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On m-quasi Einstein almost Kenmotsu manifolds

Abstract. In this article, we consider m-quasi Einstein structures on two class of almost Kenmotsu manifolds. Firstly, we study a closed m-quasi Einstein metric on a Kenmotsu manifold. Next, we proved that if a Kenmotsu manifold M admits an m-quasi Einstein metric with conformal vector field V, then M is Einstein. Finally, we prove that a non-Kenmotsu almost Kenmotsu $(\kappa, \mu)'$-manifold admitting a closed m-quasi Einstein metric is locally isometric to the Riemannian product $\mathbb{H}^{n+1} \times \mathbb{R}^n$, provided that $\frac{\lambda-\kappa(2n+m)}{2m} = 1$.

Keywords. m-quasi Einstein metric, Ricci solitons, Kenmotsu manifolds, almost Kenmotsu $(\kappa, \mu)'$-manifolds, Einstein manifolds.

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1 - Introduction

Let M be a smooth Riemannian manifold. As an extension of Einstein metric, Case et al [**4**] introduced the concept of quasi Einstein metric and this is closely related to the warped product spaces (see [**3**]). In this context, one defines m-Bakry-Emery Ricci tensor as in [**1**]

$$Ric_f^m = Ric + Hess_f - \frac{1}{m} df \otimes df, \tag{1.1}$$

namely one puts $0 < m \leq \infty$, while Ric and $Hess_f$ indicate the Ricci tensor and the Hessian of a smooth function f, respectively. The tensor Ric_f^m was

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extended by Limoncu [**19**] (see also [**2**]) for an arbitrary vector field V on M. He defines Ric_V^m as follows

$$Ric_V^m = Ric + \frac{1}{2}\pounds_V g - \frac{1}{m}V^{\#} \otimes V^{\#}, \tag{1.2}$$

where $\pounds$ indicates the Lie-derivative and $V^{\#}$ is the 1-form associated to V. In this setting, a metric g on a Riemannian manifold M will be called m-quasi Einstein metric if there exist a vector field V and constant λ such that

$$Ric + \frac{1}{2}\pounds_V g - \frac{1}{m}V^{\#} \otimes V^{\#} = \lambda g. \tag{1.3}$$

We say that an m-quasi Einstein metric is trivial when $V = 0$ and this triviality is equivalent to say that M is Einstein. The equation (1.3) reduces to a Ricci soliton when we take $m = \infty$ (for details see [**16**, **21**]). Using the terminology of Ricci soliton, an m-quasi Einstein metric is called expanding, steady, or shrinking, respectively, if $\lambda < 0$, $\lambda = 0$, or $\lambda > 0$.

Recently, many works on Ricci solitons and their various generalizations within the framework of contact metric manifolds and almost cosympletic manifolds were published (see [**5**, **6**, **8**, **11**, **12**, **14**, **22**, **23**, **29**]). Recently, the equation (1.3) has been studied by Ghosh [**13**] on contact metric manifolds and obtained several fruitful results. But, as far as we know, there are no studies on m-quasi Einstein metrics on almost Kenmotsu manifolds. So that in this paper, we want fill this gap and classify certain class of almost Kenmotsu manifolds which admits an m-quasi Einstein metric.

2 - Preliminaries

Here we review some basic notions and properties of almost Kenmotsu manifolds, see details in [**9**, **10**, **17**].

Let (M, g) be a smooth Riemannian manifold of dimension $2n + 1$. If a (1,1)-tensor field φ, a global vector field ξ (called Reeb vector field) and a global 1-form η satisfy the following tensorial equations

$$\varphi^2 X = -X + \eta(X)\xi, \quad \eta(\xi) = 1, \tag{2.1}$$

$$g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y), \tag{2.2}$$

for any vector field X Y on M, then we say that (φ, ξ, η, g) is an almost contact metric structure and this structure with M is called almost contact metric manifold. One can obtain from (2.1) and (2.2) that

$$\eta(X) = g(X, \xi), \quad \varphi\xi = 0, \quad \eta \circ \varphi = 0.$$

Define a (1,2)-type torsion tensor N_φ on M as $N_\varphi = [\varphi, \varphi] + 2d\eta \otimes \xi$, where $[\varphi, \varphi]$ is the Nijenhuis tensor. If N_φ vanishes identically, then we say that M is normal.

An almost Kenmotsu manifold is defined as an almost contact metric manifold such that $d\eta = 0$ and $d\Phi = 2\eta \wedge \Phi$, where the fundamental 2-form Φ is defined by $\Phi(X, Y) = g(X, \varphi Y)$. Following Janssens and Vanhecke [**17**], a normal almost Kenmotsu manifold is called a Kenmotsu manifold [**18**]. The study of these manifolds was developed by several authors (for instance [**15**, **20**, **24**, **27**, **28**, **30**]). Kenmotsu manifolds are characterized by

$$(\nabla_X \varphi)Y = g(\varphi X, Y)\xi - \eta(Y)\varphi X. \tag{2.3}$$

Let M be an almost Kenmotsu manifold. We consider the self-adjoint operators on M

$$h = \frac{1}{2}\pounds_\xi \varphi, \quad h' = h \circ \varphi,$$

where $\pounds$ indicates the Lie-derivative. The above defined operators satisfy the equalities (see [**10**, **17**])

$$h\xi = 0, \quad tr_g(h) = tr_g(h') = 0, \quad h\varphi + \varphi h = 0 \tag{2.4}$$

where tr_g denotes the trace operator. We also found the following formulas in [**17**]

$$\nabla_X \xi = X - \eta(X)\xi + h'X, \tag{2.5}$$

$$Ric(\xi, \xi) = g(Q\xi, \xi) = -2n - tr_g(h^2), \tag{2.6}$$

where Q is the Ricci operator associated with the Ricci tensor Ric.

Definition 2.1. A vector field V on a Riemannian manifold is said to be conformal if there exists a smooth function ν such that

$$\pounds_V g = 2\nu g. \tag{2.7}$$

If ν vanishes, then we say that V is Killing.

3 - m-quasi Einstein Kenmotsu manifolds

It is proved by Dileo and Pastore [**9**] that an almost Kenmotsu manifold is normal if and only if the foliations of the distribution $\mathcal{D}$ (where $\mathcal{D}$ is the distribution orthogonal to ξ, that is, $\mathcal{D} = Ker\eta$) are Kählerian and the tensor field h vanishes. The following formulas are valid for any Kenmotsu manifolds (see [**18**])

$$\nabla_X \xi = X - \eta(X)\xi, \tag{3.1}$$

$$R(X,Y)\xi = \eta(X)Y - \eta(Y)X, \tag{3.2}$$

$$Q\xi = -2n\xi, \tag{3.3}$$

where R is the curvature tensor. As a result of (3.1) and (3.3), one can prove (for details see [**25**])

$$(\nabla_X Q)\xi = -QX - 2nX, \tag{3.4}$$

$$(\nabla_\xi Q)X = -2QX - 4nX. \tag{3.5}$$

First we consider a closed m-quasi Einstein strucuture on Kenmotsu manifolds. It is seen that the relation (1.2) turn into (1.1), when we uptake the 1-form $V^\#$ is closed with $V = Df$. So that it is natural generalization of gradient Ricci soliton (that is, satisfying (1.3) for $V = Df$ and $m = \infty$). Before entering to the main result, we call up the following

Lemma 3.1. *For a closed m-quasi Einstein metric the following formula holds:*

$$\begin{aligned} R(X,Y)V = (\nabla_Y Q)X - (\nabla_X Q)Y + \frac{1}{m}\{V^\#(X)QY - V^\#(Y)QX\} \\ + \frac{\lambda}{m}\{V^\#(Y)X - V^\#(X)Y\}. \end{aligned} \tag{3.6}$$

Proof. Because of $V^\#$ is closed, one can write the equation (1.3) as

$$\nabla_X V + QX = \lambda X + \frac{1}{m}V^\#(X)V. \tag{3.7}$$

Applying (3.7) in the well known expression of the curvature tensor

$$R(X,Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]},$$

one easily derives (3.6). This completes the proof. □

Definition 3.2. On an almost contact metric manifold M, a vector field V is said to be infinitesimal contact transformation if $\pounds_V \eta = \sigma\eta$, for some function σ. In particular, we call V as a strict infinitesimal contact transformation if $\pounds_V \eta = 0$.

Now we are in the position to prove the following conclusion.

Theorem 3.3. *Assume that a Kenmotsu manifold M admits a closed m-quasi Einstein structure with $m \neq 1$. Then one of the following conditions occurs*

(1) *V is pointwise collinear with ξ and in such a case M is η-Einstein.*

(2) *V is strictly infinitesimal contact transformation.*

(3) *M is Einstein.*

Proof. First, replacing Y by ξ in the equation (3.6), employing (3.4) and (3.5) one has

$$R(X,\xi)V = -QX - 2nX - \frac{\lambda + 2n}{m}V^{\#}(X)\xi + \frac{\eta(V)}{m}\{\lambda X - QX\}. \tag{3.8}$$

By (3.2), one can reach at

$$g(R(X,\xi)Y,V) = g(X,Y)\eta(V) - \eta(Y)g(X,V).$$

The aforesaid equation together with (3.8) implies

$$\frac{\lambda + 2n + m}{m}V^{\#}(X)\xi - \frac{\eta(V)}{m}((\lambda + m)X - QX) + QX + 2nX = 0. \tag{3.9}$$

The, taking the scalar product with ξ and applying (3.3), one has

$$\frac{\lambda + 2n + m}{m}(g(X,V) - \eta(X)\eta(V)) = 0. \tag{3.10}$$

Since λ and m are constants, (3.10) entails that either $V = \eta(V)\xi$ or $\lambda = -(2n + m)$.

Case 1. Differentiating $V = \eta(V)\xi$ along X and making use of (3.1) one has

$$\begin{aligned}\nabla_X V &= (\nabla_X \eta)(V)\xi + g(\nabla_X V,\xi)\xi + \eta(V)\nabla_X \xi\\ &= g(\nabla_X V,\xi)\xi + \eta(V)(X - \eta(X)\xi).\end{aligned}$$

The above formula combining with (3.7) we get

$$\begin{aligned}&\lambda X - QX + \frac{1}{m}V^{\#}(X)V\\ &\qquad = 2n\eta(X)\xi + \lambda\eta(X)\xi + \frac{1}{m}V^{\#}(X)\eta(V)\xi + \eta(V)(X - \eta(X)\xi).\end{aligned}$$

It follows that

$$\varphi QX = (\lambda - \eta(V))\varphi X, \tag{3.11}$$

where we applied $\varphi V = 0$. Setting X by φX in (3.11) and recalling that Ricci operator Q and φ commutes on M (see Lemma 4.1 of [**15**]), we obtain

$$QX = (\lambda - \eta(V))X - (\lambda - \eta(V) + 2n)\eta(X)\xi. \tag{3.12}$$

Contraction of (3.12) over X, one immediately obtains

$$\lambda - \eta(V) = \frac{r}{2n} + 1.$$

Uptaking the above equation in (3.12), we reach at

$$QX = \left(\frac{r}{2n} + 1\right) X - \left(\frac{r}{2n} + 2n + 1\right) \eta(X)\xi,$$

which means that M is η-Einstein. This completes the proof of (1).

Case 2. Here $\lambda = -(2n + m)$. Making use of this, by (3.9) one has

$$\left(\frac{\eta(V)}{m} + 1\right)(QX + 2nX) = 0. \tag{3.13}$$

If we consider $\eta(V) = -m$, then from (3.7), (3.3) one can find

$$\nabla_\xi V = \eta(V)\xi - V.$$

As a result of (3.1), one can prove that $\pounds_V\xi = 2(V - \eta(V)\xi)$. Equation (1.3) implies $\pounds_V g(X, \xi) = 2(\eta(X)\eta(V) - g(X, V))$, which in turn, shows that $\pounds_V\eta = 0$, which means that V is a strictly infinitesimal contact transformation. Let us suppose that $\eta(V) \neq -m$, then (3.13) shows that M is Einstein with constant scalar curvature $-2n(2n + 1)$. □

Ghosh [**13**] proved that a K-contact manifold with conformal m-quasi Einstein metric is η-Einstein (non-trivial) and V is Killing. But, in the case of Kenmotsu manifold, it is interesting to study m-quasi Einstein metric with V as conformal vector field. In this setting, we prove

Theorem 3.4. *Let M be a Kenmotsu manifold of dimension $2n + 1$. If the metric g is m-quasi Einstein and V is a conformal vector field, then M is Einstein with constant scalar curvature $-2n(2n + 1)$.*

P r o o f. Because of V is conformal vector field, the equation (1.3) becomes

$$Ric(X,Y) = (\lambda-\nu)g(X,Y) + \frac{1}{m}V^{\#}(X)V^{\#}(Y). \tag{3.14}$$

Differentiating (3.14), one has

$$(\nabla_Z Ric)(X,Y) \\ = -(Z\nu)g(X,Y) + \frac{1}{m}\{(\nabla_Z V^{\#})(X)V^{\#}(Y) + V^{\#}(X)(\nabla_Z V^{\#})(Y)\}.$$

Taking the cyclic sum of aforementioned equation over $\{X,Y,Z\}$, since V is conformal, we obtain

$$\begin{aligned}&(\nabla_X Ric)(Y,Z) + (\nabla_Y Ric)(Z,X)\\ &\quad + (\nabla_Z Ric)(X,Y) + \{(X\nu) - \frac{2\nu}{m}V^{\#}(X)\}g(Y,Z)\\ &\quad + \{(Y\nu) - \frac{2\nu}{m}V^{\#}(Y)\}g(X,Z) + \{(Z\nu) - \frac{2\nu}{m}V^{\#}(Z)\}g(X,Y) = 0.\end{aligned} \tag{3.15}$$

Contraction of (3.15) over Y and Z entails that

$$\frac{2}{2n+3}(Xr) + (X\nu) - \frac{2\nu}{m}V^{\#}(X) = 0.$$

As a result of this, (3.15) takes the form

$$\begin{aligned}&(\nabla_X Ric)(Y,Z) + (\nabla_Y Ric)(Z,X) + (\nabla_Z Ric)(X,Y)\\ &\quad - \frac{2}{2n+3}\{(Xr)g(Y,Z) + (Yr)(Z,X) + (Zr)g(X,Y)\} = 0.\end{aligned} \tag{3.16}$$

Putting $Y = Z = \xi$, by (3.4) and (3.5) we obtain

$$(Xr) + 2(\xi r)\eta(X) = 0. \tag{3.17}$$

Trace of (3.5) yields $(\xi r) = -2(r + 2n(2n+1))$. By this equation, one can find that $\pounds_\xi r = -2(r + 2n(2n+1))$. Applying d to this equation, since $\pounds_\xi$ commutes with d, we have $\pounds_\xi dr = -2dr$. In terms of gradient operator D, the last equation can be written as $\pounds_\xi Dr = -2Dr$. This together with (3.1) implies that

$$\nabla_\xi Dr = -Dr - (\xi r)\xi.$$

As a result of (3.17), one can find that

$$(\xi r)\xi + 2\xi(\xi r)\xi = 0.$$

In view of $(\xi r) = -2(r+2n(2n+1))$, we find that $\xi(\xi r) = -2(\xi r)$ and going back to the above equation, we have $(\xi r) = 0$, which shows that $r = -2n(2n+1)$. Hence, the scalar curvature r is constant. Uptaking this constancy of r in (3.16) one has

$$(\nabla_X Ric)(Y, Z) + (\nabla_Y Ric)(Z, X) + (\nabla_Z Ric)(X, Y) = 0.$$

Substituting Z in the aforesaid equation by ξ and taking account of (3.4) and (3.5) we have $QX = -2nX$, which shows that M is Einstein with negative scalar curvature. □

4 - m-quasi Einstein almost Kenmotsu $(\kappa, \mu)'$-manifolds

If the curvature tensor R of an almost Kenmotsu manifolds M satisfies

$$R(X, Y)\xi = \kappa\{\eta(Y)X - \eta(X)Y\} + \mu\{\eta(Y)h'X - \eta(X)h'Y\}, \tag{4.1}$$

for any vector fields X, Y and κ, μ are constants, then we call M an almost Kenmotsu $(\kappa, \mu)'$-manifold. Classification of almost Kenmotsu $(\kappa, \mu)'$-manifolds have done by many geometers. In this regard, we recommend [**7**, **9**, **20**, **24**, **26**, **28**, **31**, **32**] for more information. According to the results of [**10**], any almost Kenmotsu $(\kappa, \mu)'$-manifold satisfies $\mu = -2$ and $h'^2 = (\kappa + 1)\varphi^2$. From this, we have $\kappa \leq -1$ and the equality holds only if $h = 0$. We recall the following result for our later use.

Lemma 4.1 (Lemma 3 in [**32**]). *The expression of Ricci operator Q on an almost Kenmotsu $(\kappa, \mu)'$-manifold M is of the form*

$$QX = -2nX + 2n(\kappa + 1)\eta(X)\xi - 2nh'X, \tag{4.2}$$

where $\kappa \leq -1$. Moreover, the scalar curvature of M is $2n(\kappa - 2n)$.

Lemma 4.2 (Lemma 4.1 in [**10**]). *On an almost Kenmotsu $(\kappa, \mu)'$-manifold with $\kappa \leq -1$, we have*

$$\begin{aligned}(\nabla_X h')Y = g((\kappa + 1)X - h'X, Y)\xi \\ + \eta(Y)((\kappa + 1)X - h'X) - 2(\kappa + 1)\eta(X)\eta(Y)\xi.\end{aligned}$$

As seen in Section 3, it is known that closed m-quasi Einstein metric is an extension of gradient Ricci soliton, therefore in this section we study closed m-quasi Einstein structures on an almost Kenmotsu $(\kappa, \mu)'$-manifold M and generalize the result of Wang et al [**30**].

Theorem 4.3. *Let M be a non-Kenmotsu almost Kenmotsu $(\kappa,\mu)'$-manifold of dimension $2n+1$. If a metric of M is closed m-quasi Einstein, then M is locally isometric to the Riemannian product of an $(n+1)$-dimensional manifold of constant sectional curvature -4 and a flat n-dimensional manifold, provided that* $\dfrac{\lambda-\kappa(2n+m)}{2m}=1$.

Proof. By the support of Lemma 4.1, one can prove that

$$\begin{aligned}(\nabla_Y Q)X-(\nabla_X Q)Y=&-2n\{(\nabla_Y h')X-(\nabla_X h')Y\}\\&-2n(\kappa+1)\{\eta(Y)(X+h'X)-\eta(X)(Y+h'Y)\}.\end{aligned}$$

Then, by (3.6) and Lemma 4.2 we have

$$g(R(X,Y)V,\xi)=\frac{\lambda-2n\kappa}{m}\{V^{\#}(Y)\eta(X)-V^{\#}(X)\eta(Y)\}, \tag{4.3}$$

where we have used $Q\xi=2n\kappa\xi$. Going back to (4.1), equation (4.3) implies

$$\begin{aligned}\left(\frac{\lambda-2n\kappa}{m}-\kappa\right)&\{V^{\#}(Y)\eta(X)-V^{\#}(X)\eta(Y)\}\\&-2\{V^{\#}(h'X)\eta(Y)-V^{\#}(h'Y)\eta(X)\}=0.\end{aligned}$$

Replacing X in the above equation by ξ, we obtain

$$2V^{\#}(h'Y)=\frac{1}{m}(\lambda-\kappa(2n+m))\{\eta(V)\eta(Y)-V^{\#}(Y)\}.$$

Since $\lambda-\kappa(2n+m)=2m$ and h' is a symmetric operator, the above equation implies

$$h'V=\eta(V)\xi-V. \tag{4.4}$$

Recalling the relation $h'^2=(\kappa+1)\varphi^2$, the action of h' on (4.4) yields $(\kappa+1)(V-\eta(V)\xi)=h'V$. This together with (4.4) implies that

$$(\kappa+2)(V-\eta(V)\xi)=0. \tag{4.5}$$

From this, we have either $\kappa=-2$ or $V=\eta(V)\xi=f\xi$. Now, we show that the second case cannot occur on M. Let us suppose that the second case is true, that is, $V=f\xi$, where $f=\eta(V)$ is a smooth function, the covariant derivative of this relation along X yields

$$\nabla_X V=(Xf)\xi+f(X-\eta(X)\xi+h'X).$$

Since $V^{\#}$ is closed, we have $\nabla_X V = \lambda X - QX + \frac{1}{m}V^{\#}(X)V$ and going back to the above equation we have

$$\lambda X - QX + \frac{1}{m}V^{\#}(X)V = (Xf)\xi + f(X - \eta(X)\xi + h'X). \tag{4.6}$$

Applying (4.2), by (4.6) we obtain

$$\begin{aligned}(\lambda + 2n - f)X + (2n - f)h'X + (f\eta(X) - (Xf) \\ - 2n(\kappa+1)\eta(X))\xi + \frac{1}{m}V^{\#}(X)V = 0.\end{aligned} \tag{4.7}$$

By $h'\xi = 0$, (4.7) implies

$$(\lambda + 2n - f)h'X + (2n - f)h'^2X = 0, \tag{4.8}$$

where we have employed $h'V = 0$. Calling up the relation $h'^2 = (\kappa+1)\varphi^2$, the contraction of (4.8) yields

$$2n(\kappa+1)(2n - f) = 0,$$

and this implies $f = 2n$ because of $\kappa < -1$. Substituting in (4.7) we obtain

$$\lambda m X + 2n(2n - \kappa m)\eta(X)\xi = 0. \tag{4.9}$$

Taking X orthogonal to ξ, we have $\lambda = 0$. Thus, (4.9) becomes $(2n - \kappa m)\eta(X)\xi = 0$. By virtue of this, we have $\kappa = \frac{2n}{m} > -1$, which contradicts our assumption. So that the only choice is $\kappa = -2$. By Corollary 4.2 and Proposition 4.1 [**10**] we claim that M is locally isometric to the Riemannian product $\mathbb{H}^{n+1} \times \mathbb{R}^n$. Substituting $\kappa = -2$ in the relation $\frac{\lambda - \kappa(2n+m)}{2m} = 1$, we obatin $\lambda = -4n$, which shows that a closed m-quasi Einstein metric is expanding. □

Remark 4.4. Theorem 1.2 of Wang et al [**30**] is a direct corollary of our Theorem 4.3.

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