexified generalized tangent bundle. The splitting into $\pm i$ eigenspaces of $\widehat{J}$ is denoted by $E^{\mathbb{C}} := E^{1,0}_{\widehat{J}} \oplus E^{0,1}_{\widehat{J}}$ with $E^{0,1}_{\widehat{J}} = \overline{E^{1,0}_{\widehat{J}}}$. Let $P_+ : E^{\mathbb{C}} \to E^{1,0}_{\widehat{J}}$ and $P_- : E^{\mathbb{C}} \to E^{0,1}_{\widehat{J}}$ be the projection operators $P_{\pm} = \frac{1}{2}(I \mp i\widehat{J})$. Then the following holds.

Lemma 5.4. *For any $\sigma, \tau \in C^{\infty}(E^{\mathbb{C}})$, we have:*

$$P_{\mp}[P_{\pm}(\sigma), P_{\pm}(\tau)]_{\nabla} = -\frac{1}{4} P_{\mp}(N^{\nabla}(\widehat{J})(\sigma, \tau)).$$

Corollary 5.5. *For any affine connection ∇ on M, we have that $E^{1,0}_{\widehat{J}}$ and $E^{0,1}_{\widehat{J}}$ are $[\cdot, \cdot]_{\nabla}$-involutive if and only if $N^{\nabla}(\widehat{J}) = 0$.*

Definition 5.6. A generalized almost complex structure $\widehat{J}$ on M is called ∇-*integrable* if $N^{\nabla}(\widehat{J}) = 0$.

5.4 - *Quasi-statistical Geometry fits perfectly into Generalized Geometry*

Proposition 5.7. *Let (M, g) be a pseudo-Riemannian manifold and let ∇ be an affine connection on M. Then the generalized almost complex structure defined by g:*

$$\widehat{J} := \begin{pmatrix} 0 & -g^{-1} \\ g & 0 \end{pmatrix}$$

is ∇-integrable if and only if (M, g, ∇) is a quasi-statistical manifold.

Theorem 5.8 ([**13**]). *Let (M, g, J) be a Norden manifold and let ∇ be an affine connection on M. Then the pseudo-calibrated generalized almost complex structure defined by (g, J):*

$$\widehat{J} := \begin{pmatrix} J & 0 \\ g & -J^* \end{pmatrix}$$

is ∇-integrable if and only if, for any $X, Y \in C^\infty(TM)$, the following conditions hold:

$$\begin{cases} N(J) = 0 \\ (\nabla_{JX} J) + J(\nabla_X J) = 0 \\ (d^\nabla g)(JX, Y) + (d^\nabla g)(X, JY) - g((d^\nabla J)(X, Y)) = 0. \end{cases}$$

Corollary 5.9. *Let (M, g, J) be a Norden manifold, let ∇ be the Levi-Civita connection of g and let $\tilde{g}$ be the twin metric defined by (g, J). If $(M, \tilde{g}, \nabla)$ is a statistical manifold, then $\widehat{J}$ is ∇-integrable if and only if, for any $X \in C^\infty(TM)$, the following condition holds:*

$$(\nabla_{JX} J) + J(\nabla_X J) = 0.$$

In particular, $\widehat{J} = \begin{pmatrix} J & 0 \\ g & -J^* \end{pmatrix}$ is ∇-integrable if and only if $\tilde{J} = \begin{pmatrix} J & 0 \\ \tilde{g} & -J^* \end{pmatrix}$ is ∇-integrable.

Proof. We have $d^\nabla g = 0$, $d^\nabla \tilde{g} = 0$, $d^\nabla J = 0$ and $(\nabla_{JX} J) + J(\nabla_X J) = 0$ which furthermore give $N(J) = 0$. $\square$

Lemma 5.10. *For a torsion-free affine connection ∇ and an almost complex structure J,*

$$\begin{cases} (\nabla_{JX} J) - J(\nabla_X J) = 0 \\ (d^\nabla J)(X, Y) = 0 \end{cases}$$

for any $X, Y \in C^\infty(TM)$ if and only if

$$\nabla J = 0.$$

Definition 5.11. ([**11**]) (M, g, J) is called a *Kähler-Norden manifold* if it is a Norden manifold such that $\nabla J = 0$, where ∇ is the Levi-Civita connection of g.

Corollary 5.12. *Let (M, g, J) be a Norden manifold, let ∇ be the Levi-Civita connection of g and let $\tilde{g}$ be the twin metric defined by (g, J). If $(M, \tilde{g}, \nabla)$ is a statistical manifold, then the following conditions are equivalent:*

1. *$\widehat{J}$ is ∇-integrable*

2. *$\tilde{J}$ is ∇-integrable;*

moreover the following are equivalent:

3. *(M, g, J) is a Kähler-Norden manifold*

4. *$(M, \tilde{g}, J)$ is a Kähler-Norden manifold*

and 3., *or* 4., *implies* 1. *and* 2..

5.5 - $\nabla^{(\alpha)}$*-integrability of generalized almost complex structures*

Let (M, g, ∇) be a statistical manifold, let ∇^* be the dual connection of ∇ and let $[\cdot,\cdot]_{\nabla^{(\alpha)}}$ be the bracket defined by $\nabla^{(\alpha)}$. Then:

$$[X+\eta, Y+\beta]_{\nabla^{(\alpha)}} := [X,Y] + \nabla^{(\alpha)}_X \beta - \nabla^{(\alpha)}_Y \eta$$

$$= [X+\eta, Y+\beta]_{\nabla} - \frac{1-\alpha}{2}\{(\nabla_X g)(g^{-1}(\beta)) - (\nabla_Y g)(g^{-1}(\eta))\},$$

for any $X, Y \in C^\infty(TM)$ and $\eta, \beta \in C^\infty(T^*M)$.

Proposition 5.13. *Let (M, g) be a pseudo-Riemannian manifold and let ∇ be an affine connection on M. Then the generalized almost complex structure defined by g:*

$$\hat{J} := \begin{pmatrix} 0 & -g^{-1} \\ g & 0 \end{pmatrix}$$

is $\nabla^{(\alpha)}$-integrable for any $\alpha \in \mathbb{R}$ if and only if (M, g, ∇) is a statistical manifold.

Proof. For $X, Y \in C^\infty(TM)$ we have:

$$N^{\nabla^{(\alpha)}}(\hat{J})(X,Y) = T^{\nabla}(X,Y) + \frac{1+\alpha}{2} g^{-1}\{(\nabla_X g)(Y) - (\nabla_Y g)(X)\}$$

$$N^{\nabla^{(\alpha)}}(\hat{J})(X, g(Y)) = -g(T^{\nabla}(X,Y)) - \frac{1+\alpha}{2}\{(\nabla_X g)(Y) - (\nabla_Y g)(X)\}$$

$$N^{\nabla^{(\alpha)}}(\hat{J})(g(X), g(Y)) = -T^{\nabla}(X,Y) - \frac{1+\alpha}{2} g^{-1}\{(\nabla_X g)(Y) - (\nabla_Y g)(X)\},$$

thus the statement. □

Let (M, g, J) be a Norden manifold and let ∇ be a torsion-free affine connection on M. We consider the tensor field F defined by:

$$F(X,Y,Z) := g((\nabla_X J)Y, Z),$$

for $X, Y, Z \in C^\infty(TM)$, which is very important in the classification of almost complex structures and it is related to the theory of α-connections.

Remark 5.14. ∇J is g-symmetric, i.e., $g((\nabla_X J)Y, Z) = g(Y, (\nabla_X J)Z)$, for any $X, Y, Z \in C^\infty(TM)$, if and only if F satisfies:

$$F(\cdot, X, Y) = F(\cdot, Y, X),$$

for any $X, Y \in C^\infty(TM)$.

Proposition 5.15 ([**5**]). *Let (M, g, J, ∇) be a Norden and statistical manifold. Let $\widehat{J} = \begin{pmatrix} J & 0 \\ g & -J^* \end{pmatrix}$ be the generalized almost complex structure induced by (g, J). Then the Nijenhuis tensor field of $\widehat{J}$ with respect to $[\cdot,\cdot]_{\nabla^{(\alpha)}}$ satisfies:*

$$N^{\nabla^{(\alpha)}}(\widehat{J}) = N^{\nabla}(\widehat{J}),$$

for any $\alpha \in \mathbb{R}$, if and only if the tensor field F satisfies:

$$F(X, Y, Z) + F(Y, Z, X) - F(X, Z, Y) - F(Y, X, Z) = 0,$$

for any $X, Y, Z \in C^\infty(TM)$.

Corollary 5.16. *Let (M, g, J, ∇) be a Norden and statistical manifold such that ∇J is g-symmetric. Then the Nijenhuis tensor field of the generalized almost complex structure $\widehat{J}$ with respect to $[\cdot,\cdot]_{\nabla^{(\alpha)}}$ satisfies:*

$$N^{\nabla^{(\alpha)}}(\widehat{J}) = N^{\nabla}(\widehat{J}),$$

for any $\alpha \in \mathbb{R}$.

In particular, the definition of integrability for $\widehat{J}$ is α-invariant.

Proposition 5.17. *Let (M, g, J) be a Norden manifold such that $(M, \tilde{g}, \nabla)$ is a statistical manifold, where ∇ is the Levi-Civita connection of g and $\tilde{g}$ is the twin metric defined by (g, J). Let $\tilde{J} := \begin{pmatrix} J & 0 \\ \tilde{g} & -J^* \end{pmatrix}$ be the generalized almost complex structure induced by $(\tilde{g}, J)$. Then the Nijenhuis tensor field of $\tilde{J}$ with respect to $[\cdot,\cdot]_{\nabla^{(\alpha)}}$ satisfies:*

$$N^{\nabla^{(\alpha)}}(\tilde{J}) = N^{\nabla}(\tilde{J})$$

if and only if F satisfies:

$$F(X, \cdot, Y) = F(Y, \cdot, X),$$

for any $X, Y \in C^\infty(TM)$.

In particular, this holds if and only if (M, g, J) is a Kähler-Norden manifold, namely, under the above hypothesis, a direct computation gives the following.

Remark 5.18.

$$F(X, \cdot, Y) = F(Y, \cdot, X) \iff J((\nabla_X J)Y) = (\nabla_X J)JY,$$

or equivalently, if and only if

$$\nabla J = 0,$$

i.e., if and only if (M, g, J) is a Kähler-Norden manifold.

6 - Generalized statistical structures

Let ∇ be a torsion-free affine connection on M and let $\check{\nabla}$ be the affine connection on $TM \oplus T^*M$ defined by:

$$\check{\nabla}_{X+\eta}(Y+\beta) := \nabla_X Y + \nabla_X \beta,$$

for $X, Y \in C^\infty(TM)$ and $\eta, \beta \in C^\infty(T^*M)$.

Remark 6.1. For any $\sigma, \tau \in C^\infty(TM \oplus T^*M)$, $T^{\check{\nabla}}(\sigma, \tau) := \check{\nabla}_\sigma \tau - \check{\nabla}_\tau \sigma - [\sigma, \tau]_\nabla = 0$.

Definition 6.2. Let $\hat{g}$ be a non degenerate bilinear form on $TM \oplus T^*M$. Then $(\hat{g}, \check{\nabla})$ is called a *generalized statistical structure* if $d^{\check{\nabla}}\hat{g} = 0$, where

$$(d^{\check{\nabla}}\hat{g})(\sigma, \tau, \nu) := (\check{\nabla}_\sigma \hat{g})(\tau, \nu) - (\check{\nabla}_\tau \hat{g})(\sigma, \nu),$$

for $\sigma, \tau, \nu \in C^\infty(TM \oplus T^*M)$.

Proposition 6.3. *Let $\hat{g}$ be the indefinite metric $< \cdot, \cdot >$ or the natural symplectic structure $(\cdot, \cdot)$ on $TM \oplus T^*M$. Then $\check{\nabla}\hat{g} = 0$. As a consequence, $(\hat{g}, \check{\nabla})$ is a generalized statistical structure.*

Proof. Let $\sigma = X + \eta$, $\tau = Y + \beta$, $\nu = Z + \gamma$, where $X, Y, Z \in C^\infty(TM)$ and $\eta, \beta, \gamma \in C^\infty(T^*M)$. We have:

$$-2(\check{\nabla}_\sigma \hat{g})(\tau, \nu) = -2\{X(\hat{g}(\tau, \nu)) - \hat{g}(\check{\nabla}_X \tau, \nu) - \hat{g}(\tau, \check{\nabla}_X \nu)\}$$

$$= X(\beta(Z) \pm \gamma(Y)) - \{(\nabla_X \beta)(Z) \pm \gamma(\nabla_X Y)\} - \{\beta(\nabla_X Z) \pm (\nabla_X \gamma)(Y)\}$$

$$= \{X(\beta(Z)) - \beta(\nabla_X Z) - (\nabla_X \beta)(Z)\} \pm \{X(\gamma(Y)) - \gamma(\nabla_X Y) - (\nabla_X \gamma)(Y)\} = 0. \square$$

7 - Generalized quasi-statistical structures

Let g be a pseudo-Riemannian metric on M. We define the bilinear form $\breve{g}$ on $TM \oplus T^*M$ by:

$$\breve{g}(X + \eta, Y + \beta) := g(X, Y) + g(g^{-1}(\eta), g^{-1}(\beta)),$$

for $X, Y \in C^\infty(TM)$ and $\eta, \beta \in C^\infty(T^*M)$.

Furthermore, given an affine connection ∇ on M, we define the affine connection $\hat{\nabla}$ on $TM \oplus T^*M$ by:

$$\hat{\nabla}_{X+\eta}(Y + \beta) := \nabla_X Y + g(\nabla_X(g^{-1}(\beta))).$$

Definition 7.1. Let g be a pseudo-Riemannian metric and let ∇ be an affine connection on M. Let $\hat{\nabla}$ be the affine connection on $TM \oplus T^*M$ induced by (g, ∇) and let $\hat{g}$ be a non degenerate bilinear form on $TM \oplus T^*M$. Then $(\hat{g}, \hat{\nabla})$ is called a *generalized quasi-statistical structure* if $d^{\hat{\nabla}}\hat{g} = 0$, where

$$(d^{\hat{\nabla}}\hat{g})(\sigma, \tau, \nu) := (\hat{\nabla}_\sigma \hat{g})(\tau, \nu) - (\hat{\nabla}_\tau \hat{g})(\sigma, \nu) + \hat{g}(T^{\hat{\nabla}}(\sigma, \tau), \nu),$$

for $\sigma, \tau, \nu \in C^\infty(TM \oplus T^*M)$ and $T^{\hat{\nabla}}(\sigma, \tau) := \hat{\nabla}_\sigma \tau - \hat{\nabla}_\tau \sigma - [\sigma, \tau]_\nabla$.

Theorem 7.2 ([**4**]). *Let $\hat{g}$ be the indefinite metric $< \cdot, \cdot >$ or the symplectic structure $(\cdot, \cdot)$ on $TM \oplus T^*M$. Then $(\hat{g}, \hat{\nabla})$ is a generalized quasi-statistical structure if and only if (M, g, ∇) is a quasi-statistical manifold.*

Proposition 7.3 ([**4**]). *$(\check{g}, \hat{\nabla})$ is a generalized quasi-statistical structure if and only if (M, g, ∇) is a quasi-statistical manifold.*

Now we can define the dualistic structure $(\check{g}, \hat{\nabla}, \hat{\nabla}^*)$ and the family of connections $\{\hat{\nabla}^{(\alpha)}\}_{\alpha \in \mathbb{R}}$ on $TM \oplus T^*M$, called *generalized α-connections*:

$$\hat{\nabla}^{(\alpha)} := \frac{1+\alpha}{2}\hat{\nabla} + \frac{1-\alpha}{2}\hat{\nabla}^*.$$

We immediately have that the dual connection of $\hat{\nabla}^{(\alpha)}$ is $\hat{\nabla}^{(-\alpha)}$.

Definition 7.4. Let (M, g) be a pseudo-Riemannian manifold with a dualistic structure (g, ∇, ∇^*). Then (M, g, ∇, ∇^*) is called *conjugate Ricci-symmetric* if $\mathrm{Ric}^\nabla = \mathrm{Ric}^{\nabla^*}$, where Ric is the Ricci curvature tensor of g.

After computing the Ricci curvature tensors, we get the following.

Proposition 7.5 ([**5**]). *Let g be a pseudo-Riemannian metric on M, let ∇ be an affine connection and let $(\check{g}, \hat{\nabla})$ be the generalized structure on $TM \oplus T^*M$ induced by (g, ∇). If $\nabla g = 0$, then*

$$\mathrm{Ric}^{\hat{\nabla}^{(\alpha)}}(X + \eta, Y + \beta) = \mathrm{Ric}^\nabla(X, Y) = \mathrm{Ric}^{\hat{\nabla}^{(-\alpha)}}(X + \eta, Y + \beta),$$

for any $X, Y \in C^\infty(TM)$, $\eta, \beta \in C^\infty(T^*M)$ and any $\alpha \in \mathbb{R}$, i.e., $(TM \oplus T^*M, \check{g}, \hat{\nabla}^{(\alpha)}, \hat{\nabla}^{(-\alpha)})$ is conjugate Ricci-symmetric.

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