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Quasiselective and weakly Ramsey ultrafilters

Dedicated to Roberto Dvornicich
on his seventieth birthday

Abstract. *Selective (Ramsey)* ultrafilters are characterized by many equivalent properties. Natural weakenings of these properties led to the inequivalent notions of *weakly Ramsey* and of *quasi-selective* ultrafilter, introduced and studied in [**1**] and [**4**], respectively. Call $\mathcal{U}$ *weakly Ramsey* if for every finite colouring of $[\mathbb{N}]^2$ there is $U \in \mathcal{U}$ s.t. $[U]^2$ has *only two colours*, and call $\mathcal{U}$ f*-quasi-selective* if every function $g \leq f$ is *nondecreasing* on some $U \in \mathcal{U}$. (So the *quasi-selective* ultrafilters of [**4**] are id-quasi selective.) In this paper we characterize those weakly Ramsey ultrafilters that are isomorphic to a quasi-selective ultrafilter by analyzing the relations between various natural cuts of the ultrapowers of $\mathbb{N}$ modulo these ultrafilters.

Keywords. Selective ultrafilters, quasi-selective ultrafilters, weakly Ramsey ultrafilters, interval P-points.

Mathematics Subject Classification: 03E02, 03E05, 03E20, 03E65.

Introduction

Special classes of ultrafilters over $\mathbb{N}$ have been introduced and studied in the literature, starting from the pioneering work by G. Choquet [**8**, **9**] in the sixties (see *e.g.* [**5**]). Particular attention was received by the class of *selective* (also called *Ramsey*, or in French *absolute*) ultrafilters. It is well known that the ultrafilter $\mathcal{U}$ is selective if and only if every finite colouring of $[\mathbb{N}]^2$ has a homogeneous set $U \in \mathcal{U}$ (*i.e.* $[U]^2$ is monochromatic), or equivalently if and only if every function $f : \mathbb{N} \to \mathbb{N}$ is nondecreasing on some $U \in \mathcal{U}$.

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Allowing sets U such that $[U]^2$ is *dichromatic* in the first characterization led to the notion of *weakly Ramsey* ultrafilter over $\mathbb{N}$, introduced and studied in [**1**] (see also [**13**]). On the other hand, restricting the second characterization to functions *bounded by the identity* defines the *quasi-selective* ultrafilters over $\mathbb{N}$, introduced and studied in [**4**]. Quasi-selective ultrafilters have independent interest, because they are necessary in modelling the "Euclidean numerosities" of point sets considered in [**4**], as well as in providing the so called "fine densities" of sets of natural numbers in [**10**].

In this paper we make a comparative study of *weakly Ramsey* and *f-quasi-selective* ultrafilters, the latter class being the natural parametric generalization of quasi-selective ultrafilters, where a function $f : \mathbb{N} \to \mathbb{N}$ replaces the identity in the original definition of [**4**].

It is worth mentioning that, on the one hand, selective ultrafilters are simultaneously weakly Ramsey and quasi-selective, while in turn both the latter classes are P-points. On the other hand, these classes are distinct, once there exist a selective and a non-selective quasi-selective ultrafilter. The existence of these ultrafilters is not provable in ZFC, but follows from mild set theoretical hypotheses, *e.g.* the Continuuum Hypothesis CH, or Martin's Axiom MA. The study of weak sufficient conditions for the existence of these kinds of ultrafilters seems to be an interesting field of set theoretic research, very little explored up to now.

The paper is organized as follows. In Section 1 we introduce the class of f-quasi-selective ultrafilters on $\mathbb{N}$, and we study their properties, generalizing some results of [**4**]. In Section 2 we study the weakly Ramsey ultrafilters introduced in [**1**], and we give a complete classification in terms of the mutual ordering of three natural cuts of the corresponding ultrapowers of $\mathbb{N}$, simultaneously specifying their corresponding properties of "quasi-selectivity". Final remarks and open questions may be found in the concluding Section 3.

In general, we refer to [**6**] and [**3**] for definitions and basic facts concerning ultrafilters and ultrapowers.

1 - f-quasi-selective ultrafilters

Throughout this paper $\mathcal{U}$ denotes a nonprincipal ultrafilter on $\mathbb{N}$, and all functions are $\mathbb{N} \to \mathbb{N}$, unless different mention is made explicitly. Recall that two functions f, g are $\mathcal{U}$-equivalent (written $g \equiv_{\mathcal{U}} f$) if there exists $U \in \mathcal{U}$ such that $f(u) = g(u)$ for all $u \in U$. In general we say that a function f is increasing, unbounded, one-to-one, *etc.*, *modulo* $\mathcal{U}$ if there exists $U \in \mathcal{U}$ such that the restriction of f to U is increasing, unbounded, one-to-one, *etc.*

Definition 1.1. Let $\mathcal{U}$ be a nonprincipal ultrafilter on $\mathbb{N}$, and let $f : \mathbb{N} \to \mathbb{N}$ be unbounded modulo $\mathcal{U}$. Then

- $\mathcal{U}$ is *f-quasi-selective* (shortly f-QS) if, for all $g : \mathbb{N} \to \mathbb{N}$,

$$\exists U \in \mathcal{U}\, \forall x \in U\, g(x) \leq f(x) \implies g \text{ nondecreasing mod } \mathcal{U}.$$

- $\mathcal{U}$ is *quasi-selective* (shortly QS) if it is id-QS, where $id : \mathbb{N} \to \mathbb{N}$ is the identity.
- $\mathcal{U}$ is *properly quasi-selective* (shortly PQS) if it is f-QS for some, but not for all functions f.
- $\mathcal{U}$ is *strongly quasi-selective* (shortly SQS) if it is f-QS for some 1-1 function f. $\mathcal{U}$ is *weakly quasi-selective* (shortly WQS) if it is PQS, but not SQS.

Clearly the ultrafilter $\mathcal{U}$ is selective if and only if it is f-QS for all f.

Recall that the ultrafilter $f\mathcal{U}$ is defined by $f\mathcal{U} = \{V \mid f^{-1}[V] \in \mathcal{U}\}$. Useful relations between QS ultrafilters and generic f-QS ultrafilters are given in the following proposition:

Proposition 1.1.

1. *If $\mathcal{U}$ is f-QS, then $f\mathcal{U}$ is QS.*
2. *If f is increasing modulo $\mathcal{U}$, then $\mathcal{U}$ is $(g \circ f)$-QS if and only if $f\mathcal{U}$ is g-QS; in particular $\mathcal{U}$ is f-QS if and only if $f\mathcal{U}$ is QS.*

Proof.

1. Let $\mathcal{U}$ be f-QS, with f nondecreasing on $U \in \mathcal{U}$. Assume that $h(x) \leq x$ for $x \in f[V], V \in \mathcal{U}$, so that $h \circ f \leq f$ on $U \cap V$. Then both f and $h \circ f$ are nondecreasing on $U \cap V$. Suppose by contradiction that there exist $x, y \in U \cap V$ such that $f(x) < f(y)$, but $h(f(x)) > h(f(y))$: the first inequality implies $x < y$, whereas the second implies $x > y$, contradiction. Therefore h is nondecreasing on $f[U \cap V] \in f\mathcal{U}$.

2. Pick $U \in \mathcal{U}$ such that, for all $x, y \in U$, $x < y \iff f(x) < f(y)$. Then, for every function h,

$$\forall x, y \in U\ (\, x < y \implies h(x) \leq h(y)\,)$$

is equivalent to

$$\forall z, w \in f[U]\ (\, z < w \implies h(f^{-1}(z)) \leq h(f^{-1}(w))\,).$$

Moreover

$$\forall x \in U\ (h(x) < g(f(x))) \iff \forall z \in f[U]\ (h(f^{-1}(z)) < g(z)).$$

So, if $f\mathcal{U}$ is g-QS and $h < g \circ f$ on U, then $h \circ f^{-1} < g$ on $f[U]$, and hence $h \circ f^{-1}$ is nondecreasing on $f[U]$, which in turn is equivalent to h nondecreasing on U.

Similarly, if $\mathcal{U}$ is $(g \circ f)$-QS and $h < g$ on $f[U]$, then $h \circ f < g \circ f$ on U, so $h \circ f$ is nondecreasing on U, and $h = h \circ f \circ f^{-1}$ is nondecreasing on $f[U]$.

The last assertion is the case $g = id$. □

It is proved in [**4**] that, when $\mathcal{U}$ is QS, every function is $\mathcal{U}$-equivalent either to a constant, or to an "interval-to-one" function, *i.e.* a function g such that, for all n, $g^{-1}(n)$ is a (finite, possibly empty) *interval* of $\mathbb{N}$. A weaker property, still sufficient to imply P-pointness, holds for all PQS ultrafilters, namely:

Proposition 1.2. *Let $\mathcal{U}$ be a PQS ultrafilter and let $\langle X_n \mid n \in \mathbb{N}\rangle$ be a partition of $\mathbb{N}$ such that no part X_n is in $\mathcal{U}$. Then there exists an* interval partition *$\langle Y_m \mid m \in \mathbb{N}\rangle$ and a set $U \in \mathcal{U}$ such that*

$$\forall n\, \exists m\ X_n \cap U \subseteq Y_m.$$

In particular every function is either constant or "finite-to-one" modulo $\mathcal{U}$. Hence all PQS ultrafilters are nonselective P-points.

Proof. Let f be a nondecreasing unbounded function such that $\mathcal{U}$ is f-QS. Define the function g by

$$g(x) = f(\min X_n) = \min f(X_n) \text{ for all } x \in X_n.$$

Then $g \le f$, so there exists a nondecreasing function h that is equal to g on some set $U \in \mathcal{U}$. The partition $\langle Y_m = h^{-1}(m) \mid m \in \mathbb{N}\rangle$ is an interval partition that satisfies the wanted condition, because h is constant on $X_n \cap U$. □

We remark that if f is one-to-one, then each nonempty $Y_m \cap U$ is equal to one $X_n \cap U$. In particular, modulo a SQS ultrafilter, every non-constant function is interval-to-one.

Recall that the ultrafilter $\mathcal{U}$ is *rapid* if for every increasing function g there exists $U = \{u_1 < u_2 < \ldots < u_n < \ldots\} \in \mathcal{U}$ such that $u_n > g(n)$. If moreover $\mathcal{U}$ is a P-point, then $\mathcal{U}$ is rapid if and only if the functions that are 1-to-1 modulo $\mathcal{U}$ are coinitial in the nonstandard part of the ultrapower $\mathbb{N}^{\mathbb{N}}_{\mathcal{U}}$ (see *e.g.* [**2**]). It is well known that the existence of nonselective rapid P-points is consistent, see *e.g.* [**7**]. However these ultrafilters cannot be PQS ultrafilters, since we have

Proposition 1.3. *Let $\mathcal{U}$ be f-QS: then $\mathcal{U}$ is rapid if and only if it is selective.*

Proof. Every selective ultrafilter is rapid, so we have to prove the 'only if' part. Let $\mathcal{U}$ be f-QS and let $\mathcal{P} = \{[p_n, p_{n+1}) \mid n \in \mathbb{N}\}$ be an interval partition of $\mathbb{N}$. By possibly unifying some intervals, we may assume w.l.o.g. that $f(p_n) > n$. By rapidity, there is a set $U = \{u_1 < u_2 < \ldots < u_n < \ldots\} \in \mathcal{U}$ such that $u_n > p_n$. Define the function g by

$$g(x) = |\{m \leq n \mid x \leq u_m < p_{n+1}\}| \quad \text{for } x \in [p_n, p_{n+1}).$$

Then g takes on decreasing values on $U \cap [p_n, p_{n+1})$, and $g \leq f$, because $|U \cap [p_n, p_{n+1})| \leq n < f(p_n)$. Let $V \in \mathcal{U}$ be a set on which g is nondecreasing: clearly $U \cap V$ has at most one point in each interval $[p_n, p_{n+1})$. □

Following [**4**], let us consider the following families of functions

$$\mathcal{S}^{\mathcal{U}} = \{f \mid \exists U \in \mathcal{U} \text{ s.t. } f \text{ 1-1 on } U\}, \quad \mathcal{F}_{\mathcal{U}} = \{f \mid \mathcal{U} \text{ is } f\text{-QS}\}, \quad \text{and}$$

$$\mathcal{G}_{\mathcal{U}} = \{g \mid \exists U \in \mathcal{U}\, \forall x, y \in U\ (x < y \implies g(x) < y)\}.$$

Recall the following facts, that represent three important features of QS ultrafilters, extensively used in [**4**]:

Fact 1. ([**4**, Theorem 1.1]) *If $\mathcal{U}$ is QS, then $\mathcal{F}_{\mathcal{U}} = \mathcal{G}_{\mathcal{U}}$.*

Fact 2. ([**4**, Proposition 1.5]) *Let g be interval-to-one, and put $g^+(x) = \max\{y \mid g(y) = g(x)\}$. Then $g \in \mathcal{S}^{\mathcal{U}}$ if and only if $g^+ \in \mathcal{G}_{\mathcal{U}}$.*

Fact 3. ([**4**, Propositions 1.4 and 1.7]) *$\mathcal{F}_{\mathcal{U}}$ is closed under sums, products, powers and compositions. Moreover $\mathcal{G}_{\mathcal{U}}$ has uncountable cofinality.*

For general PQS ultrafilters we can prove both Facts 2 and 3, but only one half of Fact 1, namely:

Proposition 1.4. *Let $\mathcal{U}$ be PQS. Then*

1. *$\mathcal{F}_{\mathcal{U}} \subseteq \mathcal{G}_{\mathcal{U}}$, and equality holds if and only if $\mathcal{U}$ is QS.*

2. *For g finite-to-one, put $g^+(x) = \max\{y \mid g(y) = g(x)\}$; then $g \in \mathcal{S}^{\mathcal{U}}$ if and only if $g^+ \in \mathcal{G}_{\mathcal{U}}$.*

3. *$\mathcal{F}_{\mathcal{U}}$ is closed under sums, products, powers and compositions; moreover $\mathcal{G}_{\mathcal{U}}$ has uncountable cofinality.*

Proof.

1. Assume that $\mathcal{U}$ is f-QS, with nondecreasing $f \in \mathcal{F}_{\mathcal{U}}$, and pick any sequence $\langle x_n \mid n \in \mathbb{N} \rangle$ s.t. $x_{n+1} = f(x_n) + x_n$. Define the function h by $h(x_n + j) = f(x_n) - j$ for $0 \leq j < f(x_n)$. Then there is a set in $A \in \mathcal{U}$ which meets each interval $[x_n, x_{n+1})$ in one point a_n. So by putting either $u_n = a_{2n}$ or $u_n = a_{2n+1}$ we obtain a set $U \in \mathcal{U}$ witnessing that $g = id + f$ belongs to $\mathcal{G}_{\mathcal{U}}$. Namely, in the even case we have

$$u_{n+1} - u_n > x_{2n+2} - x_{2n+1} = f(x_{2n+1}) \geq f(u_n),$$

and similarly in the odd case.

The equality $\mathcal{F}_{\mathcal{U}} = \mathcal{G}_{\mathcal{U}}$ has been proved for QS ultrafilters in Theorem 1.1 of [**4**]. Finally, the function g has be choosen greater than the identity, so if $\mathcal{U}$ is not QS, then $g \notin \mathcal{F}_{\mathcal{U}}$, and the inclusion is *proper*.

2. Observe first that g^+ depends only on the partition induced by g, and not on its actual values. Moreover, if h is any interval-to-one function inducing a coarser partition than g, then $h^+ \geq g^+$. Hence we may assume w.l.o.g. that g is interval-to-one.

Assume $g^+ \in \mathcal{G}_{\mathcal{U}}$, and pick $U = \{u_n \mid n \in \mathbb{N}\} \in \mathcal{U}$ such that $u_{n+1} > g^+(u_n)$. Suppose that $g(u_n) = g(u_{n+1})$ for some n: then $g^+(u_n) \geq u_{n+1} > g^+(u_n)$, a contradiction. Hence g is one-to-one on U.

The reverse implication follows from the fact that $g^{++} = g^+$.

3. We prove first that if every function $g < f$ is $\mathcal{U}$-equivalent to a nondecreasing one, then the same property holds for every function $g < f^2$.

Given g, let h be the integral part of the square root of g. So $g < h^2 + 2h + 1$, hence $g = h^2 + h_1 + h_2$ for suitable functions $h_1, h_2 \leq h < f$. By hypothesis we can pick nondecreasing functions h', h'_1, h'_2 that are $\mathcal{U}$-equivalent to h, h_1, h_2, respectively. Then clearly g is $\mathcal{U}$-equivalent to the nondecreasing function $h'^2 + h'_1 + h'_2$. So $\mathcal{F}_{\mathcal{U}}$ is closed under squares, and hence also under sums, products and powers. To settle compositions, observe first that, if $g, h \leq id$, then $g \circ h \leq h$, and the thesis is trivial. On the other hand, if $id \in \mathcal{F}_{\mathcal{U}}$, then $\mathcal{U}$ is QS, and we refer to the proof of Fact 3. given *sub* Proposition 1.5 of [**4**].

Finally, the proof of $\operatorname{cof} \mathcal{G}_{\mathcal{U}} > \omega$ given *sub* Proposition 1.7 of [**4**] relies solely on the fact that $\mathcal{U}$ is a P-point, so it works here as well. □

CAVEAT: When $\mathcal{U}$ is not QS, we may not state point 2 for $\mathcal{F}_{\mathcal{U}}$, as it is done in [**4**], because $\mathcal{G}_{\mathcal{U}}$ is greater than $\mathcal{F}_{\mathcal{U}}$.

The main tool in the study of PQS ultrafilters (and especially of PWR ultrafilters in the next section) is the relative position of particular cuts in the corresponding ultrapowers of $\mathbb{N}$.

Given a non-Q-point ultrafilter $\mathcal{U}$, let $\mathcal{P} = \langle [p_n, p_{n+1}) \mid n \in \mathbb{N}\rangle$ be an interval partition witnessing the non-Q-pointness of $\mathcal{U}$, *i.e.* such that there is no $U \in \mathcal{U}$ with $|U \cap [p_n, p_{n+1})| \le 1$ for all $n \in \mathbb{N}$. For $U \in \mathcal{U}$ and $p_n \le x < p_{n+1}$ define the functions a_p^U, b_p^U, and c_p^U by

$$a_p^U(x) = |U \cap [p_n, x)|, \quad b_p^U(x) = |U \cap [x, p_{n+1})|, \quad c_p^U(x) = |U \cap [p_n, p_{n+1})|,$$

and consider the corresponding families of functions

$$\mathcal{A}_p^{\mathcal{U}} = \{a_p^U \mid U \in \mathcal{U}\}, \quad \mathcal{B}_p^{\mathcal{U}} = \{b_p^U \mid U \in \mathcal{U}\}, \quad \mathcal{C}_p^{\mathcal{U}} = \{c_p^U \mid U \in \mathcal{U}\}.$$

Put $\mathcal{E}^{\mathcal{U}} = \{f \mid f \text{ increasing} \bmod \mathcal{U}\}$, and recall that $\mathcal{S}^{\mathcal{U}} = \{f \mid f \text{ 1-1} \bmod \mathcal{U}\}$.

We have

Theorem 1.5. *Let $\mathcal{U}$ be a* PQS *ultrafilter, and let $\mathcal{P}$ be an interval partition without selection set in $\mathcal{U}$. Let $F_{\mathcal{U}}$ be the cut of the ultrapower $\mathbb{N}_{\mathcal{U}}^{\mathbb{N}}$ whose left part is generated by $\mathcal{F}_{\mathcal{U}}$; let $E^{\mathcal{U}}, S^{\mathcal{U}}, A_p^{\mathcal{U}}, B_p^{\mathcal{U}}$, and $C_p^{\mathcal{U}}$ be the cuts of the ultrapower $\mathbb{N}_{\mathcal{U}}^{\mathbb{N}}$ whose right parts are generated by $\mathcal{E}^{\mathcal{U}}, \mathcal{S}^{\mathcal{U}}, \mathcal{A}_p^{\mathcal{U}}, \mathcal{B}_p^{\mathcal{U}}$, and $\mathcal{C}_p^{\mathcal{U}}$ respectively.*

Then all cuts, but possibly $A_p^{\mathcal{U}}$, are greater than $\mathbb{N}$, and

$$A_p^{\mathcal{U}}, S^{\mathcal{U}} \le E^{\mathcal{U}}, \quad F_{\mathcal{U}} \le B_p^{\mathcal{U}}, \quad \textit{and} \quad \max\{A_p^{\mathcal{U}}, B_p^{\mathcal{U}}\} = C_p^{\mathcal{U}}.$$

Moreover $\mathcal{U}$ is SPS if and only if $E^{\mathcal{U}} < F_{\mathcal{U}}$, and in this case

$$A_p^{\mathcal{U}} = S^{\mathcal{U}} = E^{\mathcal{U}} < F_{\mathcal{U}} \le B_p^{\mathcal{U}} = C_p^{\mathcal{U}}.$$

Proof. For $U \in \mathcal{U}$ put $e^U(x) = |U \cap [0, x)|$, so every function increasing on U is not smaller than e^U. Hence the cut $\mathcal{E}^{\mathcal{U}}$ is generated also by the set $\{e^U \mid U \in \mathcal{U}\}$. Since $a_p^U \le e^U$, one gets $A_p^{\mathcal{U}} \le E^{\mathcal{U}}$. The inequality $S^{\mathcal{U}} \le E^{\mathcal{U}}$ is trivial, and $F_{\mathcal{U}} \le B_p^{\mathcal{U}}$ holds because every $U \in \mathcal{U}$ intersects some interval $[p_n, p_{n+1})$ in more than one point, and hence no function b_p^U is nondecreasing modulo $\mathcal{U}$.

Moreover, for all $U \in \mathcal{U}$,

$$a_p^U, b_p^U \le c_p^U = a_p^U + b_p^U, \quad \text{whence} \quad \frac{1}{2}c_p^U \le \max\{a_p^U, b_p^U\} \le c_p^U.$$

Hence $\max\{A_p^{\mathcal{U}}, B_p^{\mathcal{U}}\} = C_p^{\mathcal{U}}$, because for all $U \in \mathcal{U}$ there exists $V \in \mathcal{U}$ s.t. $c_p^V \le \frac{1}{2}c_p^U$.

One has $\mathbb{N} < F_{\mathcal{U}}, S_{\mathcal{U}}$ because $\mathcal{U}$ is PQS, so it cannot be rapid. It follows that only $A_p^{\mathcal{U}}$ might possibly be equal to $\mathbb{N}$.

Finally, if $E^{\mathcal{U}} < F_{\mathcal{U}}$, then obviously $\mathcal{S}^{\mathcal{U}} \cap \mathcal{F}_{\mathcal{U}} \neq \emptyset$. Conversely, $f \in \mathcal{S}^{\mathcal{U}} \cap \mathcal{F}_{\mathcal{U}}$ implies $f \in \mathcal{E}^{\mathcal{U}}$, and hence $A_p^{\mathcal{U}} \leq S^{\mathcal{U}} = E^{\mathcal{U}} < F_{\mathcal{U}} \leq B_p^{\mathcal{U}} = C_p^{\mathcal{U}}$. Moreover if $a_p^U \in \mathcal{F}_{\mathcal{U}}$, then it is nondecreasing on some $V \in \mathcal{U}$. It follows that a_p^U becomes increasing by taking off at most one point from each interval $V \cap [p_n, p_{n+1})$, and the resulting set V' belongs to $\mathcal{U}$, too. So $a_p^U \in \mathcal{E}^{\mathcal{U}}$, and also $A_p^{\mathcal{U}} = E^{\mathcal{U}}$. $\square$

We conclude this section by extending Proposition 1.9 of [**4**] to arbitrary PQS ultrafilters, thus obtaining that the class of f-QS ultrafilters can be closed under isomorphisms only in the trivial case when every P-point is selective.

Proposition 1.6. *Assume that the ultrafilter $\mathcal{U}$ is not a Q-point, and let f be an arbitrary nondecreasing unbounded function. Then there exists an increasing function φ such that the ultrafilter $\varphi\mathcal{U} \cong \mathcal{U}$ is not f-QS.*

Proof. Let $\mathcal{P} = \langle [p_n, p_{n+1}) \mid n \in \mathbb{N} \rangle$ be an interval partition witnessing the non-Q-pointness of $\mathcal{U}$, *i.e.* such that there is no $U \in \mathcal{U}$ with $|U \cap [p_n, p_{n+1})| \leq 1$ for all $n \in \mathbb{N}$. Pick a sequence b_n such that $f(b_n) > p_{n+1}$ and $b_{n+1} - b_n > p_{n+1} - p_n$. Define the function φ by

$$\varphi(p_n + j) = b_n + j \quad \text{for} \quad 0 \leq j < p_{n+1} - p_n.$$

So the points $\varphi(p_n) = b_n$ determine an interval partition that has no selection set in $\varphi\mathcal{U}$. Moreover $f(b_n) > p_{n+1}$, hence any function g such that

$$g(b_n + j) = p_{n+1} - j \quad \text{for} \quad 0 \leq j < a_{n+1} - a_n$$

is positive and not greater than f on $\varphi[\mathbb{N}]$, but cannot be nondecreasing modulo $\varphi\mathcal{U}$. $\square$

2 - Weakly Ramsey ultrafilters

An interesting weakening of the Ramsey property of selective ultrafilters has been considered by A. Blass in [**1**]:

Definition 2.1. The ultrafilter $\mathcal{U}$ on $\mathbb{N}$ is *weakly Ramsey* (shortly WR) if for every finite colouring of $[\mathbb{N}]^2$ there is $U \in \mathcal{U}$ s.t. $[U]^2$ has *only two colours.* $\mathcal{U}$ is *properly weakly Ramsey* (abbreviated PWR) if it is WR but not selective.

Throughout this section we assume that $\mathcal{U}$ is a PWR ultrafilter, and that $\mathcal{P} = \langle [p_n, p_{n+1}) \mid n \in \mathbb{N} \rangle$ is an interval partition witnessing the non-selectivity of $\mathcal{U}$, so there is no $U \in \mathcal{U}$ with $|U \cap [p_n, p_{n+1})| \leq 1$ for all $n \in \mathbb{N}$.

The behaviour of functions modulo a PWR ultrafilter $\mathcal{U}$ is subject to severe constraints, which recall those given by selectivity; namely every function f is

$\mathcal{U}$-equivalent either to a 1-to-1 function, or to a function that is constant on each interval $[p_n, p_{n+1})$, independently of the choice of the interval partition $\mathcal{P}$. More precisely (see Theorem 5 of [**1**]):

Lemma 2.1. *Let $f : \mathbb{N} \to \mathbb{N}$ and an interval partition $\mathcal{P}$ be given. Let $\mathcal{U}$ be a PWR ultrafilter: then there exists $U \in \mathcal{U}$ such that exactly one of the following cases occurs:*

(*i*) *f is constant on U;*

(*ii*) *f is increasing on U;*

(*iii*) *$f(x) < f(y)$ whenever $x, y \in U$ and there is n such that $x < p_n \leq y$, and f is constant on $U \cap [p_n, p_{n+1})$ for all $n \in \mathbb{N}$;*

(*iv*) *$f(x) < f(y)$ whenever $x, y \in U$ and there is n such that $x < p_n \leq y$, and f is decreasing on $U \cap [p_n, p_{n+1})$ for all $n \in \mathbb{N}$.*

In particular, the ultrafilter $f\mathcal{U}$ is selective if and only if f is constant on each interval $[p_n, p_{n+1})$, i.e. *of type (iii).*

Proof. Put $p(x) = n$ if $x \in [p_n, p_{n+1})$, and identify $[\mathbb{N}]^2$ with the set of pairs $\{(x, y) \in \mathbb{N}^2 \mid x < y\}$. Define the 6-colouring of $[\mathbb{N}]^2$ according to all possible combinations of $p(x) \lneqq p(y)$ and $f(x) \gtreqless f(y)$.

By the choice of the interval partition, any 2-coloured set $[U]^2$ with $U \in \mathcal{U}$ must comprehend both pairs with $p(x) = p(y)$ and pairs with $p(x) < p(y)$. Now, when both are paired with $f(x) = f(y)$, then case (i) occurs, whereas case (ii) occurs when both are paired with $f(x) < f(y)$; case (iii) and (iv) occur when $p(x) < p(y)$ is paired with $f(x) < f(y)$ and $p(x) = p(y)$ with either $f(x) = f(y)$, or $f(x) > f(y)$, respectively. It is easily seen that no one of the remaining cases can occur. *E.g.*, pairing $p(x) = p(y)$ with $f(x) < f(y)$ and $p(x) < p(y)$ with $f(x) = f(y)$ yields a contradiction by taking $p(x) = p(y) < p(z)$, *etc.*.

All functions of type (*ii*) and (*iv*) are 1-1 modulo $\mathcal{U}$, so $f\mathcal{U}$ is isomorphic to $\mathcal{U}$. On the other hand, if f is constant on each interval, then $g \circ f$ is non decreasing modulo $\mathcal{U}$ for all g. Hence all functions are nondecreasing modulo $f\mathcal{U}$, which is therefore selective. □

In order to classify the different types of PWR ultrafilters, we recall the notation of Section 1. For $U \in \mathcal{U}$ and $p_n \leq x < p_{n+1}$ let

$$a_p^U(x) = |U \cap [p_n, x)|, \quad b_p^U(x) = |U \cap [x, p_{n+1})|, \quad c_p^U(x) = |U \cap [p_n, p_{n+1})|;$$

$$\mathcal{A}_p^{\mathcal{U}} = \{a_p^U \mid U \in \mathcal{U}\}, \quad \mathcal{B}_p^{\mathcal{U}} = \{b_p^U \mid U \in \mathcal{U}\}, \quad \mathcal{C}_p^{\mathcal{U}} = \{c_p^U \mid U \in \mathcal{U}\};$$

$$\mathfrak{S}^{\mathcal{U}} = \{f \mid f \text{ 1-1 } \bmod \mathcal{U}\}, \quad \text{and} \quad \mathcal{E}^{\mathcal{U}} = \{f \mid f \text{ increasing } \bmod \mathcal{U}\}.$$

Then we have

Theorem 2.2. *Let $\mathcal{U}$ be a* PWR *ultrafilter. Let $A_p^{\mathcal{U}}, B_p^{\mathcal{U}}, C_p^{\mathcal{U}}, E^{\mathcal{U}}$, and $S^{\mathcal{U}}$ be the cuts of the ultrapower $\mathbb{N}_{\mathcal{U}}^{\mathbb{N}}$ whose right parts are generated by $\mathcal{A}_p^{\mathcal{U}}, \mathcal{B}_p^{\mathcal{U}}, \mathcal{C}_p^{\mathcal{U}}, \mathcal{E}^{\mathcal{U}}$, and $\mathfrak{S}^{\mathcal{U}}$ respectively. Let $F_{\mathcal{U}}$ be the cut of the ultrapower $\mathbb{N}_{\mathcal{U}}^{\mathbb{N}}$ whose left part is generated by $\mathcal{F}_{\mathcal{U}} = \{f \mid \mathcal{U}\ f\text{-}QS\}$. Then, independently of the chosen interval partition,*

$$\min\{A_p^{\mathcal{U}}, B_p^{\mathcal{U}}\} = S^{\mathcal{U}} \leq \begin{cases} A_p^{\mathcal{U}} = E^{\mathcal{U}} \\ B_p^{\mathcal{U}} = F_{\mathcal{U}} \end{cases} \leq C_p^{\mathcal{U}} = \max\{A_p^{\mathcal{U}}, B_p^{\mathcal{U}}\}.$$

Moreover a PWR ultrafilter $\mathcal{U}$ is rapid if and only if $\mathbb{N} = F^{\mathcal{U}}$, and then all considered cuts coincide with $\mathbb{N}$.

Proof. According to Lemma 2.1, all functions are nondecreasing modulo $\mathcal{U}$, except those of type (iv). Moreover every function of type (iv) w.r.t. $U \in \mathcal{U}$ is greater than b_p^U, so the cuts $F_{\mathcal{U}}$ and $B_p^{\mathcal{U}}$ coincide.

Similarly a function is 1-1 on some $U \in \mathcal{U}$ if and only if its type is either (ii) or (iv). All functions of the former type are not less than the corresponding function a_p^U, while those of the latter type are not less than the corresponding function b_p^U. Hence the cut $S^{\mathcal{U}}$ coincides with the smaller between $A_p^{\mathcal{U}}$ and $B_p^{\mathcal{U}}$.

The equality $\max\{A_p^{\mathcal{U}}, B_p^{\mathcal{U}}\} = C_p^{\mathcal{U}}$ has been proved in Theorem 1.5, without any use of quasi-selectivity, as well as the trivial inequality $A_p^{\mathcal{U}} \leq E^{\mathcal{U}}$. On the other hand, each function a_p^U is increasing modulo $\mathcal{U}$, so for all $U \in \mathcal{U}$ there is $V \in \mathcal{U}$ such that $a_p^U \geq e^V$ on V, and the converse inequality $A_p^{\mathcal{U}} \geq E^{\mathcal{U}}$ follows.

Finally, $\mathcal{U}$ being a P-point, it is rapid if and only if the functions that are 1-1 modulo $\mathcal{U}$ are coinitial in $\mathbb{N}_{\mathcal{U}}^{\mathbb{N}} \setminus \mathbb{N}$, *i.e.* $\mathbb{N} = S^{\mathcal{U}}$. But then also $F_{\mathcal{U}}$ has to be equal to $\mathbb{N}$, otherwise $\mathcal{U}$ would be f-QS for some f, and so selective by Proposition 1.3. So it remains to prove that $\mathbb{N} = F_{\mathcal{U}}$ implies $\mathbb{N} = C_p^{\mathcal{U}}$. Assume the contrary: then $C_p^{\mathcal{U}} = A_p^{\mathcal{U}} > B_p^{\mathcal{U}} = \mathbb{N}$. Define the bijection σ of $\mathbb{N}$ by

$$\sigma(x) = p_n + p_{n+1} - x - 1 \text{ for } \ p_n \leq x < p_{n+1}.$$

Then clearly

$$a_p^U >_{\mathcal{U}} b_p^U \iff a_p^{\sigma U} <_{\sigma\mathcal{U}} b_p^{\sigma U}.$$

So $A_p^{\sigma\mathcal{U}} < B_p^{\sigma\mathcal{U}} = F_{\sigma\mathcal{U}}$, and $\sigma\mathcal{U} \cong \mathcal{U}$ would be simultaneously rapid and PQS, against Proposition 1.3. $\square$

It follows immediately that a PWR ultrafilter $\mathcal{U}$ is QS if and only if the identity is less than the cut $B^{\mathcal{U}}$. More generally, the above theorem allows for a complete specification of the "quasi-selectivity" properties of PWR ultrafilters. Namely

Corollary 2.3. *Let $\mathcal{U}$ be a* PWR *ultrafilter, and let $A_p^{\mathcal{U}}, B_p^{\mathcal{U}}$, and $C_p^{\mathcal{U}}$ be the cuts of the ultrapower $\mathbb{N}_{\mathcal{U}}^{\mathbb{N}}$ whose right parts are generated by $\mathcal{A}_p^{\mathcal{U}}, \mathcal{B}_p^{\mathcal{U}}$, and $\mathcal{C}_p^{\mathcal{U}}$ respectively. Then*

1. *$\mathcal{U}$ is PQS if and only if $\mathbb{N} \neq B_p^{\mathcal{U}}$, or equivalently if and only if $\mathcal{U}$ is not rapid;*
2. *$\mathcal{U}$ is SQS if and only if $A_p^{\mathcal{U}} < B_p^{\mathcal{U}}$, or equivalently $A_p^{\mathcal{U}} \neq C_p^{\mathcal{U}}$; (in particular $\mathcal{U}$ is QS if and only if $id < C_p^{\mathcal{U}}$)*
3. *$\mathcal{U}$ is isomorphic to a QS ultrafilter if and only if $A_p^{\mathcal{U}} \neq B_p^{\mathcal{U}}$.*

Proof.
1. Any unbounded function $f < B^{\mathcal{U}} = F_{\mathcal{U}}$ witnesses that $\mathcal{U}$ is f-QS, and the last assertion of Theorem 2.2 implies that such a function f exists unless $\mathcal{U}$ is rapid.

2. We have $C_p^{\mathcal{U}} = \max\{A_p^{\mathcal{U}}, B_p^{\mathcal{U}}\}$, hence $A_p^{\mathcal{U}} \neq C_p^{\mathcal{U}}$ is equivalent to $S^{\mathcal{U}} = A_p^{\mathcal{U}} < B_p^{\mathcal{U}} = F_{\mathcal{U}}$, by Theorem 2.2. So there is $U \in \mathcal{U}$ s.t. $a_p^U < B_p^{\mathcal{U}}$: then a_p^U is increasing modulo $\mathcal{U}$, and $\mathcal{U}$ is a_p^U-QS.

3. If $A_p^{\mathcal{U}} < B_p^{\mathcal{U}}$, then $\mathcal{U}$ is SQS; so there is a function f increasing modulo $\mathcal{U}$ such that $\mathcal{U}$ is f-QS. Then $\mathcal{U} \cong f\mathcal{U}$, and $f\mathcal{U}$ is QS by Proposition 1.1.

If $A_p^{\mathcal{U}} > B_p^{\mathcal{U}}$, define the bijection σ of $\mathbb{N}$ by $\sigma(x) = p_n + p_{n+1} - x - 1$ for $p_n \leq x < p_{n+1}$. Then clearly

$$a_p^U >_{\mathcal{U}} b_p^U \iff a_p^{\sigma U} <_{\sigma\mathcal{U}} b_p^{\sigma U}.$$

So $A_p^{\sigma\mathcal{U}} < B_p^{\sigma\mathcal{U}}$, and $\sigma\mathcal{U}$ is isomorphic to a QS ultrafilter by the preceeding case.

Conversely, let φ be a 1-1 function, which we may assume of type (ii) or (iv), according to Lemma 2.1. In both cases there is an interval partition $\mathcal{P}'$ such that $\varphi[p_n, p_{n+1}) \subseteq [p'_n, p'_{n+1})$ for all $n \in \mathbb{N}$. Then one has

$$a_p^U >_{\mathcal{U}} b_p^U \iff a_{p'}^{\varphi U} <_{\varphi\mathcal{U}} b_{p'}^{\varphi U}, \quad \text{when } \varphi \text{ is of type } (iv);$$

whereas

$$a_p^U <_{\mathcal{U}} b_p^U \iff a_{p'}^{\varphi U} <_{\varphi\mathcal{U}} b_{p'}^{\varphi U}, \quad \text{when } \varphi \text{ is of type } (ii).$$

It follows that the equality $A_p^{\mathcal{U}} = B_p^{\mathcal{U}}$ is preserved under isomorphism, and such ultrafilters cannot be QS (nor SQS). □

3 - Final remarks and open questions

Recall that both PWR and PQS ultrafilters are nonselective P-points, so the above results are nontrivial only when such ultrafilters exist. (And their existence is independent of ZFC by a celebrated result of Shelah's, see *e.g.* [**15**].) However mild hypotheses, like CH or MA, suffice in making both classes rich and distinct (see [**1**, **4**]). In fact these classes are already different unless both are empty, because the former is closed under isomorphism, whereas the latter is not, by Proposition 1.6.

In ZFC, one can draw the following diagram of implications

$$\begin{tikzcd}
& \text{QS} \longrightarrow \exists f.\, f\text{-QS} \arrow[dr] & \\
\text{Selective} \arrow[ur] \arrow[dr] & & \text{P-point} \\
& \text{Weakly Ramsey} \arrow[ur] &
\end{tikzcd}$$

Recall that, assuming CH, the following facts hold:

(A) there exist PWR ultrafilters $\mathcal{U}$ such that the cut induced by $C_p^{\mathcal{U}}$ in the ultrapower $\mathbb{N}^{\mathbb{N}}_{p\mathcal{U}}$ is arbitrarily chosen among those having left part closed under exponentiation and right part of uncountable coinitiality (Theorem 4 of [**1**]);[1]

(B) there are non-WR P-points (Theorem 2 of [**1**]);

(C) there exist P-points that are not QS, and QS ultrafilters that are not selective (Theorem 1.2 of [**4**]).

It follows from (A) that there exist rapid PWR ultrafilters, necessarily not PQS, and also that for every f there exist f-QS PWR ultrafilters, necessarily non-g-QS for suitable g.

So, considering also (B-C), we may conclude that, in the diagram above, no arrow can be reversed nor inserted, except compositions.

Many weaker conditions than the Continuum Hypothesis have been considered in the literature, in order to get more information about special classes of ultrafilters on $\mathbb{N}$. Of particular interest are (in)equalities among the so called "combinatorial cardinal characteristics of the Continuum". *E.g.* one has that

[1] It is worth mentioning that, according to Theorem 2.2, if $C_p^{\mathcal{U}}$ is taken to be $\mathbb{N}$, then $\mathcal{U}$ is a *rapid nonselective P-point*. Thus one has a non-forcing proof of the consistency of the existence of such ultrafilters.

P-points or selective ultrafilters are generic if $\mathfrak{c} = \mathfrak{d}$ or $\mathfrak{c} = \mathbf{cov}(\mathcal{B})$, respectively. Moreover if $\mathbf{cov}(\mathcal{B}) < \mathfrak{d} = \mathfrak{c}$ then there are filters that are included in P-points, but cannot be extended to selective ultrafilters. See the comprehensive survey [**3**].

We remark that both QS and WR ultrafilters are P-points of a special kind, since they share the property that every function is equivalent to an *interval-to-one* function. So the question naturally arises as to whether this class of "interval P-points" is distinct from either one of the other three classes.[2]

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