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Structural invariants of L-functions and applications: a survey

Dedicated to Roberto Dvornicich
on the occasion of his seventieth birthday

Abstract. This is a survey of the structural invariants of the L-functions in the extended Selberg class $\mathcal{S}^\sharp$, covering some of their applications. In particular, we deal with the applications to the functional equation of the standard twist and to the classification of the functions in $\mathcal{S}^\sharp$ of degree $d = 2$ and conductor $q = 1$. Moreover, we give a new, purely algebraic, definition of the structural invariants and provide an explicit expression for them.

Keywords. Invariants of L-functions, standard twist, Selberg class, functional equation, modular forms.

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1 - Structural invariants

For a function $f(s)$ of a complex variable $s = \sigma + it$ we write $\overline{f}(s) = \overline{f(\overline{s})}$. Every L-function F from the *extended Selberg class* $\mathcal{S}^\sharp$ (see next section for definitions) satisfies a general functional equation with multiple gamma factors

$$\gamma(s)F(s) = \omega\overline{\gamma}(1-s)\overline{F}(1-s), \tag{1.1}$$

where $|\omega| = 1$ and

$$\gamma(s) = Q^s \prod_{j=1}^{r} \Gamma(\lambda_j s + \mu_j) \tag{1.2}$$

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with $Q > 0$, $r \geq 0$, $\lambda_j > 0$ and $\Re(\mu_j) \geq 0$. We refer to Selberg [**23**], Conrey-Ghosh [**2**] and to our survey papers [**3**], [**5**], [**19**], [**20**], [**21**], [**22**] for further definitions, examples and the basic theory of the Selberg class.

The *γ-factor* in (1.2) is uniquely determined by F up to a multiplicative constant, see [**2**], but the *data* r, Q, λ_j's and μ_j's are not unique and their particular values can vary due to identities satisfied by the Euler gamma function. For instance the classical γ-factor of the Riemann zeta function,

$$\gamma(s) = \pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right),$$

can also be written as

$$\gamma(s) = Q^s \prod_{j=1}^{M} \Gamma\left(\frac{s}{2m_j} + \frac{2l_j}{2m_j}\right), \tag{1.3}$$

where

$$Q = \left(\frac{1}{\pi}\prod_{j=1}^{M} m_j^{\frac{1}{m_j}}\right)^{1/2}$$

and (l_j, m_j), $j = 1, \ldots, M$, is any *exact covering system*, i.e. a family of pairs of positive integers such that for every integer n there exists a unique $1 \leq j \leq M$ with $n \equiv l_j (\mathrm{mod} m_j)$. It can be proved that (1.3) exhaust all admissible forms of the γ-factor of the Riemann zeta function (see [**6**], Proposition 2.1).

By an *invariant* of $F \in \mathcal{S}^\sharp$ we mean an expression formed with the data of the functional equation (1.1)-(1.2) which is independent of their particular values. Among the most important invariants we have the *degree* $d_F = 2\sum_{j=1}^r \lambda_j$, introduced by Selberg [**23**], the *conductor* $q_F = (2\pi)^d Q^2 \prod_{j=1}^r \lambda_j^{2\lambda_j}$, the *root number* $\omega_F = \omega \prod_{j=1}^r \lambda_j^{-2i\Im(\mu_j)}$ and the *H-invariants*, introduced in [**6**] and [**7**]; the latter are defined as

$$H_F(n) = 2\sum_{j=1}^{r} \frac{B_n(\mu_j)}{\lambda_j^{n-1}}, \tag{1.4}$$

where $B_n(z)$ is the nth Bernoulli polynomial. In this paper we deal only with $F \in \mathcal{S}^\sharp$ of positive degree, hence with $r \geq 1$. We shall often denote d_F and q_F simply by d and q; observe that $H_F(0) = d_F$. An important role is also played by $H_F(1)$; we call it the *ξ-invariant* of F. So, by definition, $\xi_F = H_F(1)$ and, moreover, we write

$$\xi_F = \eta_F + i d_F \theta_F$$

with real η_F and θ_F. We call θ_F the *internal shift* of F; the classical L-functions have $\theta_F = 0$. The importance of these invariants is illustrated by the following theorem, see Theorem 1 in [**7**].

Theorem 1. *Two L-functions F and G from the extended Selberg class $\mathcal{S}^\sharp$ share the same γ-factor if and only if $q_F = q_G$ and $H_F(n) = H_G(n)$ for every $n \geq 0$. If in addition $\omega_F = \omega_G$, then F and G satisfy the same functional equation.*

Because of this result, q_F, ω_F and $H_F(n)$, $n \geq 0$, are called the *basic invariants* of $F \in \mathcal{S}^\sharp$. We also say that q_F and $H_F(n)$, $n \geq 0$, are the basic invariants of the γ-factor of F.

So far we referred to the symmetric form (1.1) of the functional equation. The latter can easily be transformed to the asymmetric *invariant form*

$$F(s) = S_F(s)h_F(s)\overline{F}(1-s), \tag{1.5}$$

where

$$S_F(s) := 2^r \prod_{j=1}^{r} \sin(\pi(\lambda_j s + \mu_j)) = \sum_{j=-N}^{N} a_j e^{i\pi d_F \omega_j s} \tag{1.6}$$

with certain $N \in \mathbb{N}$, $a_j \in \mathbb{C}$ and $-1/2 = \omega_{-N} < \cdots < \omega_N = 1/2$, and

$$h_F(s) = \frac{\omega}{(2\pi)^r} Q^{1-2s} \prod_{j=1}^{r} \big(\Gamma(\lambda_j(1-s) + \overline{\mu}_j)\Gamma(1-\lambda_j s - \mu_j)\big).$$

The above two functions are called the *S-function* and the *h-function*, respectively; it can be shown that both are invariants.

The *structural invariants* $d_F(\ell)$, $\ell \geq 0$, which we are going to define now and which are in the focus of this survey, are certain invariants which together with degree, conductor and root number uniquely determine the h-function in the asymmetric functional equation (1.5). So they play, in the case of (1.5), a role similar to the H-invariants in the case of (1.1). Here we define, for integer $\ell \geq 0$, the structural invariants by

$$d_F(\ell) = \Psi_\ell\big(H_F(2), \ldots, H_F(\ell+1), \overline{H_F(1)}, \ldots, \overline{H_F(\ell+1)}\big), \tag{1.7}$$

where $\Psi_\ell(X_1, \ldots, X_\ell, Y_1, \ldots, Y_{\ell+1})$ are the polynomials defined in the Appendix, see (5.5) with parameters $d = d_F$ and $\theta = \theta_F$. From (1.7) we plainly see that the structural invariants $d_F(\ell)$ are in fact invariants; we refer to the discussion after Theorem 4 for further information on the definition of the $d_F(\ell)$'s. The following result is the analog of Theorem 1 in terms of structural invariants; it follows from Theorem 4.

Theorem 2. *Two L-functions F and G from the extended Selberg class $\mathcal{S}^\sharp$ share the same h-function if and only if $d_F = d_G$, $q_F = q_G$, $\omega_F = \omega_G$ and $d_F(\ell) = d_G(\ell)$ for every $\ell \geq 0$. If in addition $S_F = S_G$, then F and G satisfy the same functional equation.*

The definition of H-invariants and structural invariants may seem complicated at first sight, especially in the latter case. Nevertheless, it will become clear later on that they are very useful in the Selberg class theory. These invariants are also natural objects from the point of view of certain asymptotic expansions, as the next two theorems show.

Theorem 3. *Let $F \in \mathcal{S}^\sharp$ with degree $d > 0$ and γ-factor $\gamma(s)$. Then for $|\arg(s)| < \pi - \delta$ with any fixed $0 < \delta < \pi$ we have*

$$\log\gamma(s) \approx \frac{1}{2}ds\log s + \frac{1}{2}\left(\log q - d\log(2\pi e)\right)s + \frac{1}{2}\xi_F\log s + c(\gamma) + \frac{1}{2}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n(n+1)}H_F(n+1)s^{-n},$$

where

$$c(\gamma) = \sum_{j=1}^{r}\left(\mu_j - \frac{1}{2}\right)\log\lambda_j + \frac{r}{2}\log(2\pi) \tag{1.8}$$

and $\approx$ means that cutting the sum at $\ell = M$ one gets a meromorphic remainder which is $\ll$ than the modulus of the M-th term times $1/|s|$ as $|s| \to \infty$.

We refer to equation (2.8) in [**7**] for Theorem 3. We remark that $c(\gamma)$ in (1.8) depends on the particular form of the γ-factor and is not an invariant of F. This agrees with the fact that the γ-factor of F is determined only up to a multiplicative constant. Thanks to Theorem 3 we may conclude that the *modified γ-factor*

$$\widetilde{\gamma}_F(s) := (2\pi)^{-r/2}Q^s\prod_{j=1}^{r}\left(\lambda_j^{\frac{1}{2}-\mu_j}\Gamma(\lambda_j s + \mu_j)\right)$$

is an invariant, but here we shall not use nor prove this fact.

Theorem 4. *Let $F \in \mathcal{S}^\sharp$ with degree $d \geq 1$. Then for $|\arg(-s)| < \pi - \delta$ with any fixed $0 < \delta < \pi$ we have*

$$h_F(s) \approx \frac{\omega_F}{\sqrt{2\pi}}\left(\frac{q^{1/d}}{2\pi d}\right)^{d(\frac{1}{2}-s)}\sum_{\ell=0}^{\infty}d_F(\ell)\Gamma\big(d(s_\ell^* - s)\big), \tag{1.9}$$

where $\approx$ *has the same meaning as in Theorem* 3 *and for* $\ell = 0, 1, \dots$ *we write*

$$s_\ell^* := s_\ell - i\theta_F \quad \textit{with} \quad s_\ell := \frac{d+1}{2d} - \frac{\ell}{d}. \tag{1.10}$$

Expansion (1.9) was proved in Section 3.2 of [**16**] with unspecified coefficients $d_F(\ell)$, there denoted by $d_\ell(F)$, and was used to define the structural invariants. The approach in this paper is different, but of course it leads to the same objects. Indeed, our present definition of the structural invariants is purely algebraic, and the fact that the coefficients of asymptotic expansion (1.9) are values of certain polynomials at H-invariants as in (1.7) requires justification. Actually, (1.7) is a closed formula for the coefficients eventually arising from the various expansions involved in the proof of (1.9). A sketch of the proof of Theorem 4 is given in Section 4 below.

The asymptotic expansion in Theorem 4 is somehow non-standard and is crucial for the applications we are going to present here. The original motivation for such an expansion came from an attempt to extend to the whole class $\mathcal{S}^\sharp$ a technique employed in [**15**] to obtain, in a special case, the functional equation of the standard twist. The goal was achieved in [**16**], and it turns out that the structural invariants are important in a number of other questions.

The present survey is organized as follows. In Section 2 we recall some basic definitions and facts on the Selberg class. In the subsequent section we discuss the applications of structural invariants to the functional equation of the standard twist and to our recent result giving a full description of L-functions from $\mathcal{S}^\sharp$ of degree 2 and conductor 1. In Section 4 we present sketches of some proofs, while the Appendix contains the construction of the polynomials Ψ_ℓ in (1.7).

2 - Definitions and basic facts

We first recall several definitions which we shall use later on; we write $f(s) \equiv c$ to mean that $f(s) = c$ identically. The *extended Selberg class* $\mathcal{S}^\sharp$ consists of the Dirichlet series

$$F(s) = \sum_{n=1}^{\infty} \frac{a(n)}{n^s},$$

$F(s) \not\equiv 0$ and absolutely convergent for $\sigma > 1$, such that $(s-1)^m F(s)$ is entire of finite order for some integer $m \geq 0$, and satisfying a functional equation of type (1.1). Note that the *conjugate function* $\overline{F}$ has conjugated coefficients $\overline{a(n)}$. The *Selberg class* $\mathcal{S}$ is, roughly speaking, the subclass of $\mathcal{S}^\sharp$ of the functions with

Euler product and satisfying the Ramanujan conjecture $a(n) \ll n^\varepsilon$. As already pointd out in the previous section, we refer to Selberg [**23**], Conrey-Ghosh [**2**] and to our survey papers [**3**], [**5**], [**19**], [**20**], [**21**], [**22**] for further definitions, examples and the basic theory of the Selberg class.

We write m_F for the order of pole of F at $s=1$ and

$$\sum_{m=1}^{m_F} \frac{\gamma_m}{(s-1)^m} \tag{2.1}$$

for its polar part. We remark that since the H-invariants depend only on the data r, Q, λ_j, μ_j of γ-factors (1.2), we may define such invariants for any γ-factor, without referring to functions $F \in \mathcal{S}^\sharp$. More generally, the same holds for any invariant depending only on the data of γ-factors. Clearly, the invariants of a γ-factor $\gamma(s)$ coincide with the corresponding invariants of any $F \in \mathcal{S}^\sharp$ having $\gamma(s)$ as γ-factor. The invariants of γ-factors are usually denoted replacing the suffix F by γ.

For $\alpha > 0$ and $\sigma > 1$ the *standard twist* of $F \in \mathcal{S}^\sharp$ is defined by

$$F(s,\alpha) := \sum_{n=1}^{\infty} \frac{a(n)}{n^s} e(-\alpha n^{1/d}), \qquad e(x) = e^{2\pi i x},$$

and the *spectrum* of F is

$$\mathrm{Spec}(F) = \{\alpha > 0 : a(n_\alpha) \neq 0\} = \Big\{ d\big(\frac{m}{q}\big)^{1/d} : m \in \mathbb{N} \text{ with } a(m) \neq 0 \Big\},$$

where

$$n_\alpha := q d^{-d} \alpha^d \quad \text{and} \quad a(n_\alpha) := 0 \text{ if } n_\alpha \notin \mathbb{N}. \tag{2.2}$$

Finally we recall some basic results from the Selberg class theory. Every $F \in \mathcal{S}^\sharp$ has polynomial growth on vertical strips. Moreover, the standard twist $F(s,\alpha)$ is entire if $\alpha \notin \mathrm{Spec}(F)$, while for $\alpha \in \mathrm{Spec}(F)$ it is meromorphic on $\mathbb{C}$ with at most simple poles at the points $s = s_\ell^*$ in (1.10) with residue denoted by $\rho_\ell(\alpha)$. It is known that all $F \in \mathcal{S}^\sharp$ have $\rho_0(\alpha) \neq 0$ for every $\alpha \in \mathrm{Spec}(F)$. Further, $F(s,\alpha)$ has polynomial growth on every vertical strip. We refer to [**9**], [**12**] and [**13**] for these and other results on $F(s,\alpha)$ and, more generally, on the nonlinear twists of the functions in $\mathcal{S}^\sharp$. We finally recall that $\mathrm{Spec}(F)$ is an infinite set, since the functions F with positive degree cannot be Dirichlet polynomials, and that there are no functions in $\mathcal{S}^\sharp$ with degree $0 < d < 1$, see [**4**]; hence the functions of positive degree actually have $d \geq 1$.

3 - Applications of structural invariants

The first application of the structural invariants deals with the functional equation of the standard twist; we need further notation to state the results. With γ_m as in (2.1) and with the notation in (2.2) and (1.6), we consider the functions

$$R(s,\alpha) = (2\pi i\alpha)^{-ds} \sum_{m=1}^{m_F} d^m \gamma_m \sum_{h=0}^{m-1} \frac{(-1)^h \log^h(2\pi i\alpha)\Gamma^{(m-1-h)}(ds)}{h!(m-1-h)!},$$

hence $R(s,\alpha) \equiv 0$ if $F(s)$ is entire, and for $\ell = 0,1,\dots$

$$\overline{F}_\ell(s,\alpha) = \sum_{j=-N}^{N} a_j e^{i\pi d\omega_j(1-s)} \sum_{n\geq 1}^{\flat} \frac{\overline{a(n)}}{n^s} \left(1 + e^{i\pi(\frac{1}{2}-\omega_j)} \left(\frac{n_\alpha}{n}\right)^{1/d}\right)^{d(1-s-s_\ell^*)}, \tag{3.1}$$

where the symbol $^\flat$ in the inner sum indicates that the term $n = n_\alpha$ is omitted if $j = -N$. Hence $\overline{F}_\ell(s,\alpha)$ is well defined since $1 + e^{i\pi(\frac{1}{2}-\omega_j)} \left(\frac{n_\alpha}{n}\right)^{1/d} \neq 0$ always. Note that if $\alpha \notin \mathrm{Spec}(F)$ we may omit $^\flat$, since $a(n_\alpha) = 0$ in this case. Note also that the inner sum in (3.1) is a general Dirichlet series with complex frequencies, absolutely convergent for $\sigma > 1$, and

$$\left(1 + e^{i\pi(\frac{1}{2}-\omega_j)} \left(\frac{n_\alpha}{n}\right)^{1/d}\right)^{d(1-s-s_\ell^*)} = e^{d(1-s-s_\ell^*)\log\left(1+e^{i\pi(\frac{1}{2}-\omega_j)}\left(\frac{n_\alpha}{n}\right)^{1/d}\right)} \tag{3.2}$$

where the branch of $\log(z)$ for $z \in \overline{\mathbb{H}} \setminus \{0\}$ has argument in $[0,\pi]$, $\mathbb{H}$ being the upper half-plane $\{z \in \mathbb{C} : \Im(z) > 0\}$. Let finally

$$e_\ell = \frac{d-1}{2} + \ell + i\theta_F.$$

With the above notation, the functional equation of the standard twist is given by the following theorem.

Theorem 5. *Let $F \in \mathcal{S}^\sharp$ with $d \geq 1$ and let $\alpha > 0$. Then for any integer $k \geq 0$ and s in the strip $s_{k+1} < \sigma < s_k$ we have*

$$\begin{aligned} F(s,\alpha) = \frac{\omega_F}{\sqrt{2\pi}} \left(\frac{q^{1/d}}{2\pi d}\right)^{d(\frac{1}{2}-s)} &\sum_{\ell=0}^{k} d_F(\ell)\Gamma\big(d(1-s)-e_\ell\big)\overline{F}_\ell(1-s,\alpha) \\ &+ R(1-s,\alpha) + H_k(s,\alpha), \end{aligned} \tag{3.3}$$

where the function $H_k(s,\alpha)$ is holomorphic in the above strip and meromorphic over $\mathbb{C}$ with all poles in a horizontal strip of bounded height. Moreover, there exists $\theta = \theta(d) > 0$ such that for any $\sigma \in [s_{k+1}, s_k] \cap (-\infty, 0)$ we have

$$H_k(s,\alpha) \ll |t|^{-\theta} \qquad\qquad \text{as} \quad |t| \to \infty.$$

Note that functional equation (3.3) is a kind of general form of the functional equation satisfied by the Hurwitz-Lerch zeta functions; indeed, the standard twist of the Riemann zeta function corresponds to a special case of such zeta functions. Actually, such a similarity holds for all $F \in \mathcal{S}^\sharp$ of degree $d = 1$, since their standard twists are again related with the Hurwitz-Lerch zeta functions. However, the proof of Theorem 5 goes along different lines, and will be outlined in the next section. The function $R(1-s,\alpha)$ is present in (3.3) only if F has a pole at $s = 1$, while the terms $H_k(s,\alpha)$ may be regarded as error terms. These terms are not present in degree 1 as well as in other special cases with degree $d > 1$, for example for the L-functions associated with half-integral weight cusp forms; see [**15**], which also includes a discussion of the cases where such a simpler form holds. Actually, if there exists an integer $h \geq 0$ such that $H_k(s,\alpha) \equiv 0$ for every $k \geq h$ and $\alpha > 0$ we say that $F(s,\alpha)$ satisfies a *strict functional equation.* In Theorem 4 of [**16**] we showed that a strict functional equation exists if and only if there are only finitely many nonvanishing structural invariants or, equivalently, if and only if F has a γ-factor of type

$$\gamma(s) = Q^s \prod_{j=1}^{N} \Gamma\Big(\frac{d}{2N}s + \frac{2n_j - d - 1}{4N}\Big)$$

with $Q > 0$, $N \geq 1$ and suitable integers n_j. It is an interesting open question to establish if there exist L-functions, other than the above mentioned ones, having a γ-factor of this type.

Theorem 5 is complemented by the following result, giving the properties of the functions $\overline{F}_\ell(s,\alpha)$ involved in (3.3). As one can guess from the definition in (3.1), after expanding the left hand side of (3.2) these functions are close to suitable "stratifications" of $\overline{F}(s)$.

Theorem 6. *Let $F \in \mathcal{S}^\sharp$ with degree $d \geq 1$, $\alpha > 0$ and $\ell = 0,1,2,\dots$ Then $\overline{F}_\ell(s,\alpha)$ is an entire function, not identically vanishing. Moreover, uniformly for σ in any bounded interval, as $|t| \to \infty$ we have*

$$\overline{F}_\ell(s,\alpha) \ll e^{\frac{\pi}{2}d|t|}|t|^{c(\sigma)}$$

with a certain $c(\sigma) \geq 0$ independent of ℓ and α, satisfying $c(\sigma) = 0$ for $\sigma > 1$.

We are not going to present the proof of Theorem 6, for which we refer to Section 2 of [**16**]. From the proof of Theorem 5 we obtain the following explicit expression for the residues $\rho_\ell(\alpha)$ of $F(s,\alpha)$ at the potential poles s_ℓ^* in (1.10), when $\alpha \in \mathrm{Spec}(F)$.

Theorem 7. *Let $F \in \mathcal{S}^\sharp$ with $d \geq 1$, $\alpha \in \mathrm{Spec}(F)$ and $\ell = 0, 1, \ldots$. Then*

$$\rho_\ell(\alpha) = \frac{d_F(\ell)}{d} \frac{\omega_F}{\sqrt{2\pi}} e^{-i\frac{\pi}{2}(\xi_F + ds_\ell^*)} \left(\frac{q^{1/d}}{2\pi d}\right)^{\frac{d}{2}-ds_\ell^*} \frac{\overline{a(n_\alpha)}}{n_\alpha^{1-s_\ell^*}}.$$

In particular, the set of poles of $F(s,\alpha)$ is independent of α and equals $\{s_\ell^* : d_F(\ell) \neq 0\}$.

Although obtained as a by-product of the proof of Theorem 5, the explicit expression of $\rho_\ell(\alpha)$ in Theorem 7 is important in various situations, since it connects the polar structure of the standard twist to the structural invariants. Again, we are not going to present the proof of Theorem 7, since it requires entering the details of the proof of Theorem 5; the interested reader is referred to Section 3.7 of [**16**].

Finally we turn to the application of structural invariants to the classification of the Selberg class. It is generally expected that the class $\mathcal{S}$ coincides with the class of automorphic L-functions, but a proof of this statement appears to be very difficult at present. In particular, it is expected that there exist no $F \in \mathcal{S}$ with degree $d \notin \mathbb{N}$; this is known as the *degree conjecture* and is expected to hold in the wider setting of the extended class $\mathcal{S}^\sharp$. The degree conjecture for $\mathcal{S}^\sharp$ is known for degrees $0 < d < 1$, see e.g. Conrey-Ghosh [**2**] and our paper [**4**]. A short proof in this case follows at once from the polar structure of the standard twist $F(s,\alpha)$. Indeed, for $\alpha \in \mathrm{Spec}(F)$, $F(s,\alpha)$ has a pole on the line $\sigma = (d+1)/(2d) > 1$ for $0 < d < 1$, a contradiction. The next case $1 < d < 2$ is definitely more difficult and was first settled in [**10**], after partial results in [**8**]; a shorter proof has been recently devised by Balasubramanian-Raghunathan [**1**]. The classification of the functions of degree $= 1$ in $\mathcal{S}$ and $\mathcal{S}^\sharp$ has been obtained in [**4**]; in the case of $\mathcal{S}$, it turns out that such functions are the Riemann zeta function and the shifts of the Dirichlet L-functions with primitive characters. Another proof was obtained by Soundararajan [**24**]. The next open case is $d = 2$. Here one expects that the members of $\mathcal{S}$ are the L-functions associated with the Hecke and Maass eigenforms of any level, while there is no standard guess on the nature of the functions of degree $d = 2$ in $\mathcal{S}^\sharp$; note that the level of a form coincides with the conductor of the associated L-function. Recently, in [**17**] we classified the functions in $\mathcal{S}^\sharp$ with degree $d = 2$ and conductor $q = 1$; a first step, dealing only with the class $\mathcal{S}$, was taken in [**11**] using different arguments.

In order to state our last result we need to introduce a normalization. We say that a function $F \in \mathcal{S}^\sharp$ is *normalized* if its internal shift θ_F vanishes and the first nonvanishing Dirichlet coefficient equals 1. Normalized functions have two nice properties. Indeed, on the one hand every $F \in \mathcal{S}^\sharp$ with $d = 2$ and $q = 1$ can be normalized by means of a simple procedure, so we may consider only such functions without loosing generality. On the other hand, it turns out that normalized functions have real coefficients, hence the functional equation reflects $F(s)$ into $F(1-s)$ rather than into $\overline{F}(1-s)$ as in the general case. This is important, since the functional equation of the L-functions $L(s)$ of the level 1 forms, once suitably normalized to fit $\mathcal{S}^\sharp$, reflects $L(s)$ into $L(1-s)$.

The classification of the functions in $\mathcal{S}^\sharp$ with degree $d = 2$ and conductor $q = 1$ is carried out by means of the numerical invariant

$$\chi_F = H_F(1) + H_F(2) + 2/3,$$

where the $H_F(n)$ are the H-invariants defined by (1.4). For example, a simple computation shows that

$$\chi_F = \frac{(k-1)^2}{2} \quad \text{or} \quad \chi_F = -2\kappa^2$$

when, respectively, $F(s) = L(s + \frac{k-1}{2}, f)$ with a holomorphic cusp form f of level 1, weight k and first nonvanishing Fourier coefficient 1, or $F(s) = L(s, u)$ with a Maass form u of level 1, weight 0, eigenvalue $1/4 + \kappa^2$ and first Fourier-Bessel coefficient 1. Our result shows, conversely, that the value of χ_F detects the nature of any normalized $F \in \mathcal{S}^\sharp$ of degree 2 and conductor 1.

Theorem 8. *Let $F \in \mathcal{S}^\sharp$ of degree 2 and conductor 1 be normalized. Then $\chi_F \in \mathbb{R}$ and*

(i) *if $\chi_F > 0$ then there exists a holomorphic cusp form f of level 1 and even integral weight $k = 1 + \sqrt{2\chi_F}$ such that $F(s) = L(s + \frac{k-1}{2}, f)$;*

(ii) *if $\chi_F = 0$ then $F(s) = \zeta(s)^2$;*

(iii) *if $\chi_F < 0$ then there exists a Maass form u of level 1, weight 0 and with eigenvalue $1/4 + \kappa^2 = (1 - 2\chi_F)/4$ such that $F(s) = L(s, u)$.*

In case (iii) we can specify the parity ε of u by means of the root number ω_F of F, namely

$$\varepsilon = \frac{1 - \omega_F}{2}.$$

Clearly, if $F \in \mathcal{S}^\sharp$ with $d = 2$ and $q = 1$ is not normalized, we may first normalize it and then use Theorem 8 to detect its nature. Therefore, every such F is closely related to one of the L-functions in Theorem 8. Moreover, it follows that every $F \in \mathcal{S}$ as in Theorem 8 is an automorphic L-function. We also remark that quite possibly the method of proof of Theorem 8 can be extended to cover the case of other small integer moduli $q > 1$.

As already pointed out, the structural invariants $d_F(\ell)$ and the explicit expression for the residues $\rho_\ell(\alpha)$ in Theorem 7 play an important role in the proof of Theorem 8, an outline of which is given in the next section. We conclude by remarking that a new and crucial ingredient in the proof of Theorem 8 is that the structural invariants lie, in the special case at hand, on a certain *universal* family of algebraic varieties; see Section 4.3 for some details. We guess that a similar phenomenon should hold in general for the functions $F \in \mathcal{S}^\sharp$, i.e. the invariants $d_F(\ell)$ should lie on certain algebraic varieties to a large extent independent of F. If true, this could explain why L-functions satisfy only functional equations with very special Γ-factors, and in particular could shed some light on the general structure of the Selberg class.

4 - Outline of the proof of Theorems 4, 5 and 8

In this section we give a sketch of the proof of Theorem 4 and outline those of Theorems 5 and 8. We refer to the original papers for detailed proofs of Theorems 5 and 8; precisely, Theorem 5 is Theorem 2 in [**16**] while Theorem 8 is Theorem 1.1 in [**17**]. Since in this paper we changed the definition of the structural invariants $d_F(\ell)$, using (1.7) rather than the asymptotic expansion (1.9) as in [**16**], strictly speaking Theorem 4 is a new result. However, such expansion of $h_F(s)$ is actually the same as that in Section 1.3 of [**16**].

4.1 - *Sketch of the proof of Theorem 4*

The proof is based on Stirling's formula, which we write in the form

$$\log\Gamma(s+a) \approx (s+a-\frac{1}{2})s\log s - s + \frac{1}{2}\log(2\pi) + \sum_{\nu=1}^{\infty}\frac{(-1)^{\nu+1}B_{\nu+1}(a)}{\nu(\nu+1)}\frac{1}{s^\nu}$$

uniformly for $|\arg(s)| < \pi-\delta$ for every $0<\delta<\pi$, where $\approx$ has the same meaning as in Theorem 4. Hence for $|\arg(-s)| < \pi-\delta$ we have

(4.1)

$$\begin{aligned}\log(\overline{\omega}h_F(s)) &= (1-2s)\log Q - r\log(2\pi)\\ &\quad+\sum_{j=1}^{r}\big(\log\Gamma(-\lambda_j s+\lambda_j+\overline{\mu}_j)+\log\Gamma(-\lambda_j s+1-\mu_j)\big)\\ &\approx (\frac{1}{2}-s)\log\left(\frac{q}{(2\pi)^d}\right)-2i\Im\sum_{j=1}^{r}\mu_j\log\lambda_j+(-ds+\frac{1}{2}d-id\theta_F)\log(-s)+ds\\ &\quad-\sum_{\nu=1}^{\infty}\frac{1}{\nu(\nu+1)}\sum_{j=1}^{r}\left(\frac{B_{\nu+1}(\lambda_j+\overline{\mu}_j)}{\lambda_j^{\nu}}+\frac{B_{\nu+1}(1-\mu_j)}{\lambda_j^{\nu}}\right)\frac{1}{s^{\nu}}.\end{aligned}$$

Using the well-known formulae

$$B_\nu(x+y)=\sum_{k=0}^{\nu}\binom{\nu}{k}B_k(x)y^{\nu-k}\quad\text{and}\quad B_\nu(1-x)=(-1)^\nu B_\nu(x)$$

we see that the double sum in (4.1) equals

$$\frac{1}{2}\sum_{\nu=1}^{\infty}\frac{1}{\nu(\nu+1)}\left((-1)^{\nu+1}H_F(\nu+1))+\sum_{k=1}^{\nu+1}\binom{\nu+1}{k}\overline{H_F(k)}+d\right)\frac{1}{s^{\nu}}. \tag{4.2}$$

Moreover we have the expansion

$$\begin{aligned}&\log\Gamma(-ds+\frac{d+1}{2}-id\theta)\\ &\approx(-ds+\frac{d}{2}-id\theta)\log(-s)+ds(-ds+\frac{d}{2}-id\theta)\log d+\frac{1}{2}\log(2\pi)\\ &\quad-\sum_{\nu=1}^{\infty}\frac{1}{\nu(\nu+1)}\frac{B_{\nu+1}\left(\frac{d+1}{2}-id\theta\right)}{d^{\nu}}\frac{1}{s^{\nu}}.\end{aligned}\tag{4.3}$$

Gathering (4.1)-(4.3) we obtain the formula

$$\begin{aligned}h_F(s)&\approx\frac{\omega_F d^{id\theta}}{\sqrt{2\pi}}\left(\frac{q}{(2\pi d)^d}\right)^{\frac{1}{2}-s}\\ &\quad\Gamma(d(s_d^*-s))\exp\left(\sum_{\nu=1}^{\infty}\frac{P_\nu(H_F(\nu+1),\overline{H_F(1)},\dots,\overline{H_F(\nu+1)})}{s^{\nu}}\right),\end{aligned}$$

where the polynomials P_ν are defined in (5.3) below and have hidden parameters $d=d_F$ and $\theta=\theta_F$. Applying the power series expansion of the exponential function we rewrite the last formula as

(4.4)
$$h_F(s) \approx \frac{\omega_F d^{id\theta}}{\sqrt{2\pi}} \left(\frac{q}{(2\pi d)^d}\right)^{\frac{1}{2}-s} \Gamma(d(s_d^* - s))$$
$$\times \sum_{N=0}^{\infty} \frac{V_N(H_F(2),\dots,H_F(N+1),\overline{H_F(1)},\dots,\overline{H_F(N+1)})}{s^N},$$

where $V_0 \equiv 1$ and for $N \geq 1$ the polynomials V_N are defined as in (5.4) below, again with hidden parameters $d = d_F$ and $\theta = \theta_F$. Using the Lemma in the Appendix we obtain

$$\sum_{N=1}^{\infty} \frac{V_N}{s^N} \approx \sum_{\ell=1}^{\infty} \frac{(-1)^\ell R_\ell(V_1,\dots,V_\ell)}{(ds+1-ds_0^*)_\ell} \tag{4.5}$$

with polynomials R_ℓ as in such Lemma, with parameters $a = d$ and $b = 1 - ds_0^*$. Moreover,

$$\Gamma(d(s_\ell^* - s)) = \frac{(-1)^\ell \Gamma(d(s_0^* - s))}{(ds+1-ds_0^*)_\ell}. \tag{4.6}$$

Finally, gathering (4.4)-(4.6) and recalling the definition of the polynomials Ψ_ℓ in (5.5) and of the structural invariants in (1.7), we arrive at (1.9) and the result follows. □

4.2 - *Outline of the proof of Theorem 5*

For simplicity we assume that $\theta_F = 0$. We start with the smoothed standard twist

$$F_X(s,\alpha) = \sum_{n=1}^{\infty} \frac{a(n)}{n^s} e^{-n^{1/d} z_X(\alpha)},$$

where $X > 1$ is sufficiently large, $\alpha > 0$ and

$$z_X(\alpha) = \frac{1}{X} + 2\pi i\alpha. \tag{4.7}$$

Clearly, $F_X(s,\alpha)$ is absolutely convergent over $\mathbb{C}$ and for every $\alpha > 0$ we have

$$\lim_{X\to\infty} F_X(s,\alpha) = F(s,\alpha) \qquad \text{for } \sigma > 1.$$

Our aim is to obtain a suitable expression for $F_X(s,\alpha)$ and then to investigate the limit as $X \to \infty$ for s in certain regions inside the half-plane $\sigma < 1$.

For $-c<\sigma<2$, where $c>0$ is sufficiently large, by Mellin's transform we have

$$F_X(s,\alpha)=\frac{1}{2\pi i}\int_{(d(c+2))}F(s+\frac{w}{d})\Gamma(w)z_X(\alpha)^{-w}\mathrm{d}w.$$

For $k\geq 0$ we consider the ranges

$$\sigma\in\mathcal{I}_k,\ \text{where } \mathcal{I}_k \text{ is an arbitrary compact subinterval of } (-c,s_k). \tag{4.8}$$

For s as in (4.8) and a suitably chosen $u_k\in\mathbb{R}$, we shift the line of integration in the above integral to $\Re(w)=u_k$, apply the functional equation of F in the form (1.5), use the Dirichlet series expansion of $\overline{F}(1-s-w/d)$ and switch summation and integration, thus obtaining that

$$\begin{aligned}F_X(s,\alpha)=\sum_{n=1}^{\infty}\frac{\overline{a(n)}}{n^{1-s}}\frac{1}{2\pi i}\int_{(u_k)}h_F(s+\frac{w}{d})S_F(s+\frac{w}{d})\Gamma(w)\big(\frac{z_X(\alpha)}{n^{1/d}}\big)^{-w}\mathrm{d}w\\ +R_X(1-s,\alpha)+R_{k,X}(s,\alpha),\end{aligned}\tag{4.9}$$

where $R_X(1-s,\alpha)$ is the residue at $w=d(1-s)$ and $R_{k,X}(s,\alpha)$ is the sum of the residues at $w=-\nu$ with $0\leq\nu<ds_k$.

Next we plug into the integral (4.9) the asymptotic expansion of $h_F(s)$ in Theorem 1 and get

$$\begin{aligned}F_X(s,\alpha)=\frac{\omega_F}{\sqrt{2\pi}}\left(\frac{q^{1/d}}{2\pi d}\right)^{\frac{d}{2}-ds}\sum_{\ell=0}^{M}d_F(\ell)\sum_{j=-N}^{N}a_je^{i\pi d\omega_j s}\sum_{n=1}^{\infty}\frac{\overline{a(n)}}{n^{1-s}}\\ \times\frac{1}{2\pi i}\int_{(u_k)}\Gamma\big(d(s_\ell-s)-w\big)\Gamma(w)z_{j,X}(\alpha,n)^{-w}\mathrm{d}w\\ +H_{M,X}^{(u_k)}(s,\alpha)+R_X(1-s,\alpha)+R_{k,X}(s,\alpha)\end{aligned}\tag{4.10}$$

where, recalling (4.7),

$$z_{j,X}(\alpha,n)=\frac{q^{1/d}z_X(\alpha)e^{-i\pi\omega_j}}{2\pi d n^{1/d}}\tag{4.11}$$

and $H_{M,X}^{(u_k)}(s,\alpha)$ is the term coming from the error arising after cutting at $\ell=M$ the asymptotic expansion of $h_F(s)$. Note that the integral in (4.10) is of type

$$\frac{1}{2\pi i}\int_{(c)}\Gamma(\xi-w)\Gamma(w)\eta^{-w}\mathrm{d}w=\Gamma(\xi)(1+\eta)^{-\xi},\tag{4.12}$$

valid under the conditions

$$0<c<\Re(\xi)\quad\text{and}\quad|\arg\eta|<\pi.$$

Actually, the asymptotic expansion of $h_F(s)$ was specially designed to fit (4.12), which represents one of the few cases of a Mellin-Barnes integral with an explicit expression in terms of elementary functions. Plugging (4.12) into (4.10), after a series of computations we arrive to the following expression

$$F_X(s,\alpha) = \frac{\omega_F}{\sqrt{2\pi}}\left(\frac{q^{1/d}}{2\pi d}\right)^{\frac{d}{2}-ds}\sum_{\ell=0}^{k} d_F(\ell)\Gamma\big(d(s_\ell - s)\big)\overline{F}^*_{\ell,X}(1-s,\alpha) + R_X(1-s,\alpha) + H_{k,X}(s,\alpha) \tag{4.13}$$

valid for $k \geq 0$ and $\sigma \in \mathcal{I}_k \cap (-\infty, -2\delta)$, $\delta > 0$ being sufficiently small, where

$$\overline{F}^*_{\ell,X}(1-s,\alpha) = \sum_{j=-N}^{N} a_j e^{i\pi d\omega_j s}\sum_{n=1}^{\infty}\frac{\overline{a(n)}}{n^{1-s}}(1+z_{j,X}(\alpha,n))^{d(s-s_\ell)} \tag{4.14}$$

and $H_{k,X}(s,\alpha)$ comes from a suitable treatment of error terms. This is the expression for $F_X(s,\alpha)$ alluded to at the beginning of this proof.

The next step is to let $X \to \infty$ in (4.13), but this requires some care. Indeed, as we already pointed out, the limit of $F_X(s,\alpha)$ is $F(s,\alpha)$ when $\sigma > 1$, but (4.13) holds in the range $\mathcal{I}_k \cap (-\infty,-2\delta)$. Moreover, in view of (4.11), the limit of the terms $(1+z_{j,X}(\alpha,n))^{d(s-s_\ell)}$ in (4.14) is not always well defined, since the term $1+z_{j,X}(\alpha,n)$ vanishes as $X\to\infty$ when $\alpha \in \mathrm{Spec}(F)$, $n = n_\alpha$ and $j = -N$; we call it the *critical term*. To overcome these problems we first compute $F_X(s,\alpha)$ in a different way, looking at it as the twist of $F(s,\alpha)$ by $e^{-n^{1/d}/X}$ and using its expression by means of Mellin's transform. After a suitable shift of the integration line, this gives

$$F_X(s,\alpha) = F(s,\alpha) + \Sigma_X(s,\alpha) + I_X(s,\alpha), \tag{4.15}$$

where $I_X(s,\alpha)$ is a harmless term and

$$\Sigma_X(s,\alpha) = \sum_{\ell=0}^{k} d\rho_\ell(\alpha)\Gamma(d(s_\ell - s))X^{d(s_\ell - s)}. \tag{4.16}$$

The term (4.16) is very important since eventually will cancel the contribution of the critical term in (4.14), thus allowing to let $X\to\infty$. Indeed, comparing (4.13) with (4.15) and computing separately the contribution of the critical term in (4.14), we obtain

$$F(s,\alpha) = \frac{\omega_F}{\sqrt{2\pi}}\left(\frac{q^{1/d}}{2\pi d}\right)^{\frac{d}{2}-ds}\sum_{\ell=0}^{k} d_F(\ell)\Gamma(d(s_\ell - s))\overline{F}_{\ell,X}(1-s,\alpha) + R_X(1-s,\alpha) + H_{k,X}(s,\alpha) - I_X(s,\alpha) + \widetilde{\Sigma}_X(s,\alpha) - \Sigma_X(s,\alpha), \tag{4.17}$$

where $\widetilde{\Sigma}_X(s,\alpha)$ comes from the critical term and $\overline{F}_{\ell,X}(1-s,\alpha)$ equals $\overline{F}^*_{\ell,X}(1-s,\alpha)$ minus the critical term.

Now we are ready to let $X\to\infty$. It is not difficult to show that, as $X\to\infty$, the sum on the right hand side of (4.17) tends to the corresponding sum in (3.3), $I_X(s,\alpha)\to 0$ and $R_X(1-s,\alpha)\to R(1-s,\alpha)$. Moreover, two technical lemmas, see Lemmas 3.1 and 3.2 in [**16**], show that $H_{k,X}(s,\alpha)\to H_k(s,\alpha)$ and $H_k(s,\alpha)$ satisfies the required bounds. Finally, the remaining two terms in (4.17) are first rewritten in the form

$$\Sigma_X(s,\alpha)=\sum_{\ell=0}^{k} a_\ell(s,\alpha)X^{d(s_\ell-s)} \quad\text{and}\quad \widetilde{\Sigma}_X(s,\alpha)=\sum_{\ell=0}^{k}\widetilde{a}_\ell(s,\alpha)X^{d(s_\ell-s)},$$

and then shown to be equal by an inductive argument. Hence these terms cancel in (4.17), thus concluding the proof of Theorem 5. □

4.3 - *Outline of the proof of Theorem 8*

A good deal of information about invariants comes from the transformation formula for nonlinear twists, which we studied in [**10**], [**11**], [**13**], [**14**]. Indeed, roughly speaking, the transformation formula gives different outputs if the same nonlinear twist of a function $F\in\mathcal{S}^\sharp$ is written in formally different ways, and this phenomenon imposes constraints on the invariants. So we start the proof of Theorem 8 with a fully explicit version of such transformation formula for the following nonlinear twist

$$F(s;f)=\sum_{n=1}^{\infty}\frac{a(n)}{n^s}e(-f(n,\alpha)), \qquad f(n,\alpha)=n+\alpha\sqrt{n},$$

in the special case of normalized $F\in\mathcal{S}^\sharp$ with $d=2$ and $q=1$. This is done in Lemma 4.2 of [**17**] and requires delicate computations. Since $F(s;f)$ coincides with the standard twist $F(s,\alpha)$ thanks to the periodicity of the complex exponential, the output of the transformation formula is an identity of type

$$F(s,\alpha)=\sum_{m=0}^{M}W_m(s,\alpha)F\big(s+\frac{m}{2},\alpha\big)+H_M(s,\alpha), \tag{4.18}$$

where $\alpha\in\mathrm{Spec}(F)$, $M\geq 0$ is any integer and $H_M(s,\alpha)$ is holomorphic for $\sigma>-(M-1)/2$. Moreover, the functions $W_m(s,\alpha)$ with $m\geq 1$ are quite complicated but explicit polynomials in the (s,α)-variables whose coefficients involve, among others, the structural invariants $d_F(\ell)$, while $W_0(s,\alpha)\equiv 1$.

As a consequence, (4.18) shows that the sum of the terms from 1 to M is holomorphic for $\sigma > -(M-1)/2$. Thus, in particular, its residue at $s = s_M$ (see (1.10)) vanishes. Recalling the polar structure of the standard twist, this gives at once that

$$\sum_{m=1}^{M} W_m(s_M, \alpha)\rho_{M-m}(\alpha) = 0. \tag{4.19}$$

But, thanks to Theorem 7, in the case at hand the residues $\rho_{M-m}(\alpha)$ have a simple explicit expression, again involving the structural invariants. Therefore, after a careful computation, from (4.19) we deduce that for any $N \geq 2$ the structural invariants satisfy

$$Q_N\big(d_F(0), \dots, d_F(N)\big) = 0, \tag{4.20}$$

where $Q_N(X_0, \dots, X_N)$ are certain quadratic forms independent of F; see Proposition 4.1 in [**17**]. This is the universal family of algebraic varieties alluded to in the last remark of Section 3. Since $d_F(0) = 1$, (4.20) and the special shape of these quadratic forms allow to express any $d_F(\ell)$ with $\ell \geq 2$ in terms of $d_F(1)$, by an algorithm completely independent of F. Hence all $d_F(\ell)$ are determined by $d_F(1)$, and a further computation shows that

$$d_F(1) = \chi_F - \frac{1}{8}. \tag{4.21}$$

Moreover, $d_F(\ell) \in \mathbb{R}$ for every ℓ in the case at hand.

The next step of the proof is the introduction of the *virtual γ-factors*

$$\gamma(s) = \begin{cases} (2\pi)^{-s}\Gamma(s+\mu) & \text{with } \mu > 0 \\ \pi^{-s}\Gamma\Big(\dfrac{s+\varepsilon+i\kappa}{2}\Big)\Gamma\Big(\dfrac{s+\varepsilon-i\kappa}{2}\Big) & \text{with } \varepsilon \in \{0,1\} \text{ and } \kappa \geq 0, \end{cases} \tag{4.22}$$

which clearly are the γ-factors of the L-functions associated with the normalized Hecke and Maass forms, respectively. Denoting the invariants related to such γ-factors by means of the suffix γ, a computation shows that

$$\chi_\gamma = \begin{cases} 2\mu^2 \\ -2\kappa^2. \end{cases} \tag{4.23}$$

Although not every virtual γ-factor corresponds to an existing L-function, it turns out that the invariants $d_\gamma(\ell)$ satisfy the same properties of the $d_F(\ell)$. Hence by (4.23) and the γ-analog of (4.21) we have that the set $\{d_\gamma(1)$:

γ virtual γ-factor} coincides with $\mathbb{R}$. Thus to any F we associate a unique virtual γ-factor such that $d_\gamma(1) = d_F(1)$, and hence by the above reported properties we have that $d_\gamma(\ell) = d_F(\ell)$ for every $\ell \geq 0$. In turn, this implies that $h_F(s) = \omega_F h_\gamma(s)$. Thus, in view of (1.5), F satisfies the functional equation

$$\gamma(s)F(s) = \omega_F R(s)\gamma(1-s)F(1-s), \qquad R(s) = \frac{S_F(s)}{S_\gamma(s)}, \tag{4.24}$$

where γ is the virtual γ-factor associated with F. Now we observe that if $R(s) \equiv 1$, then (4.24) becomes a functional equation of Hecke or Maass type, thus Theorem 8 follows at once from the classical converse theorems of Hecke and Maass.

The last step is therefore proving that $R(s) \equiv 1$. This step is quite technical and involves a non-standard use of certain *period functions*, in the sense of Lewis-Zagier [**18**]. Indeed, given F and its virtual γ-factor, we consider the Fourier series

$$f(z) = \sum_{n=1}^{\infty} a(n)n^\lambda e(nz), \qquad z \in \mathbb{H},$$

where $\lambda = \mu$ or $\lambda = i\kappa$ according to (4.22), and its period function

$$\psi(z) = f(z) - z^{-2\lambda-1} f(-1/z).$$

Then we proceed to the analysis of the function $f(z)$, involving the use the properties of certain Mittag-Leffler functions and of the three-term functional equation

$$\psi(z) = \psi(z+1) + (z+1)^{-2\lambda-1}\psi\big(\frac{z}{z+1}\big)$$

satisfied by $\psi(z)$. We refer to Sections 6 and 7 of [**17**] for the rather tricky and delicate arguments leading to the conclusion of the proof. Here we only recall that, assuming by contradiction $R(s) \not\equiv 1$, the aim of such arguments is to obtain the analytic continuation of $f(z)$ to a region of type $-\delta_0 < \arg(z) < \pi$ with some $\delta_0 > 0$. But $f(z)$ is periodic of period 1, so it is in fact an entire function. On the other hand it is not difficult to prove that $f(z)$ cannot be entire, and this contradiction concludes the proof of Theorem 8. □

5 - Appendix: construction of the polynomials Ψ_ℓ

We need the following lemma. Note that, notwithstanding a similar notation, the polynomials Q_N in the lemma are different from the quadratic forms in (4.20). We shall use the Pochhammer symbol $(z)_\ell$, defined by

$$(z)_\ell = \begin{cases} 1 & \text{if } \ell = 0 \\ z(z+1)\dots(z+\ell-1) & \text{if } \ell > 0. \end{cases}$$

Lemma. *Let a and b be fixed and not simultaneously vanishing. Then for every $N \geq 1$ there exist polynomials*

$$R_\ell(X_1, \dots, X_\ell) \qquad (1 \leq \ell \leq N)$$

and a polynomial

$$Q_N(X_1, \dots, X_N, T) \quad \textit{with} \quad \deg_T Q_N(X_1, \dots, X_N, T) \leq N-1,$$

such that

$$\sum_{k=1}^{N} \frac{X_k}{T^k} = \sum_{\ell=1}^{N} \frac{(-1)^\ell R_\ell(X_1, \dots, X_\ell)}{(aT+b)_\ell} + \frac{Q_N(X_1, \dots, X_N, T)}{T^N (aT+b)_N}. \tag{5.1}$$

The polynomials R_ℓ and Q_N depend only on a and b and are uniquely determined.

Proof. We define the polynomials R_ℓ, $1 \leq \ell \leq N$, and Q_N inductively. For N=1 we set

$$R_1(X_1) := -aX_1 \quad \text{and} \quad Q_1(X_1, T) := bX_1.$$

Now suppose, for $N \geq 2$, that polynomials $R_\ell(X_1, \dots, X_\ell)$, $1 \leq \ell \leq N-1$, and

$$Q_{N-1}(X_1, \dots, X_{N-1}, T) = E_{N-1}(X_1, \dots, X_{N-1}) T^{N-2} + \dots + E_1(X_1, \dots, X_{N-1})$$

satisfying (5.1) are already defined. Then we put

$$R_N(X_1, \dots, X_N) := (-1)^N \left(a^N X_N + aE_{N-1}(X_1, \dots, X_{N-1})\right)$$

and

$$\begin{aligned} Q_N(X_1, \dots, X_N, T) := {} & Q_{N-1}(X_1, \dots, X_{N-1}, T)T(aT+b+N-1) \\ & + X_N(aT+b)_N - (-1)^N R_N(X_1, \dots, X_N)T^N. \end{aligned}$$

Then $\deg_T Q_N(X_1, \dots, X_N, T) \leq N-1$ and (5.1) holds. This shows the existence of R_ℓ, $1 \leq \ell \leq N$, and Q_N for all $N \geq 1$.

To prove the uniqueness let us assume by contradiction that for a certain N we have two representations of type (5.1), say

$$\sum_{k=1}^{N}\frac{X_k}{T^k}=\sum_{\ell=1}^{N}\frac{(-1)^\ell R_\ell^{(j)}(X_1,\ldots,X_\ell)}{(aT+b)_\ell}+\frac{Q_N^{(j)}(X_1,\ldots,X_N,T)}{T^N(aT+b)_N}, \tag{5.2}$$

with $j=1,2$. We may suppose that N is minimal with this property. Then $R_\ell^{(1)}(X_1,\ldots,X_\ell)=R_\ell^{(2)}(X_1,\ldots,X_\ell)$ for $1\le\ell\le N-1$ but $R_N^{(1)}(X_1,\ldots,X_N)\neq R_N^{(2)}(X_1,\ldots,X_N)$. Subtracting both sides of (5.2) we therefore obtain

$$\begin{aligned}Q_N^{(1)}&(X_1,\ldots,X_N,T)-Q_N^{(2)}(X_1,\ldots,X_N,T)\\&=(-1)^NT^N\left(R_N^{(2)}(X_1,\ldots,X_N)-R_N^{(1)}(X_1,\ldots,X_N)\right).\end{aligned}$$

Thus

$$\begin{aligned}N=\deg_T\left((-1)^NT^N(R_N^{(2)}-R_N^{(1)})\right)&=\deg_T(Q_N^{(1)}-Q_N^{(2)})\\&\le\max(\deg_T Q_N^{(1)},\deg_T Q_N^{(2)})\le N-1,\end{aligned}$$

a contradiction proving the lemma. □

If we want to stress dependence on a and b we write

$$R_\ell(X_1,\ldots X_\ell)=R_\ell(X_1,\ldots X_\ell;a,b).$$

Let now $d>0$ and θ be real parameters. For $\nu\ge1$ we define the following polynomials, depending on d and θ,

$$\begin{aligned}P_\nu&(X,Y_1,\ldots,Y_{\nu+1})=P_\nu(X,Y_1,\ldots,Y_{\nu+1};d,\theta)\\&:=\frac{1}{2\nu(\nu+1)}\left((-1)^\nu X-\sum_{k=1}^{\nu+1}\binom{\nu+1}{k}X_k+c_0(\nu,d,\theta)\right),\end{aligned}\tag{5.3}$$

where

$$c_0(\nu,d,\theta):=\frac{2B_{\nu+1}(ds_0^*)}{d^\nu}-d\quad\text{and}\quad s_0^*=\frac{d+1}{2d}-i\theta.$$

Moreover, for $N\ge1$ let

$$\begin{aligned}V_N&(X_1,\ldots,X_N,Y_1,\ldots,Y_{N+1})=V_N(X_1,\ldots,X_N,Y_1,\ldots,Y_{N+1};d,\theta)\\&:=\sum_{1\le m\le N}\frac{1}{m!}\sum_{\substack{\nu_1\ge1\\ \nu_1+\ldots+\nu_m=N}}\cdots\sum_{\nu_m\ge1}\prod_{j=1}^{m}P_{\nu_j}(X_{\nu_j},Y_1,\ldots,Y_{\nu_j+1};d,\theta).\end{aligned}\tag{5.4}$$

With this notation we can finally define the polynomials Ψ_ℓ, $\ell \geq 0$, setting

$$\Psi_0(Y_1) := d^{id\theta}$$

and for $\ell \geq 1$

$$\begin{aligned} \Psi_\ell(X_1,\dots,X_\ell,Y_1,\dots,Y_{\ell+1}) &= \Psi_\ell(X_1,\dots,X_\ell,Y_1,\dots,Y_{\ell+1},d,\theta) \\ &:= d^{id\theta} R_\ell(V_1(X_1,Y_1,Y_2),\dots,V_\ell(X_1,\dots,X_\ell,Y_1,\dots,Y_{\ell+1})), \end{aligned} \tag{5.5}$$

where the R_ℓ's are as in the Lemma with parameters $a = d$ and $b = 1 - ds_0^*$, whereas the polynomials V_k, $1 \leq k \leq \ell$, are computed with parameters d and θ.

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References

[**1**] R. Balasubramanian and R. Raghunathan, *Beyond the extended Selberg class:* $1 < d_F < 2$, arXiv:2011.07525, preprint, 2020.

[**2**] J. B. Conrey and A. Ghosh, *On the Selberg class of Dirichlet series: small degrees*, Duke Math. J. **72** (1993), 673–693.

[**3**] J. Kaczorowski, *Axiomatic theory of L-functions: the Selberg class*, in "Analytic Number Theory", C.I.M.E. Summer School, (Cetraro, Italy 2002), A. Perelli and C. Viola, eds., Lecture Notes in Math., **1891**, Springer, Berlin, 2006, 133–209.

[**4**] J. Kaczorowski and A. Perelli, *On the structure of the Selberg class, I,* $0 \leq d \leq 1$, Acta Math. **182** (1999), 207–241.

[**5**] J. Kaczorowski and A. Perelli, *The Selberg class: a survey*, in "Number Theory in Progress, vol. 2", Proc. conf. in honor of A. Schinzel, de Gruyter, Berlin, 1999, 953–992.

[**6**] J. Kaczorowski and A. Perelli, *On the structure of the Selberg class, II, Invariants and conjectures*, J. Reine Angew. Math. **524** (2000), 73–96.

[**7**] J. Kaczorowski and A. Perelli, *On the structure of the Selberg class, IV, Basic invariants*, Acta Arith. **104** (2002), 97–116.

[**8**] J. Kaczorowski and A. Perelli, *On the structure of the Selberg class, V,* $1 < d < 5/3$, Invent. Math. **150** (2002), 485–516.

[**9**] J. Kaczorowski and A. Perelli, *On the structure of the Selberg class, VI, Non-linear twists*, Acta Arith. **116** (2005), 315–341.

[**10**] J. Kaczorowski and A. Perelli, *On the structure of the Selberg class, VII,* $1 < d < 2$, Ann. of Math. (2) **173** (2011), 1397–1441.

[**11**] J. Kaczorowski and A. Perelli, *Twists, Euler products and a converse theorem for L-functions of degree* 2, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) **14** (2015), 441–480.

[**12**] J. Kaczorowski and A. Perelli, *Twists and resonance of L-functions, I*, J. Eur. Math. Soc. (JEMS) **18** (2016), 1349–1389.

[**13**] J. Kaczorowski and A. Perelli, *Twists and resonance of L-functions, II*, Int. Math. Res. Not. IMRN **2016**, 7637–7670.

[**14**] J. Kaczorowski and A. Perelli, *A weak converse theorem for degree* 2 *L-functions with conductor* 1, Publ. Res. Inst. Math. Sci. **53** (2017), 337–347.

[**15**] J. Kaczorowski and A. Perelli, *On the standard twist of the L-functions of half-integral weight cusp forms*, Nagoya Math. J. **240** (2020), 150–180.

[**16**] J. Kaczorowski and A. Perelli, *The standard twist of L-functions revisited*, Acta Arith. **201** (2021), 281–328.

[**17**] J. Kaczorowski and A. Perelli, *Classification of L-functions of degree* 2 *and conductor* 1, Adv. Math., to appear.

[**18**] J. Lewis and D. Zagier, *Period functions for Maass wave forms, I*, Ann. of Math. (2) **153** (2001), 191–258.

[**19**] A. Perelli, *A survey of the Selberg class of L-functions, Part I*, Milan J. Math. **73** (2005), 19–52.

[**20**] A. Perelli, *A survey of the Selberg class of L-functions, part II*, Riv. Mat. Univ. Parma (7) **3*** (2004), 83–118.

[**21**] A. Perelli, *Non-linear twists of L-functions: a survey*, Milan J. Math. **78** (2010), 117–134.

[**22**] A. Perelli, *Converse theorems: from the Riemann zeta function to the Selberg class*, Boll. Unione Mat. Ital. **10** (2017), 29–53.

[**23**] A. Selberg, *Old and new conjectures and results about a class of Dirichlet series*, in “Proc. Amalfi Conf. Analytic Number Theory”, E. Bombieri et al., eds., Univ. Salerno, Salerno, 1992, 367–385; Collected Papers, vol. II, Springer Verlag 1991, 47–63.

[**24**] K. Soundararajan, *Degree* 1 *elements of the Selberg class*, Expo. Math. **23** (2005), 65–70.

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