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On generalized matrix algebras having multiplicative generalized Lie type derivations

Abstract. Let $\mathfrak{R}$ be a commutative ring with unity. The $\mathfrak{R}$-algebra $\mathfrak{G} = \mathfrak{G}(\mathrm{A}, \mathrm{M}, \mathrm{N}, \mathrm{B})$ is a generalized matrix algebra defined by the Morita context $(\mathrm{A}, \mathrm{B}, \mathrm{M}, \mathrm{N}, \xi_{\mathrm{MN}}, \Omega_{\mathrm{NM}})$. In this article, we study multiplicative generalized Lie type derivations on generalized matrix algebras and prove that it has the standard form.

Keywords. Generalized matrix algebras, generalized derivation, generalized Lie derivation.

Mathematics Subject Classification: 16W25, 47L35, 15A78.

1 - Historical Development

There has been a great deal of work concerning characterizations of Lie derivations on rings. The first characterization of Lie derivations was obtained by Martindale [**16**] in 1964 who proved that every Lie derivation on primitive ring can be written as a sum of derivation and an additive mapping of ring to its center that maps commutators into zero, i.e, Lie derivation has the standard form. Chen and Zhang [**4**] described multiplicative Lie derivation on upper triangular matrix algebra. Further, the authors [**2**] characterized multiplicative generalized Lie triple derivations on triangular algebras. Following the well-established approach and the sophisticated computational method by Cheung [**5**], several authors studied the different linear mappings on generalized matrix algebras for example [**6**, **7**, **10**, **11**, **12**, **13**, **17**, **23**] and the bibliographic content existing therein.

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Recently, many authors studied Lie $\mathtt{n}$-derivation on various kind of algebras [**3**, **8**, **9**, **15**, **17**] and references therein. In the year 2014, Wang and Wang [**22**] studied multiplicative Lie $\mathtt{n}$-derivation on generalized matrix algebras and proved that it has standard form under certain assumptions. Also, Wang [**21**] investigated Lie $\mathtt{n}$-derivation on unital algebras with idempotents and obtained that every Lie $\mathtt{n}$-derivation can be written as a sum of derivation, singular Jordan derivation and anti-derivation on the unital algebras with idempotents. Furthermore, Qi [**19**] characterized Lie $\mathtt{n}$-derivation on reflexive algebras and obtained that it has the standard form, i.e, it can be expressed as the sum of linear derivation and linear functional vanishing at every $(\mathtt{n-1})$-th commutator on reflexive algebras. Lin [**14**] carried out the study of multiplicative generalized Lie $\mathtt{n}$-derivation on triangular algebras and proved that every multiplicative Lie $\mathtt{n}$-derivation can be written as sum of additive generalized derivation and a central mapping annihilating $(\mathtt{n-1})$-th commutator on triangular algebras under some limitations.

Motivated by these studies our main purpose is to characterize multiplicative generalized Lie $\mathtt{n}$-derivation on generalized matrix algebra and show that every multiplicative generalized Lie $\mathtt{n}$-derivation on generalized matrix algebra can be written as the sum of an additive generalized derivation and a central mapping annihilating $(\mathtt{n-1})$-th commutator with some limitations. We also study some direct implications of our main result.

2 - Basic Definitions & Preliminaries

Let $\mathfrak{R}$ be a commutative ring with unity and $\mathfrak{U}$ be an $\mathfrak{R}$-algebra having center $\mathfrak{Z}(\mathfrak{U})$. A map $\mathrm{L} : \mathfrak{U} \to \mathfrak{U}$ (not necessarily linear) is called a multiplicative derivation (resp. multiplicative Lie derivation) on $\mathfrak{U}$ if $\mathrm{L}(ab) = \mathrm{L}(a)b + a\mathrm{L}(b)$ (resp. $\mathrm{L}([a, b]) = [\mathrm{L}(a), b] + [a, \mathrm{L}(b)]$) holds for all $a, b \in \mathfrak{U}$. In addition, if L is linear on $\mathfrak{U}$, then L is said to be a derivation (resp. Lie derivation) on $\mathfrak{U}$. A map $\mathrm{G_L} : \mathfrak{U} \to \mathfrak{U}$ (not necessarily linear) is called a multiplicative generalized derivation (resp. multiplicative generalized Lie derivation) on $\mathfrak{U}$ associated with a multiplicative derivation (resp. multiplicative Lie derivation) L on $\mathfrak{U}$ if $\mathrm{G_L}(ab) = \mathrm{G_L}(a)b + a\mathrm{L}(b)$ (resp. $\mathrm{G_L}([a, b]) = [\mathrm{G_L}(a), b] + [a, \mathrm{L}(b)]$) holds for all $a, b \in \mathfrak{U}$. In addition, if $\mathrm{G_L}$ is linear associated with a derivation (resp. Lie derivation) L on $\mathfrak{U}$, then $\mathrm{G_L}$ is said to be a generalized derivation (resp. generalized Lie derivation) on $\mathfrak{U}$.

For a broad scope of maps. Define the family of polynomials:

$$\begin{aligned} \mathfrak{p}_1(x_1) &= x_1 \\ \mathfrak{p}_2(x_1, x_2) &= [\mathfrak{p}_{\mathtt{n}}(x_1), x_2] = [x_1, x_2] \end{aligned}$$

$$\begin{aligned}
\mathfrak{p}_3(x_1, x_2, x_3) &= [\mathfrak{p}_{\mathtt{n}}(x_1, x_2), x_3] = [[x_1, x_2], x_3] \\
&\vdots \\
\mathfrak{p}_{\mathtt{n}}(x_1, x_2, \cdots, x_{\mathtt{n}}) &= [\mathfrak{p}_{\mathtt{n}}(x_1, x_2, \cdots, x_{\mathtt{n-1}}), x_{\mathtt{n}}].
\end{aligned}$$

The polynomial $\mathfrak{p}_{\mathtt{n}}(x_1, x_2, \cdots, x_{\mathtt{n}})$ is called $\mathtt{n}$-th commutator where $\mathtt{n} \geq 2$. A map (not necessarily linear) $\mathrm{L} : \mathfrak{U} \to \mathfrak{U}$ is said to be a multiplicative Lie $\mathtt{n}$-derivation on $\mathfrak{U}$ if

$$\mathrm{L}(\mathfrak{p}_{\mathtt{n}}(x_1, x_2, \cdots, x_{\mathtt{n}})) = \sum_{\mathtt{i=1}}^{\mathtt{i=n}} \mathfrak{p}_{\mathtt{n}}(x_1, x_2, \cdots, x_{\mathtt{i-1}}, \mathrm{L}(x_{\mathtt{i}}), x_{\mathtt{i+1}}, \cdots, x_{\mathtt{n}})$$

for all $x_1, x_2, \cdots, x_{\mathtt{n}} \in \mathfrak{U}$. This notion of Lie $\mathtt{n}$-derivation developed on certain von Neumann algebras by Abdullaev in [**1**]. Undoubtedly, any multiplicative Lie $\mathtt{2}$-derivation is multiplicative Lie derivation and multiplicative Lie $\mathtt{3}$-derivation is multiplicative Lie triple derivation and so on.

Further, A map (not necessarily linear) $\mathrm{G_L} : \mathfrak{U} \to \mathfrak{U}$ is said to be a multiplicative generalized Lie $\mathtt{n}$-derivation on $\mathfrak{U}$ if there exists a multiplicative Lie $\mathtt{n}$-derivation L such that

$$\begin{aligned}
\mathrm{G_L}(\mathfrak{p}_{\mathtt{n}}(x_1, x_2, \cdots, x_{\mathtt{n}})) &= \mathfrak{p}_{\mathtt{n}}(\mathrm{G_L}(x_1), x_2, \cdots, x_{\mathtt{i-1}}, x_{\mathtt{i}}, x_{\mathtt{i+1}}, \cdots, x_{\mathtt{n}}) \\
&\quad + \sum_{\mathtt{i=2}}^{\mathtt{i=n}} \mathfrak{p}_{\mathtt{n}}(x_1, x_2, \cdots, x_{\mathtt{i-1}}, \mathrm{L}(x_{\mathtt{i}}), x_{\mathtt{i+1}}, \cdots, x_{\mathtt{n}})
\end{aligned}$$

for all $x_1, x_2, \cdots, x_{\mathtt{n}} \in \mathfrak{U}$. Obviously, any multiplicative generalized Lie $\mathtt{2}$-derivation is multiplicative generalized Lie derivation and multiplicative generalized Lie $\mathtt{3}$-derivation is multiplicative generalized Lie triple derivation and so on. These maps collectively known as multiplicative generalized Lie type derivations on $\mathfrak{U}$.

A *Morita context* consists of two unital $\mathfrak{R}$-algebras A and B, two bimodules (A, B)-bimodule M and (B, A)-bimodule N, and two bimodule homomorphisms called the bilinear pairings $\xi_{\mathrm{MN}} : \mathrm{M} \underset{\mathrm{B}}{\otimes} \mathrm{N} \longrightarrow \mathrm{A}$ and $\Omega_{\mathrm{NM}} : \mathrm{N} \underset{\mathrm{A}}{\otimes} \mathrm{M} \longrightarrow \mathrm{B}$ satisfying the following commutative diagrams:

$$\begin{array}{ccc}
\mathrm{M} \underset{\mathrm{B}}{\otimes} \mathrm{N} \underset{\mathrm{A}}{\otimes} \mathrm{M} & \xrightarrow{\xi_{\mathrm{MN}} \otimes I_{\mathrm{M}}} & \mathrm{A} \underset{\mathrm{A}}{\otimes} \mathrm{M} \\
\Big\downarrow {\scriptstyle I_{\mathrm{M}} \otimes \Omega_{\mathrm{NM}}} & & \Big\downarrow {\scriptstyle \cong} \\
\mathrm{M} \underset{\mathrm{B}}{\otimes} \mathrm{B} & \xrightarrow{\cong} & \mathrm{M}
\end{array}
\quad \text{and} \quad
\begin{array}{ccc}
\mathrm{N} \underset{\mathrm{A}}{\otimes} \mathrm{M} \underset{\mathrm{B}}{\otimes} \mathrm{N} & \xrightarrow{\Omega_{\mathrm{NM}} \otimes I_{\mathrm{N}}} & \mathrm{B} \underset{\mathrm{B}}{\otimes} \mathrm{N} \\
\Big\downarrow {\scriptstyle I_{\mathrm{N}} \otimes \xi_{\mathrm{MN}}} & & \Big\downarrow {\scriptstyle \cong} \\
\mathrm{N} \underset{\mathrm{A}}{\otimes} \mathrm{A} & \xrightarrow{\cong} & \mathrm{N}.
\end{array}$$

If $(\mathrm{A},\mathrm{B},\mathrm{M},\mathrm{N},\ \xi_{\mathrm{MN}},\Omega_{\mathrm{NM}})$ is a Morita context, then the set

$$\begin{bmatrix} \mathrm{A} & \mathrm{M} \\ \mathrm{N} & \mathrm{B} \end{bmatrix} = \left\{ \begin{bmatrix} a & m \\ n & b \end{bmatrix} \middle| a \in \mathrm{A}, m \in \mathrm{M}, n \in \mathrm{N}, b \in \mathrm{B} \right\}$$

forms an $\mathfrak{R}$-algebra under matrix addition and matrix-like multiplication consisting one of two bimodules M and N is nonzero. Aforesaid an $\mathfrak{R}$-algebra known as *generalized matrix algebra* of order 2 which is symbolized by

$$\mathfrak{G} = \mathfrak{G}(\mathrm{A},\mathrm{M},\mathrm{N},\mathrm{B}) = \begin{bmatrix} \mathrm{A} & \mathrm{M} \\ \mathrm{N} & \mathrm{B} \end{bmatrix}.$$

This kind of algebra was first introduced by Morita in [**18**]. All associative algebras having nontrivial idempotents are isomorphic to generalized matrix algebras. The familiar examples of generalized matrix algebras are full matrix algebras over a unital algebra and triangular algebras [**20**, **22**]. Moreover, $\mathfrak{G}$ is mentioned as triangular algebra if $\mathrm{N} = 0$.

The center of $\mathfrak{G}$ is

$$\mathfrak{Z}(\mathfrak{G}) = \left\{ \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \middle| am = mb, na = bn \text{ for all } m \in \mathrm{M}, n \in \mathrm{N} \right\}.$$

Define two natural projections $\pi_{\mathrm{A}} : \mathfrak{G} \to \mathrm{A}$ and $\pi_{\mathrm{B}} : \mathfrak{G} \to \mathrm{B}$ by $\pi_{\mathrm{A}}\left(\begin{bmatrix} a & m \\ n & b \end{bmatrix}\right) = a$ and $\pi_{\mathrm{B}}\left(\begin{bmatrix} a & m \\ n & b \end{bmatrix}\right) = b$. Moreover, $\pi_{\mathrm{A}}(\mathfrak{Z}(\mathfrak{G})) \subseteq \mathfrak{Z}(\mathrm{A})$ and $\pi_{\mathrm{B}}(\mathfrak{Z}(\mathfrak{G})) \subseteq \mathfrak{Z}(\mathrm{B})$ and there exists a unique algebraic isomorphism $\xi : \pi_{\mathrm{A}}(\mathfrak{Z}(\mathfrak{G})) \to \pi_{\mathrm{B}}(\mathfrak{Z}(\mathfrak{G}))$ such that $am = m\xi(a)$ and $na = \xi(a)n$ for all $a \in \pi_{\mathrm{A}}(\mathfrak{Z}(\mathfrak{G})), m \in \mathrm{M}$ and $n \in \mathrm{N}$.

Let 1_{A} (resp.1_{B}) be the identity of the algebra A (resp. B) and let I be the identity of generalized matrix algebra $\mathfrak{G}$, $e = \begin{bmatrix} 1_{\mathrm{A}} & 0 \\ 0 & 0 \end{bmatrix}$, $f = I - e = \begin{bmatrix} 0 & 0 \\ 0 & 1_{\mathrm{B}} \end{bmatrix}$ and $\mathfrak{G}_{11} = e\mathfrak{G}e$, $\mathfrak{G}_{12} = e\mathfrak{G}f$, $\mathfrak{G}_{21} = f\mathfrak{G}e$, $\mathfrak{G}_{22} = f\mathfrak{G}f$. Thus $\mathfrak{G} = e\mathfrak{G}e + e\mathfrak{G}f + f\mathfrak{G}e + f\mathfrak{G}f = \mathfrak{G}_{11} + \mathfrak{G}_{12} + \mathfrak{G}_{21} + \mathfrak{G}_{22}$ where $\mathfrak{G}_{11}$ is subalgebra of $\mathfrak{G}$ isomorphic to A, $\mathfrak{G}_{22}$ is subalgebra of $\mathfrak{G}$ isomorphic to B, $\mathfrak{G}_{12}$ is $(\mathfrak{G}_{11}, \mathfrak{G}_{22})$-bimodule isomorphic to M and $\mathfrak{G}_{21}$ is $(\mathfrak{G}_{22}, \mathfrak{G}_{11})$-bimodule isomorphic to N.

Now we mention some results which will be used subsequently in developing the proof of our results:

Lemma 2.1 ([**2**, Proposition 4.3]). *Every generalized derivation d on $\mathfrak{G}$ has the form*

$$d\left(\begin{bmatrix} a & m \\ 0 & b \end{bmatrix}\right) = \begin{bmatrix} \mathcal{U}(a) & an_0 - n_0 b + \mathcal{W}(m) \\ 0 & \mathcal{V}(b) \end{bmatrix},$$

where $a \in \mathrm{A}, m, n_0 \in \mathrm{M}, b \in \mathrm{B}$ *and* $\mathcal{U} : \mathrm{A} \to \mathrm{A}, \mathcal{W} : \mathrm{M} \to \mathrm{M}, \mathcal{V} : \mathrm{B} \to \mathrm{B}$ *are* $\mathfrak{R}$*-linear mappings satisfying*

(i) $\mathcal{U}$ *is a generalized derivation on* A *and* $\mathcal{W}(am) = \mathcal{U}(a)m + a\mathcal{W}(m)$ *for all* $a \in \mathrm{A}$ *and* $m \in \mathrm{M}$,

(ii) $\mathcal{V}$ *is a generalized derivation on* B *and* $\mathcal{W}(mb) = m\mathcal{V}(b) + \mathcal{W}(m)b$ *for all* $b \in \mathrm{B}$ *and* $m \in \mathrm{M}$.

Lemma 2.2 ([**14**, Theorem 3.3]). *Let* $\mathfrak{T}$ *be a* $(\mathfrak{n}\text{-}1)$*-torsion free triangular algebra such that*

1. $\mathfrak{Z}(\mathrm{A}) = \pi_{\mathrm{A}}(\mathfrak{Z}(\mathfrak{T}))$ *and* $\mathfrak{Z}(\mathrm{B}) = \pi_{\mathrm{B}}(\mathfrak{Z}(\mathfrak{T}))$,
2. *For any* $a \in \mathrm{A}$, *if* $[a, \mathrm{A}] \in \mathfrak{Z}(\mathrm{A})$, *then* $a \in \mathfrak{Z}(\mathrm{A})$.

If $\mathrm{G_L} : \mathfrak{T} \to \mathfrak{T}$ *is a multiplicative generalized Lie* $\mathfrak{n}$*-derivation, then there exists an additive generalized derivation* d *of* $\mathfrak{T}$ *and a map* $\tau : \mathfrak{T} \to \mathfrak{Z}(\mathfrak{T})$ *vanishing at* $\mathfrak{p}_{\mathfrak{n}}(x_1, x_2, \ldots, x_{\mathfrak{n}})$ *for all* $x_1, x_2, \ldots, x_{\mathfrak{n}} \in \mathfrak{T}$ *such that* $\mathrm{G_L} = d + \tau$.

3 - Key Content

The main result of the this paper states as follows:

Theorem 3.1. *Let* $\mathfrak{G}$ *be a* $(\mathfrak{n}\text{-}1)-$*torsion free generalized matrix algebra such that*

1. $\mathfrak{Z}(\mathrm{A}) = \pi_{\mathrm{A}}(\mathfrak{Z}(\mathfrak{G}))$ *and* $\mathfrak{Z}(\mathrm{B}) = \pi_{\mathrm{B}}(\mathfrak{Z}(\mathfrak{G}))$,
2. A *or* B *does not contain nonzero central ideals,*
3. *for* $\mathfrak{n} \geq 3$ *and for any* $a \in \mathrm{A}$, *if* $[a, \mathrm{A}] \in \mathfrak{Z}(\mathrm{A})$, *then* $a \in \mathfrak{Z}(\mathrm{A})$,
4. *if* $\mathfrak{n}$ *is odd and* $\mathrm{N} \neq 0$, *then for each* $m \in \mathrm{M}$ *the condition* $m\mathrm{N} = 0 = \mathrm{N}m$ *implies* $m = 0$,
5. *if* $\mathfrak{n}$ *is odd and* $\mathrm{M} \neq 0$, *then for each* $n \in \mathrm{N}$ *the condition* $n\mathrm{M} = 0 = \mathrm{M}n$ *implies* $n = 0$.

If $\mathrm{G_L} : \mathfrak{G} \to \mathfrak{G}$ *is a multiplicative generalized Lie* $\mathfrak{n}$*-derivation, then there exists an additive generalized derivation* δ *of* $\mathfrak{G}$ *and a map* $\phi : \mathfrak{G} \to \mathfrak{Z}(\mathfrak{G})$ *vanishing at* $\mathfrak{p}_{\mathfrak{n}}(x_1, x_2, \ldots, x_{\mathfrak{n}})$ *for all* $x_1, x_2, \ldots, x_{\mathfrak{n}} \in \mathfrak{G}$ *such that* $\mathrm{G_L} = \delta + \phi$.

Let us assume that $\mathrm{G_L} : \mathfrak{G} \to \mathfrak{G}$ is a generalized Lie $\mathtt{n}$-derivation with associated Lie $\mathtt{n}$-derivation L. Set

$$\mathfrak{T} = e\mathfrak{G}e + e\mathfrak{G}f + f\mathfrak{G}f.$$

Then $\mathfrak{T}$ is a triangular algebra itself. Also note that $\mathfrak{Z}(\mathfrak{T}) = \mathfrak{Z}(\mathfrak{G})$. Now we define maps $\delta_1, \mathrm{G}'_\mathrm{L} : \mathfrak{G} \to \mathfrak{G}$ by

$$\delta_1(x) = [\mathrm{G_L}(f), x] \text{ and } \mathrm{G}'_\mathrm{L}(x) = \mathrm{G_L}(x) - \delta_1(x).$$

It is easy to verify that δ_1 is an inner derivation and G'_L is a multiplicative generalized Lie $\mathtt{n}$-derivation. Since

$$\mathrm{G}'_\mathrm{L}(f) = \mathrm{G_L}(f) - \delta_1(f) = \mathrm{G_L}(f) - [\mathrm{G_L}(f), f] = \mathrm{G_L}(f) - \mathrm{G_L}(f)f + f\mathrm{G_L}(f),$$

we find that $e\mathrm{G}'_\mathrm{L}(f)f = 0$.

Now we consider only the multiplicative generalized Lie $\mathtt{n}$-derivation which satisfies $e\mathrm{G_L}(f)f = 0$. The proof of the theorem will be organized in a series of lemmas:

Lemma 3.1. $\mathrm{G_L}(0) = 0$.

Proof. Obviously true. □

Lemma 3.2. $e\mathrm{G_L}(x)f = 0$ *for all* $x \in \mathrm{A} \cup \mathrm{B}$.

Proof. Using $[x, f] = 0$, we find that

$$\begin{aligned} 0 &= \mathrm{G_L}(\mathfrak{p}_\mathtt{n}(x, f, \ldots, f)) \\ &= \mathfrak{p}_\mathtt{n}(\mathrm{G_L}(x), f, \ldots, f) + \mathfrak{p}_\mathtt{n}(x, \mathrm{L}(f), \ldots, f). \end{aligned}$$

Multiplying by e from the left and by f from the right, we obtain that

$$0 = e\mathrm{G_L}(x)f + e[x, \mathrm{L}(f)]f. \tag{3.1}$$

Also, we have

$$\begin{aligned} 0 &= \mathrm{G_L}(\mathfrak{p}_\mathtt{n}(f, f, \ldots, f)) \\ &= \mathfrak{p}_\mathtt{n}(\mathrm{G_L}(f), f, \ldots, f) + \mathfrak{p}_\mathtt{n}(f, \mathrm{L}(f), \ldots, f) \\ &= e\mathrm{G_L}(f)f - e\mathrm{L}(f)f \\ &= -e\mathrm{L}(f)f. \end{aligned}$$

Hence in view of (3.1), we find that $e\mathrm{G_L}(x)f = 0$ for all $x \in \mathrm{A} \cup \mathrm{B}$. □

Now we define a map $\delta_2 : \mathfrak{G} \to \mathfrak{G}$ by

$$\delta_2(x) = [\mathrm{G_L}(e), x] \text{ and } \mathrm{G}''_{\mathrm{L}}(x) = \mathrm{G_L}(x) - \delta_2(x).$$

Obviously, δ_2 is an inner derivation and $\mathrm{G}''_{\mathrm{L}}$ is a multiplicative generalized Lie $\mathtt{n}$-derivation. In view of Lemma 3.2, we find that

$$e\mathrm{G}''_{\mathrm{L}}(x)f = e\mathrm{G_L}(x)f - e\delta_2(x)f = e\mathrm{G_L}(x)f - e[\mathrm{G_L}(e), x]f = 0$$

for all $x \in \mathrm{A} \cup \mathrm{B}$. Also, we have

$$\mathrm{G}''_{\mathrm{L}}(e) = \mathrm{G_L}(e) - \delta_2(e) = \mathrm{G_L}(e) - [\mathrm{G_L}(e), e] = \mathrm{G_L}(e) - f\mathrm{G_L}(e)e.$$

This gives $f\mathrm{G}''_{\mathrm{L}}(e)e = 0$. Now consider only those multiplicative generalized Lie $\mathtt{n}$-derivation which satisfies $f\mathrm{G_L}(e)e = 0$. In particular, we have $f\mathrm{L}(e)e = 0$.

Lemma 3.3. *$f\mathrm{G_L}(x)e = 0$ and $\mathrm{G_L}(x) = e\mathrm{G_L}(x)e + f\mathrm{G_L}(x)f$ for all $x \in \mathrm{A} \cup \mathrm{B}$.*

Proof. Since $[x, e] = 0$ for all $x \in \mathrm{A} \cup \mathrm{B}$, we have

$$\begin{aligned} 0 &= \mathrm{G_L}(\mathfrak{p}_{\mathtt{n}}(x, e, \ldots, e)) \\ &= \mathfrak{p}_{\mathtt{n}}(\mathrm{G_L}(x), e, \ldots, e) + \mathfrak{p}_{\mathtt{n}}(x, \mathrm{L}(e), \ldots, e) \\ &= f\mathrm{G_L}(x)e - f[x, \mathrm{L}(e)]e \\ &= f\mathrm{G_L}(x)e. \end{aligned}$$

Using Lemma 3.2, the above yields that $\mathrm{G_L}(x) = e\mathrm{G_L}(x)e + f\mathrm{G_L}(x)f$ for all $x \in \mathrm{A} \cup \mathrm{B}$. □

Lemma 3.4. *$\mathrm{G_L}(m) = e\mathrm{G_L}(m)f$ for all $m \in \mathrm{M}$.*

Proof. For any $m \in \mathrm{M}$, we have

$$\begin{aligned} \mathrm{G_L}(m) &= \mathrm{G_L}(\mathfrak{p}_{\mathtt{n}}(m, -e, \ldots, -e)) \\ &= \mathfrak{p}_{\mathtt{n}}(\mathrm{G_L}(m), -e, \ldots, -e) + \sum_{\mathtt{k}=2}^{\mathtt{n}} \mathfrak{p}_{\mathtt{n}}(m, -e, \ldots, \underbrace{\mathrm{L}(-e)}_{\mathtt{k}th-place}, \ldots, -e) \\ &= e\mathrm{G_L}(m)f + (-1)^{\mathtt{n}-1} f\mathrm{G_L}(m)e + (\mathtt{n}-1)[m, \mathrm{L}(-e)] \end{aligned} \tag{3.2}$$

Multiplying by e from the left and by f from the right, we obtain that $(\mathtt{n}-1)[m, \mathrm{L}(-e)] = 0$ and hence $[m, \mathrm{L}(-e)] = 0$ for all $m \in \mathrm{M}$. This implies that $\mathrm{L}(-e) \in \mathfrak{Z}(\mathfrak{G})$. From (3.2), we obtain that

$$\mathrm{G_L}(m) = e\mathrm{G_L}(m)f + (-1)^{\mathtt{n}-1} f\mathrm{G_L}(m)e. \tag{3.3}$$

Now if $\mathtt{n}$ is even, then form (3.3) we find that $\mathrm{G_L}(m) = e\mathrm{G_L}(m)f - f\mathrm{G_L}(m)e$. On multiplying by f from the left and by e from the right, we get $f\mathrm{G_L}(m)e = 0$. Finally, $\mathrm{G_L}(m) = e\mathrm{G_L}(m)f$.

If $\mathtt{n}$ is odd, then form (3.3) we find that $\mathrm{G_L}(m) = e\mathrm{G_L}(m)f + f\mathrm{G_L}(m)e$. Since $[m_1, m_2] = 0$ for all $m_1, m_2 \in \mathrm{M}$. Now we have

$$\begin{aligned} 0 &= \mathrm{G_L}(\mathfrak{p}_{\mathtt{n}}(m, m_1, m_2, -e, \ldots, -e)) \\ &= \mathfrak{p}_{\mathtt{n}}(\mathrm{G_L}(m), m_1, m_2, -e \ldots, -e) + \mathfrak{p}_{\mathtt{n}}(m, \mathrm{L}(m_1), m_2, -e, \ldots, -e) \\ &= [[f\mathrm{G_L}(m)e, m_1] + [m, f\mathrm{L}(m_1)e], m_2]. \end{aligned}$$

This leads to $[f\mathrm{G_L}(m)e, m_1] + [m, f\mathrm{L}(m_1)e] \in \mathfrak{Z}(\mathfrak{G})$. On the other way,

$$\begin{aligned} 0 &= \mathrm{G_L}(\mathfrak{p}_{\mathtt{n}}(m, -e, \ldots, -e, m_1)) \\ &= \mathfrak{p}_{\mathtt{n}}(\mathrm{G_L}(m), -e, \ldots, -e, m_1) + \sum_{\mathtt{k=2}}^{\mathtt{n}} \mathfrak{p}_{\mathtt{n}}(m, \mathrm{L}(-e), -e, \ldots, -e, m_1) \\ &= [-f\mathrm{G_L}(m)e, m_1] + [m, f\mathrm{L}(m_1)e]. \end{aligned}$$

On comparing the last two expressions, we find that $2[f\mathrm{G_L}(m)e, m_1] \in \mathfrak{Z}(\mathfrak{G})$ implies that

$$f\mathrm{G_L}(m)em_1 - m_1 f\mathrm{G_L}(m)e \in \mathfrak{Z}(\mathfrak{G})$$

for all $m, m_1 \in \mathrm{M}$. Hence it follows that $f\mathrm{G_L}(m)e\mathrm{M} \in \mathfrak{Z}(\mathrm{B})$ and $\mathrm{M}f\mathrm{G_L}(m)e \in \mathfrak{Z}(\mathrm{A})$. Since $\mathrm{M}f\mathrm{G_L}(m)e$ is central ideal of A and hence $\mathrm{M}f\mathrm{G_L}(m)e = 0$, then it leads to $f\mathrm{G_L}(m)e\mathrm{M} = 0$. Similarly, $f\mathrm{G_L}(m)e\mathrm{M} = 0$. Now by assumption (5), we get $f\mathrm{G_L}(m)e = 0$ for all $m \in \mathrm{M}$. Hence from (3.3), we obtain that $\mathrm{G_L}(m) = e\mathrm{G_L}(m)f$ for all $m \in \mathrm{M}$. □

Lemma 3.5. *$\mathrm{G_L}(exf) = e\mathrm{G_L}(x)f$ and $f\mathrm{G_L}(x)e = 0$ for all $x \in \mathfrak{T}$.*

Proof. For any $m \in \mathrm{M}$, it follows that

$$\begin{aligned} \mathrm{G_L}(m) &= \mathrm{G_L}(\mathfrak{p}_{\mathtt{n}}(m, f, -e, \ldots, -e)) \\ &= \mathfrak{p}_{\mathtt{n}}(\mathrm{G_L}(m), f, -e, \ldots, -e) + \mathfrak{p}_{\mathtt{n}}(m, \mathrm{L}(f), -e, \ldots, -e) \\ &\quad + \mathfrak{p}_{\mathtt{n}}(m, f, \mathrm{L}(-e), \ldots, -e) + \cdots + \mathfrak{p}_{\mathtt{n}}(m, f, -e, \ldots, \mathrm{L}(-e)) \\ &= e\mathrm{G_L}(m)f + [m, \mathrm{L}(f)]. \end{aligned}$$

On multiplying by e from the left and by f from the right, we have $[m, \mathrm{L}(f)] = 0$ and hence $\mathrm{L}(f) \in \mathfrak{Z}(\mathfrak{G})$. This leads to $\mathrm{G_L}(m) = e\mathrm{G_L}(m)f$. For any $x \in \mathfrak{T}$, we obtain that

$$\mathrm{G_L}(exf) = \mathrm{G_L}(\mathfrak{p}_{\mathtt{n}}(x, f, \ldots, f))$$

$$= \mathfrak{p}_n(G_L(x), f, \ldots, f) + \sum_{k=2}^{n} \mathfrak{p}_n(x, L(f), \ldots, f)$$

$$= eG_L(x)f + (-1)^{n-1} f G_L(x)e. \tag{3.4}$$

Since $G_L(exf) \in M$, we find that $(-1)^{n-1} f G_L(x)e = 0$ and hence $fG_L(x)e = 0$. From (3.4), we get $G_L(exf) = eG_L(x)f$ for all $x \in \mathfrak{T}$. □

Lemma 3.6. *For any* $a \in A$, $b \in B$ *and* $m \in M$, *we have*

1. $G_L(a + m) - G_L(a) - G_L(m) \in \mathfrak{Z}(\mathfrak{G})$,
2. $G_L(b + m) - G_L(b) - G_L(m) \in \mathfrak{Z}(\mathfrak{G})$.

Proof. Since $L(f) \in \mathfrak{Z}(\mathfrak{G})$, using Lemma 3.5, for any $a \in A$ and $m \in M$, we have

$$\begin{aligned} G_L(am_1) &= G_L(\mathfrak{p}_n(a + m, m_1, f, \ldots, f)) \\ &= \mathfrak{p}_n(G_L(a + m), m_1, f, \ldots, f) + \sum_{i=2}^{n} \mathfrak{p}_n(a + m, L(m_1), f, \ldots, f) \\ &= [eG_L(a + m)e + fG_L(a + m)f, m_1] + [a, L(m_1)]. \end{aligned}$$

On the other hand, we get

$$\begin{aligned} G_L(am_1) &= G_L(\mathfrak{p}_n(a, m_1, f, \ldots, f)) \\ &= \mathfrak{p}_n(G_L(a), m_1, f, \ldots, f) + \sum_{k=2}^{n} \mathfrak{p}_n(a, L(m_1), f, \ldots, f) \\ &= [G_L(a), m_1] + [a, L(m_1)]. \end{aligned}$$

Combining the above two expressions, we find that $[eG_L(a + m)e + fG_L(a + m)f - G_L(a), m_1] = 0$. Therefore, $G_L(a + m) - eG_L(a + m)f - G_L(a) \in \mathfrak{Z}(\mathfrak{G})$. This gives that $G_L(a + m) - G_L(m) - G_L(a) \in \mathfrak{Z}(\mathfrak{G})$ for all $a \in A$ and $m \in M$. Similarly, we can show the other part. □

Lemma 3.7. G_L *is additive on* M.

Proof. For any $m_1, m_2 \in M$, in view of Lemma 3.6 we have

$$\begin{aligned} G_L(m_1 + m_2) &= G_L(\mathfrak{p}_n(f + m_1, -e - m_2, f, \ldots, f)) \\ &= \mathfrak{p}_n(G_L(f + m_1), -e - m_2, f, \ldots, f) \end{aligned}$$

$$
\begin{aligned}
& +\mathfrak{p}_{\mathfrak{n}}(f+m_1, \mathrm{L}(-e-m_2), f, \ldots, f) \\
& +\mathfrak{p}_{\mathfrak{n}}(f+m_1, -e-m_2, \mathrm{L}(f), \ldots, f) \\
& +\cdots+\mathfrak{p}_{\mathfrak{n}}(f+m_1, -e-m_2, f, \ldots, \mathrm{L}(f)) \\
= \; & \mathfrak{p}_{\mathfrak{n}}(\mathrm{G_L}(m_1), -e, f, \ldots, f)+\mathfrak{p}_{\mathfrak{n}}(m_1, \mathrm{L}(-e), f, \ldots, f) \\
& +\mathfrak{p}_{\mathfrak{n}}(m_1, -e, \mathrm{L}(f), \ldots, f)+\cdots+\mathfrak{p}_{\mathfrak{n}}(m_1, -e, f, \ldots, \mathrm{L}(f)) \\
& +\mathfrak{p}_{\mathfrak{n}}(\mathrm{G_L}(m_1), -m_2, f, \ldots, f)+\mathfrak{p}_{\mathfrak{n}}(m_1, \mathrm{L}(-m_2), f, \ldots, f) \\
& +\mathfrak{p}_{\mathfrak{n}}(m_1, -m_2, \mathrm{L}(f), \ldots, f)+\cdots+\mathfrak{p}_{\mathfrak{n}}(m_1, -m_2, f, \ldots, \mathrm{L}(f)) \\
& +\mathfrak{p}_{\mathfrak{n}}(\mathrm{G_L}(f), -e, f, \ldots, f)+\mathfrak{p}_{\mathfrak{n}}(f, \mathrm{L}(-e), f, \ldots, f) \\
& +\mathfrak{p}_{\mathfrak{n}}(f, -e, \mathrm{L}(f), \ldots, f)+\cdots+\mathfrak{p}_{\mathfrak{n}}(f, -e, f, \ldots, \mathrm{L}(f)) \\
& +\mathfrak{p}_{\mathfrak{n}}(\mathrm{G_L}(f), -m_2, f, \ldots, f)+\mathfrak{p}_{\mathfrak{n}}(f, \mathrm{L}(-m_2), f, \ldots, f) \\
& +\mathfrak{p}_{\mathfrak{n}}(f, -m_2, \mathrm{L}(f), \ldots, f)+\cdots+\mathfrak{p}_{\mathfrak{n}}(f, -m_2, f, \ldots, \mathrm{L}(f)) \\
= \; & \mathrm{G_L}(\mathfrak{p}_{\mathfrak{n}}(m_1, -e, f, \ldots, f))+\mathrm{G_L}(\mathfrak{p}_{\mathfrak{n}}(m_1, -m_2, f, \ldots, f)) \\
& +\mathrm{G_L}(\mathfrak{p}_{\mathfrak{n}}(f, -e, f, \ldots, f))+\mathrm{G_L}(\mathfrak{p}_{\mathfrak{n}}(f, -m_2, f, \ldots, f)) \\
= \; & \mathrm{G_L}(m_1)+\mathrm{G_L}(m_2).
\end{aligned}
$$

□

Lemma 3.8. *$\mathrm{G_L}(a+m+b)-\mathrm{G_L}(a)-\mathrm{G_L}(m)-\mathrm{G_L}(b) \in \mathfrak{Z}(\mathfrak{G})$ for all $a \in \mathrm{A}, m \in \mathrm{M}$ and $b \in \mathrm{B}$.*

Proof. For any $a \in \mathrm{A}, b \in \mathrm{B}$ and $m_1, m \in \mathrm{M}$, we have

$$
\begin{aligned}
\mathrm{G_L}([a+m+b, m_1]) &= \mathrm{G_L}(\mathfrak{p}_{\mathfrak{n}}(a+m+b, m_1, f, \ldots, f)) \\
&= \mathfrak{p}_{\mathfrak{n}}(\mathrm{G_L}(a+m+b), m_1, f, \ldots, f) \\
&\quad +\mathfrak{p}_{\mathfrak{n}}(a+m+b, \mathrm{L}(m_1), f, \ldots, f) \\
&= [\mathrm{G_L}(a+m+b), m_1]+[a+m+b, \mathrm{L}(m_1)].
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
\mathrm{G_L}([a+m+b, m_1]) &= \mathrm{G_L}[a, m_1]+\mathrm{G_L}[b, m_1] \\
&= \mathrm{G_L}(\mathfrak{p}_n(a, m_1, f, \ldots, f)+\mathrm{G_L}(\mathfrak{p}_{\mathfrak{n}}(b, m_1, f, \ldots, f)) \\
&= \mathfrak{p}_{\mathfrak{n}}(\mathrm{G_L}(a), m_1, f, \ldots, f)+\mathfrak{p}_{\mathfrak{n}}(a, \mathrm{L}(m_1), f, \ldots, f) \\
&\quad +\mathfrak{p}_{\mathfrak{n}}(\mathrm{G_L}(b), m_1, f, \ldots, f)+\mathfrak{p}_{\mathfrak{n}}(b, \mathrm{L}(m_1), f, \ldots, f) \\
&= [\mathrm{G_L}(a), m_1]+[a, \mathrm{L}(m_1)]+[\mathrm{G_L}(b), m_1]+[b, \mathrm{L}(m_1)].
\end{aligned}
$$

Combining the above two expressions we find that $[\mathrm{G_L}(a+m+b)-\mathrm{G_L}(a)-\mathrm{G_L}(b), m_1]=0$. Therefore, $\mathrm{G_L}(a+m+b)-\mathrm{G_L}(a)-e\mathrm{G_L}(a+m+b)f-\mathrm{G_L}(b) \in$

$\mathfrak{Z}(\mathfrak{G})$. This gives that $\mathrm{G_L}(a+m+b)-\mathrm{G_L}(a)-\mathrm{G_L}(m)-\mathrm{G_L}(b)\in\mathfrak{Z}(\mathfrak{G})$ for all $a\in\mathrm{A}, b\in\mathrm{M}$ and $m\in\mathrm{M}$. □

Now from Lemma 3.8, we can conclude that $\mathrm{G_L}(\mathfrak{T})\subseteq\mathfrak{T}$. This implies that $\mathrm{G_L}|_{\mathfrak{T}}$ is a multiplicative generalized Lie $\mathtt{n}$-derivation. From Lemma 2.2, there exist an additive generalized derivation d on $\mathfrak{T}$ and a map $\tau:\mathfrak{T}\to\mathfrak{Z}(\mathfrak{T})$ such that $\mathrm{G_L}=d+\tau$. Now from Lemma 2.1, generalized derivation d on $\mathfrak{T}$ has the form

$$d\left(\begin{bmatrix} a & m\\ 0 & b\end{bmatrix}\right)=\begin{bmatrix}\mathcal{U}(a) & am_0-m_0b+\mathcal{W}(m)\\ 0 & \mathcal{V}(b)\end{bmatrix},$$

where $a\in\mathrm{A}, m,m_0\in\mathrm{M}, b\in\mathrm{B}$ and $\mathcal{U}:\mathrm{A}\to\mathrm{A}, \mathcal{W}:\mathrm{M}\to\mathrm{M}, \mathcal{V}:\mathrm{B}\to\mathrm{B}$ are $\mathfrak{R}$-linear mappings satisfying

(*i*) $\mathcal{U}$ is a generalized derivation on A and $\mathcal{W}(am)=\mathcal{U}(a)m+a\mathcal{W}(m)$ for all $a\in\mathrm{A}$ and $m\in\mathrm{M}$,

(*ii*) $\mathcal{V}$ is a generalized derivation on B and $\mathcal{W}(mb)=m\mathcal{V}(b)+\mathcal{W}(m)b$ for all $b\in\mathrm{B}$ and $m\in\mathrm{M}$.

Particularly,

$$\mathrm{G_L}(f)=d(f)+\tau(f)=\begin{bmatrix}0 & -m_0\\ 0 & 0\end{bmatrix}+\tau(f).$$

This implies that $m_0=0$ and hence

$$d\left(\begin{bmatrix} a & m\\ 0 & b\end{bmatrix}\right)=\begin{bmatrix}\mathcal{U}(a) & \mathcal{W}(m)\\ 0 & \mathcal{V}(b)\end{bmatrix}. \tag{3.5}$$

Obviously, $\mathrm{G_L}(e)=\tau(e)\in\mathfrak{Z}(\mathfrak{G})$ and $\mathrm{G_L}(-f)=\tau(-f)\in\mathfrak{Z}(\mathfrak{G})$. In particular, we have $\mathrm{L}(e)\in\mathfrak{Z}(\mathfrak{G})$ and $\mathrm{L}(-f)\in\mathfrak{Z}(\mathfrak{G})$. Now define a map $\phi_1:\mathfrak{G}\to\mathfrak{Z}(\mathfrak{G})$ by $\phi_1(a+m+n+b)=\tau(a+m+b)$ and set $\mathrm{G}'_\mathrm{L}=\mathrm{G_L}-\phi_1$. From here we observe that $\mathrm{G}'_\mathrm{L}|_{\mathfrak{T}}=d$ is an additive generalized derivation on $\mathfrak{T}$ and $\mathrm{G}'_\mathrm{L}(n)=\mathrm{G_L}(n)$ for all $n\in\mathrm{N}$.

Lemma 3.9. $\mathrm{G_L}(n)=f\mathrm{G_L}(n)e$ *for all* $n\in\mathrm{N}$.

Proof. For any $n\in\mathrm{N}$, we have

$$\begin{aligned}\mathrm{G_L}(n) &= \mathrm{G_L}(\mathfrak{p}_{\mathtt{n}}(n,-f,\dots,-f))\\ &= \mathfrak{p}_{\mathtt{n}}(\mathrm{G_L}(n),-f,\dots,-f)+\sum_{\mathtt{k}=2}^{\mathtt{n}}\mathfrak{p}_{\mathtt{n}}(n,\mathrm{L}(-f),-f,\dots,-f)\\ &= (-1)^{\mathtt{n}-1}e\mathrm{G_L}(n)f+f\mathrm{G_L}(n)e.\end{aligned}$$

If $\mathtt{n}$ is even, then $\mathrm{G_L}(n) = -e\mathrm{G_L}(n)f + f\mathrm{G_L}(n)e$ for all $n \in \mathrm{N}$. On multiplying by e from the left and by f from the right, we find that $2e\mathrm{G_L}(n)f = 0$ implies to $e\mathrm{G_L}(n)f = 0$.

If $\mathtt{n}$ is odd, then $\mathrm{G_L}(n) = e\mathrm{G_L}(n)f + f\mathrm{G_L}(n)e$ for all $n \in \mathrm{N}$. On using $[n_1, n_2] = 0$, we have

$$\begin{aligned} 0 &= \mathrm{G_L}(\mathfrak{p}_{\mathtt{n}}(n, n_1, m, f, \ldots, f)) \\ &= \mathfrak{p}_{\mathtt{n}}(\mathrm{G_L}(n), n_1, m, f, \ldots, f) + \mathfrak{p}_{\mathtt{n}}(n, \mathrm{L}(n_1), m, f, \ldots, f) \\ &= [[e\mathrm{G_L}(n)f, n_1] + [n, e\mathrm{L}(n_1)f], m]. \end{aligned}$$

This implies that $[e\mathrm{G_L}(n)f, n_1] + [n, e\mathrm{L}(n_1)f] \in \mathfrak{Z}(\mathfrak{G})$. On the other hand, we obtain that

$$\begin{aligned} 0 &= \mathrm{G_L}(\mathfrak{p}_{\mathtt{n}}(n, e, \ldots, e, n_1)) \\ &= \mathfrak{p}_{\mathtt{n}}(\mathrm{G_L}(n), e, \ldots, e, n_1) + \mathfrak{p}_{\mathtt{n}}(n, e, \ldots, e, \mathrm{L}(n_1)) \\ &= -[e\mathrm{G_L}(n)f, n_1] + [n, e\mathrm{L}(n_1)f]. \end{aligned}$$

On comparing the above two expressions, we get $2[e\mathrm{G_L}(n)f, n_1] \in \mathfrak{Z}(\mathfrak{G})$, and hence it follows that $[e\mathrm{G_L}(n)f, n_1] \in \mathfrak{Z}(\mathfrak{G})$ for all $n, n_1 \in \mathrm{N}$. Therefore, $e\mathrm{G_L}(n)fn_1 - n_1e\mathrm{G_L}(n)f_1 \in \mathfrak{Z}(\mathfrak{G})$. Hence $e\mathrm{G_L}(n)f\mathrm{N} \subseteq \mathfrak{Z}(\mathrm{A})$ and $\mathrm{N}e\mathrm{G_L}(n)f \subseteq \mathfrak{Z}(\mathrm{B})$. Now by assumption, we obtain that $e\mathrm{G_L}(n)f\mathrm{N} = 0$ and $\mathrm{N}e\mathrm{G_L}(n)f = 0$ and hence $e\mathrm{G_L}(n)f = 0$ for all $n \in \mathrm{N}$. □

Lemma 3.10. $\mathrm{G_L}(a+m+n+b) - \mathrm{G_L}(a) - \mathrm{G_L}(m) - \mathrm{G_L}(n) - \mathrm{G_L}(b) \in \mathfrak{Z}(\mathfrak{G})$ *for all* $a \in \mathrm{A}, m \in \mathrm{M}, n \in \mathrm{N}$ *and* $b \in \mathrm{B}$.

Proof. For any $a \in \mathrm{A}, b \in \mathrm{B}$ and $m_1, m \in \mathrm{M}$, we have

$$\begin{aligned} &\mathrm{G_L}(\mathfrak{p}_{\mathtt{n}}(a + m + n + b, m_1, m_2, f, \ldots, f)) \\ &\quad = \mathfrak{p}_{\mathtt{n}}(\mathrm{G_L}(a + m + n + b), m_1, m_2, f, \ldots, f) \\ &\qquad + \sum_{\mathtt{k}=2}^{\mathtt{n}} \mathfrak{p}_{\mathtt{n}}(a + m + n + b, \mathrm{L}(m_1), m_2, f, \ldots, f) \\ &\quad = [[\mathrm{G_L}(a + m + n + b), m_1], m_2] \\ &\qquad + [[a + m + n + b, \mathrm{L}(m_1)], m_2] + [[n, m_1], \mathrm{L}(m_2)]. \end{aligned}$$

On the other hand,

$$\begin{aligned} &\mathrm{G_L}(\mathfrak{p}_{\mathtt{n}}(a + m + n + b, m_1, m_2, f, \ldots, f)) \\ &\quad = \mathrm{G_L}(\mathfrak{p}_{\mathtt{n}}(n, m_1, m_2, f, \ldots, f) + \mathrm{G_L}(\mathfrak{p}_{\mathtt{n}}(b, m_1, m_2, f, \ldots, f)) \end{aligned}$$

$$\begin{aligned} &= \mathfrak{p}_{\mathtt{n}}(\mathrm{G_L}(n), m_1, m_2, f, \ldots, f) + \mathfrak{p}_{\mathtt{n}}(n, \mathrm{L}(m_1), m_2, f, \ldots, f) \\ &\quad + \mathfrak{p}_{\mathtt{n}}(n, m_1, \mathrm{L}(m_2), f, \ldots, f) \\ &= [[\mathrm{G_L}(n), m_1], m_2] + [[n, \mathrm{L}(m_1)], m_2] + [[n, m_1], \mathrm{L}(m_2)]. \end{aligned}$$

From the above two expressions, we obtain that $[[\mathrm{G_L}(a+m+n+b) - \mathrm{G_L}(n), m_1], m_2] = 0$. This implies that $[[f\mathrm{G_L}(a+m+n+b)e - \mathrm{G_L}(n), m_1], m_2] = 0$ and hence $[f\mathrm{G_L}(a+m+n+b)e - \mathrm{G_L}(n), m_1] \in \mathfrak{Z}(\mathfrak{G})$. This leads to

$$(f\mathrm{G_L}(a+m+n+b)e - \mathrm{G_L}(n))m_1 - m_1(f\mathrm{G_L}(a+m+n+b)e - \mathrm{G_L}(n)) \in \mathfrak{Z}(\mathfrak{G}).$$

Therefore, $(\mathrm{M}(f\mathrm{G_L}(a+m+n+b)e - \mathrm{G_L}(n)) \in \mathfrak{Z}(\mathrm{A})$ and $(f\mathrm{G_L}(a+m+n+b)e - \mathrm{G_L}(n))\mathrm{M} \in \mathfrak{Z}(\mathrm{B})$. Now by assumptions, we find that

$$f\mathrm{G_L}(a+m+n+b)e = \mathrm{G_L}(n). \tag{3.6}$$

For any $a \in \mathrm{A}, b \in \mathrm{B}$ and $m_1, m \in \mathrm{M}$, we have

$$\begin{aligned} &\mathrm{G_L}(\mathfrak{p}_{\mathtt{n}}(a+m+n+b, n_1, m_1, f, \ldots, f)) \\ &\quad = \mathfrak{p}_{\mathtt{n}}(\mathrm{G_L}(a+m+n+b), n_1, m_1, f, \ldots, f) \\ &\qquad + \sum_{\mathtt{k=2}}^{\mathtt{n}} \mathfrak{p}_{\mathtt{n}}(a+m+n+b, \mathrm{L}(n_1), m_1, f, \ldots, f) \\ &\quad = [[\mathrm{G_L}(a+m+n+b), n_1], m_1] \\ &\qquad + [[a+m+n+b, \mathrm{L}(n_1)], m_1] + [[m, n_1], \mathrm{L}(m_1)]. \end{aligned}$$

On the other hand,

$$\begin{aligned} &\mathrm{G_L}(\mathfrak{p}_{\mathtt{n}}(a+m+n+b, n_1, m_1, f, \ldots, f)) \\ &\quad = \mathrm{G_L}(\mathfrak{p}_{\mathtt{n}}(m, n_1, m_1, f, \ldots, f) \\ &\quad = \mathfrak{p}_{\mathtt{n}}(\mathrm{G_L}(m), n_1, m_1, f, \ldots, f) + \mathfrak{p}_{\mathtt{n}}(n, \mathrm{L}(n_1), m_1, f, \ldots, f) \\ &\qquad + \mathfrak{p}_{\mathtt{n}}(m, n_1, \mathrm{L}(m_1), f, \ldots, f) \\ &\quad = [[\mathrm{G_L}(m), n_1], m_1] + [[m, \mathrm{L}(n_1)], m_1] + [[m, n_1], \mathrm{L}(m_1)]. \end{aligned}$$

From the above two expressions we obtain that $[[\mathrm{G_L}(a+m+n+b) - \mathrm{G_L}(m), n_1], m_1] = 0$. This implies that $[[e\mathrm{G_L}(a+m+n+b)f - \mathrm{G_L}(m), n_1], m_1] = 0$ and hence $[e\mathrm{G_L}(a+m+n+b)f - \mathrm{G_L}(m), n_1] \in \mathfrak{Z}(\mathfrak{G})$. This leads to

$$(e\mathrm{G_L}(a+m+n+b)f - \mathrm{G_L}(m))n_1 - n_1(e\mathrm{G_L}(a+m+n+b)f - \mathrm{G_L}(m)) \in \mathfrak{Z}(\mathfrak{G}).$$

Therefore, $(\mathrm{N}(e\mathrm{G_L}(a+m+n+b)f - \mathrm{G_L}(m)) \in \mathfrak{Z}(\mathrm{A})$ and $(e\mathrm{G_L}(a+m+n+b)f - \mathrm{G_L}(m))\mathrm{N} \in \mathfrak{Z}(\mathrm{B})$. Now by assumptions, we find that

$$e\mathrm{G_L}(a+m+n+b)f = \mathrm{G_L}(m). \tag{3.7}$$

Now for any $m_1 \in \mathrm{M}$, we have

$$\begin{aligned}
&\mathrm{G_L}(\mathfrak{p}_{\mathfrak{n}}(a+m+n+b, m_1, f, \dots, f)) \\
&= \mathfrak{p}_{\mathfrak{n}}(\mathrm{G_L}(a+m+n+b), m_1, f, \dots, f) \\
&\quad + \mathfrak{p}_{\mathfrak{n}}(a+m+n+b, \mathrm{L}(m_1), f, \dots, f) \\
&= [e\mathrm{G_L}(a+m+n+b)e + f\mathrm{G_L}(a+m+n+b)f, m_1] \\
&\quad + [a+m+n+b, \mathrm{L}(m_1)].
\end{aligned}$$

On the other hand,

$$\begin{aligned}
&\mathrm{G_L}(\mathfrak{p}_{\mathfrak{n}}(a+m+n+b, m_1, f, \dots, f)) \\
&= \mathrm{G_L}([a+b, m_1]) \\
&= \mathrm{G_L}([a, m_1]) + \mathrm{G_L}([b, m_1]) \\
&= \mathfrak{p}_{\mathfrak{n}}(\mathrm{G_L}(a), m_1, f, \dots, f) + \mathfrak{p}_{\mathfrak{n}}(a, \mathrm{L}(m_1), f, \dots, f) \\
&\quad + \mathfrak{p}_{\mathfrak{n}}(\mathrm{G_L}(b), m_1, f, \dots, f) + \mathfrak{p}_{\mathfrak{n}}(b, \mathrm{L}(m_1), f, \dots, f) \\
&= [\mathrm{G_L}(a), m_1] + [a, \mathrm{L}(m_1)] + [\mathrm{G_L}(b), m_1] + [b, \mathrm{L}(m_1)].
\end{aligned}$$

On comparing the above two expressions, we get $[e\mathrm{G_L}(a+m+m+b)e+f\mathrm{G_L}(a+m+m+b)f - \mathrm{G_L}(a) - \mathrm{G_L}(b), m_1] = 0$. This implies that $e\mathrm{G_L}(a+m+n+b)e+f\mathrm{G_L}(a+m+n+b)f - \mathrm{G_L}(a) - \mathrm{G_L}(b) \in \mathfrak{Z}(\mathfrak{G})$. Now using (3.6) and (3.7), we find that $\mathrm{G_L}(a+m+n+b) - \mathrm{G_L}(a) - \mathrm{G_L}(m) - \mathrm{G_L}(n) - \mathrm{G_L}(b) \in \mathfrak{Z}(\mathfrak{G})$. □

Lemma 3.11. $\mathrm{G_L}$ *is additive on* N.

Proof. For any $n_1, n_2 \in \mathrm{N}$, we have $n_1+n_2 = \mathfrak{p}_{\mathfrak{n}}(e+n_1, -f-n_2, e, \dots, e)$. Then the proof follows similarly as Lemma 3.7. □

Define a map $\phi_2 : \mathfrak{G} \to \mathfrak{Z}(\mathfrak{G})$ by

$$\phi_2(a+m+n+b) = \mathrm{G}'_\mathrm{L}(a+m+n+b) - \mathrm{G}'_\mathrm{L}(a) - \mathrm{G}'_\mathrm{L}(m) - \mathrm{G}'_\mathrm{L}(n) - \mathrm{G}'_\mathrm{L}(b).$$

Now set $\mathrm{G}''_\mathrm{L} = \mathrm{G}'_\mathrm{L} - \phi_2$. Obviously, $\mathrm{G}''_\mathrm{L}(a) = \mathrm{G}'_\mathrm{L}(a), \mathrm{G}''_\mathrm{L}(m) = \mathrm{G}'_\mathrm{L}(m), \mathrm{G}''_\mathrm{L}(n) = \mathrm{G}'_\mathrm{L}(n)$ and $\mathrm{G}''_\mathrm{L}(b) = \mathrm{G}'_\mathrm{L}(b)$ for all $a \in \mathrm{A}, b \in \mathrm{B}, m \in \mathrm{M}$ and $n \in \mathrm{N}$.

Lemma 3.12. $\phi_2 \in \mathfrak{Z}(\mathfrak{G})$.

Proof. By using Lemma 3.10, we obtain that

$$\begin{aligned}
&\phi_2(a+m+n+b) \\
&= \mathrm{G}'_\mathrm{L}(a+m+m+b) - \mathrm{G}'_\mathrm{L}(a) - \mathrm{G}'_\mathrm{L}(m) - \mathrm{G}'_\mathrm{L}(n) - \mathrm{G}'_\mathrm{L}(b)
\end{aligned}$$

$$\begin{aligned} &= \mathrm{G_L}(a+m+n+b) - \phi_1(a+m+n+b) - \mathrm{G_L}(a) + \phi_1(a) \\ &\quad -\mathrm{G_L}(m) + \phi_1(m) - \mathrm{G_L}(n) + \phi_1(n) - \mathrm{G_L}(b) + \phi_1(b) \in \mathfrak{Z}(\mathfrak{G}). \end{aligned}$$

□

Proof of Theorem 2.1. In view of (3.5), we obtain that $\mathrm{G''_L}|_\mathrm{A} = \mathrm{G'_L}|_\mathrm{A} = \mathcal{U}_1, \mathrm{G''_L}|_\mathrm{B} = \mathrm{G'_L}|_\mathrm{B} = \mathcal{V}_3, \mathrm{G''_L}|_\mathrm{M} = \mathrm{G'_L}|_\mathrm{M} = \mathcal{W}_2$ and $\mathrm{G''_L}|_\mathrm{N} = \mathrm{G'_L}|_\mathrm{N} = \mathrm{G_L}|_\mathrm{N}$. Now for any $b \in \mathrm{B}$, we have

$$\begin{aligned} \mathrm{G''_L}(bn) &= \mathrm{G_L}(\mathfrak{p}_\mathfrak{n}(b, n, e, \ldots, e)) \\ &= \mathfrak{p}_\mathfrak{n}(\mathrm{G_L}(b), n, e, \ldots, e) + \mathfrak{p}_\mathfrak{n}(b, \mathrm{L}(n), e, \ldots, e) \\ &= [\mathrm{G''_L}(b), n] + [b, \mathrm{L}(n)] \\ &= \mathrm{G''_L}(b)n + b\mathrm{L}(n). \end{aligned}$$

In the similar way $\mathrm{G''_L}(na) = \mathrm{G''_L}(n)a + n\mathrm{L}(a)$. For any $x = a_1+m_1+n_1+b_1, y = a_2 + m_2 + n_2 + b_2 \in \mathfrak{G}$, we find that

$$\begin{aligned} \mathrm{G''_L}(x+y) &= \mathrm{G''_L}(a_1 + m_1 + n_1 + b_1 + a_2 + m_2 + n_2 + b_2) \\ &= \mathrm{G'_L}(a_1 + a_2) + \mathrm{G'_L}(m_1 + m_2) + \mathrm{G'_L}(n_1 + n_2) + \mathrm{G'_L}(b_1 + b_2) \\ &= \mathrm{G'_L}(a_1) + \mathrm{G'_L}(m_1) + \mathrm{G'_L}(n_1) + \mathrm{G'_L}(b_1) \\ &+ \mathrm{G'_L}(a_2) + \mathrm{G'_L}(m_2) + \mathrm{G'_L}(n_2) + \mathrm{G'_L}(b_2) \\ &= \mathrm{G'_L}(a_1 + m_1 + n_1 + b_1) + \mathrm{G'_L}(a_2 + m_2 + n_2 + b_2) \\ &= \mathrm{G''_L}(x) + \mathrm{G''_L}(y). \end{aligned}$$

This implies that $\mathrm{G''_L}$ is additive.

Now set $\mathcal{T} = \mathrm{G''_L}|_\mathrm{N}$. From above observations, we conclude that

$$\mathrm{G''_L}\left(\begin{bmatrix} a & m \\ n & b \end{bmatrix}\right) = \begin{bmatrix} \mathcal{U}(a) & \mathcal{V}(m) \\ \mathcal{T}(n) & \mathcal{W}(b) \end{bmatrix} \tag{3.8}$$

for all $a \in \mathrm{A}, m \in \mathrm{M}, n \in \mathrm{N}, b \in \mathrm{B}$ and satisfies following conditions:

(*i*) $\mathcal{U}$ is a generalized derivation on $\mathcal{A}$, $\mathcal{V}(am) = \mathcal{U}(a)m + a\mathcal{V}(m)$ and $\mathcal{T}(na) = \mathcal{T}(n)a + n\mathcal{U}(a)$,

(*ii*) $\mathcal{W}$ is a generalized derivation on $\mathcal{B}$, $\mathcal{V}(mb) = \mathcal{U}(m)b + m\mathcal{W}(m)$ and $\mathcal{T}(bn) = \mathcal{T}(b)n + b\mathcal{W}(n)$.

Let us assume $\phi = \phi_1 + \phi_2$ and $\mathrm{G''_L} = \mathrm{G_L} - \phi$. For any $n \in \mathrm{N}$, we have

$$\begin{aligned}
&\mathrm{G}''_{\mathrm{L}}(\mathfrak{p}_{\mathtt{n}}(m,n,m_1,f,\ldots,f))\\
&\quad= \mathrm{G}_{\mathrm{L}}(\mathfrak{p}_{\mathtt{n}}(m,n,m_1,f,\ldots,f)) - \phi(\mathfrak{p}_{\mathtt{n}}(m,n,m_1,f,\ldots,f))\\
&\quad= \mathfrak{p}_{\mathtt{n}}(\mathrm{G}_{\mathrm{L}}(m),n,m_1,f,\ldots,f) + \mathfrak{p}_{\mathtt{n}}(m,\mathrm{L}(n),m_1,f,\ldots,f)\\
&\qquad +\mathfrak{p}_{\mathtt{n}}(m,n,\mathrm{L}(m_1),f,\ldots,f) - \phi(\mathfrak{p}_{\mathtt{n}}(m,n,m_1,f,\ldots,f))\\
&\quad= [[\mathrm{G}''_{\mathrm{L}}(m),n] + [m,\mathrm{L}(n)],m_1] + [[m,n],\mathrm{L}(m_1)]\\
&\qquad -\phi(\mathfrak{p}_{\mathtt{n}}(m,n,m_1,f,\ldots,f)).
\end{aligned}$$

On the other hand, by (3.8)

$$\begin{aligned}
\mathrm{G}''_{\mathrm{L}}(\mathfrak{p}_{\mathtt{n}}(m,n,m_1,f,\ldots,f)) &= [\mathrm{G}''_{\mathrm{L}}(mn-nm),m_1] + [mn-nm,\mathrm{L}(m_1)]\\
&= [\mathrm{G}''_{\mathrm{L}}(mn) - \mathrm{G}''_{\mathrm{L}}(nm),m_1] + [[m,n],\mathrm{L}(m_1)].
\end{aligned}$$

Now from the above two relations, we obtain that

$$[\mathrm{G}''_{\mathrm{L}}(mn) - \mathrm{G}''_{\mathrm{L}}(nm) - [\mathrm{G}''_{\mathrm{L}}(m),n] - [m,\mathrm{L}(n)],m_1] \in \mathfrak{Z}(\mathfrak{G}).$$

This implies that $[\mathrm{G}''_{\mathrm{L}}(mn) - \mathrm{G}''_{\mathrm{L}}(nm) - [\mathrm{G}''_{\mathrm{L}}(m),n] - [m,\mathrm{L}(n)],m_1] = 0$, and hence $\mathrm{G}''_{\mathrm{L}}(mn) - \mathrm{G}''_{\mathrm{L}}(nm) - [\mathrm{G}''_{\mathrm{L}}(m),n] - [m,\mathrm{L}(n)] \in \mathfrak{Z}(\mathfrak{G})$. Now multiplying this expression by e on both sides, we obtain that $\mathrm{G}''_{\mathrm{L}}(mn) - \mathrm{G}''_{\mathrm{L}}(m)n - m\mathrm{L}(n) \in \mathfrak{Z}(\mathrm{A})$. Now without loss of generality, we assume that $\kappa(m,n) = \mathrm{G}''_{\mathrm{L}}(mn) - \mathrm{G}''_{\mathrm{L}}(m)n - m\mathrm{L}(n)$. Then we have

$$\begin{aligned}
\kappa(m,na) &= \mathrm{G}''_{\mathrm{L}}(mna) - \mathrm{G}''_{\mathrm{L}}(m)na - m\mathrm{L}(na)\\
&= \mathrm{G}''_{\mathrm{L}}(mn)a + mn\mathrm{L}(a) - \mathrm{G}''_{\mathrm{L}}(m)na - m\mathrm{L}(n)a - m\mathrm{L}(na)\\
&= \mathrm{G}''_{\mathrm{L}}(mn)a - \mathrm{G}''_{\mathrm{L}}(m)na - m\mathrm{L}(n)a\\
&= \kappa(m,n)a.
\end{aligned}$$

Since $\kappa(m,n)\mathrm{A}$ is a central ideal of A, we arrive at $\kappa(m,n) = 0$. This leads to $\mathrm{G}''_{\mathrm{L}}(mn) = \mathrm{G}''_{\mathrm{L}}(m)n + m\mathrm{L}(n)$ for all $m \in \mathrm{M}, n \in \mathrm{N}$. In the similar manner, we can show that $\mathrm{G}''_{\mathrm{L}}(nm) = \mathrm{G}''_{\mathrm{L}}(n)m + n\mathrm{L}(m)$. Set $\mathrm{G}''_{\mathrm{L}} = \delta$. Now it can be easily seen that δ is an additive generalized derivation on $\mathfrak{G}$. and $\phi(\mathfrak{p}_{\mathtt{n}}(x_1,x_2,\ldots,x_{\mathtt{n}})) = 0$ for all $x_1,x_2,\ldots,x_{\mathtt{n}} \in \mathfrak{G}$.

From above observations, we can conclude that if $\mathrm{G}_{\mathrm{L}} : \mathfrak{G} \to \mathfrak{G}$ is a multiplicative generalized Lie $\mathtt{n}$-derivation, then there exists an additive generalized derivation δ of $\mathfrak{G}$ and a map $\phi : \mathfrak{G} \to \mathfrak{Z}(\mathfrak{G})$ vanishing at $\mathfrak{p}_{\mathtt{n}}(x_1,x_2,\ldots,x_{\mathtt{n}})$ for all $x_1,x_2,\ldots,x_{\mathtt{n}} \in \mathfrak{G}$ such that $\mathrm{G}_{\mathrm{L}} = \delta + \phi$. □

4 - Applications

As a direct consequence of Theorem 3.1, we have the following results:

Corollary 4.1. *Let* $\mathfrak{G}$ *be a* $(\mathtt{n-1})-$*torsion free generalized matrix algebra such that*

1. $\mathfrak{Z}(\mathrm{A}) = \pi_{\mathrm{A}}(\mathfrak{Z}(\mathfrak{G}))$ *and* $\mathfrak{Z}(\mathrm{B}) = \pi_{\mathrm{B}}(\mathfrak{Z}(\mathfrak{G}))$,
2. A *or* B *does not contain nonzero central ideals.*

If $\mathrm{G_L} : \mathfrak{G} \to \mathfrak{G}$ *is a multiplicative generalized Lie derivation, then there exists an additive generalized derivation* δ *of* $\mathfrak{G}$ *and a map* $\phi : \mathfrak{G} \to \mathfrak{Z}(\mathfrak{G})$ *vanishing at* $[x_1, x_2]$ *for all* $x_1, x_2 \in \mathfrak{G}$ *such that* $\mathrm{G_L} = \delta + \phi$.

Corollary 4.2. *Let* $\mathfrak{A}$ *be a* $(\mathtt{n-1})$*-torsion free unital algebra and* $\mathfrak{M}_r(\mathfrak{A})$ *be full matrix algebra with* $r \geq 3$. *If* $\mathrm{G_L} : \mathfrak{M}_r(\mathfrak{A}) \to \mathfrak{M}_r(\mathfrak{A})$ *is a multiplicative generalized Lie* $\mathtt{n}$*-derivation, then there exists an additive generalized derivation* δ *of* $\mathfrak{M}_r(\mathfrak{A})$ *and a map* $\phi : \mathfrak{M}_r(\mathfrak{A}) \to \mathfrak{Z}(\mathfrak{M}_r(\mathfrak{A}))$ *vanishing at* $\mathfrak{p}_n(x_1, x_2, \ldots, x_n)$ *for all* $x_1, x_2, \ldots, x_n \in \mathfrak{M}_r(\mathfrak{A})$ *such that* $\mathrm{G_L} = \delta + \phi$.

Proof. One can directly check that $\mathfrak{M}_r(\mathfrak{A})$ satisfies all conditions of Theorem 3.1. Therefore, every multiplicative generalized Lie $\mathtt{n}$-derivation can be expressed as a sum of additive generalized derivation and a map vanishing at $(\mathtt{n-1})$-th commutator on full matrix algebras. □

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