Riv. Mat. Univ. Parma, Vol. **12** (2021), 245-266

Karim Chaira, Soumia Chaira and Samih Lazaiz

Best proximity point theorems in Banach Algebras

Abstract. The aim of this paper is to obtain best proximity point theorems for weakly sequentially continuous mappings in Banach algebras. An example has also been given to support the usability of our results.

Keywords. Weakly sequentially continuous, Weakly condensing, Measure of weak noncompactness, Best proximity point, Relatively nonexpansive.

Mathematics Subject Classification: 47H08, 47H09, 47H10, 54H25.

1 - Introduction

Let (Ω_1, Ω_2) be a pair of nonempty subsets of Banach Algebra X and A, B and C are nonlinear operators of X. Our main goal in this work is to take up the approximation problem of the form:

$$\text{(PP)} \qquad \begin{cases} \text{find } (x, y) \in \Omega_1 \times \Omega_2 \text{ such that} \\ x = A(x).B(y) + C(x) \text{ and } \|x - y\| = \operatorname{dist}(\Omega_1, \Omega_2), \end{cases}$$

where $\operatorname{dist}(\Omega_1, \Omega_2) = \inf\{\|x - y\| : x \in \Omega_1, y \in \Omega_2\}$. Note that when $\Omega_1 = \Omega_2$ this problem is already studied in the literature by several authors, see [**4**,**5**,**7**,**10**,**12**,**18**,**19**]. And, when $A \equiv 1$ we find the Krasnosel'skii-type problem see [**15**, **25**] for more details.

On the other hand, under suitable conditions on A, B and C we define the operator $T := \left(\frac{I-C}{A}\right)^{-1} \circ B : \Omega_1 \cup \Omega_2 \to \Omega_1 \cup \Omega_2$ such that $T(\Omega_1) \subset \Omega_2$ and $T(\Omega_2) \subset \Omega_1$. In contrast to the results of [**10**, **15**, **25**] where $\Omega_1 = \Omega_2$, the interested case here is when $\Omega_1 \cap \Omega_2 = \varnothing$. In the event that $\Omega_1 \cap \Omega_2$ is nonempty,

Received: September 8, 2020; accepted in revised form: April 8, 2021.

then the mapping T restricted to $\Omega_1 \cap \Omega_2$ is a self mapping and a solution of equation (PP) is a fixed point of T. However, if T is a non-self mapping, it is contemplated to explore to find an $x^* \in \Omega_1$ satisfying

$$\|x^* - T(x^*)\| = \text{dist}(\Omega_1, \Omega_2).$$

This point $x^* \in \Omega_1$ is said to be a best proximity point of T. Recall that a mapping $T : \Omega_1 \cup \Omega_2 \to \Omega_1 \cup \Omega_2$ is called relatively nonexpansive if

$$\|T(x) - T(y)\| \leq \|x - y\|,$$

for all $x \in \Omega_1$ and $y \in \Omega_2$.

In [**22**], Eldred et al. using the proximinal normal structure, they proved the existence of best proximity points for relatively nonexpansive mappings in Banach spaces. This class of mappings is much larger than nonexpansive mappings. The work of the fore-mentioned authors generalizes the notion of normal structure introduced by Brodskii and Milman [**14**]. Furthermore, Sankar and Veeramani in [**29**] have used convergence theorems to prove the existence of a best proximity point. For more, the interested reader can consult [**1**, **9**, **16**, **23**] and the references therein.

In the present paper we generalize the cornerstone of [**11**], namely Lemma 3.1, and we prove some best proximity point theorems where the involving operators are weakly sequentially continuous. Our results solve the problem (PP) for two and three operators. An example is given to support the usability of our results. Many recent results in this area have been improved.

2 - Preliminaries

Let X be a Banach space endowed with the norm $\|.\|$ and with the zero element θ. We denote by $B_r(x)$ the closed ball centered at x with radius r. For a subset Y of X, we write $\overline{Y}$, $\overline{Y}^w$, $convY$, and $\overline{conv}Y$, to denote the closure, the weak closure, the convex hull and the closed convex hull of the subset Y, respectively. If $\overline{Y}^w$ is weakly compact, the set Y is said to be relatively weakly compact. Moreover, we write $x_n \to x$ and $x_n \rightharpoonup x$ to denote, respectively, the strong convergence (with respect to the norm of X) and the weak convergence (with respect to the weak topology of X) of a sequence $(x_n)_n$ to x.

Recall that an algebra X is a vector space endowed with an internal composition law noted by "." i.e.,

$$\begin{cases} . : X \times X \to X \\ (x, y) \to x.y \end{cases}$$

which is associative and bilinear. A normed algebra is an algebra endowed with a norm satisfying for all $x, y \in X$; $\|x.y\| \leq \|x\|\|y\|$. A complete normed algebra is called a Banach algebra. In general, the product of two weakly sequentially continuous mappings on a Banach algebra X is not necessarily weakly sequentially continuous, to overcome this obstacle, the authors in [**10**] introduced new class of Banach algebras:

Definition 2.1 ([**10**]). We will say that the Banach algebra X satisfies condition $(\mathcal{P})$ if for any sequence $(x_n)_{n\geq 0}$ and $(y_n)_{n\geq 0}$ in X such that $x_n \rightharpoonup x$ and $y_n \rightharpoonup y$, then $x_n.y_n \rightharpoonup x.y$

In addition, the authors in [**10**] showed that every finite dimensional Banach algebra satisfies condition $(\mathcal{P})$. And, if E satisfies condition $(\mathcal{P})$ then $C(K, E)$ is also a Banach algebra satisfying condition $(\mathcal{P})$, where K is a compact Hausdorff space.

Theorem 2.2 ([**20**]). *Let S be a Hausdorff compact space and E be a Banach space. A bounded sequence $(f_n)_n \subseteq \mathcal{C}(S, E)$ converges weakly to $f \in \mathcal{C}(S, E)$ if and only if, for every $t \in S$, the sequence $(f_n(t))_n$ converges weakly (in E) to $f(t)$.*

On the other hand, the theory of condensing operators start with the result of Sadovskii [**28**]. Before we define condensing mappings, we need to present the concept of weak measure of noncompactness. Let $P_{bd}(X)$ denote the collection of all nonempty bounded subsets of X. We recall that a function $\eta : P_{bd}(X) \to \mathbb{R}_+$ is said to be a measure of weak non-compactness (MWNC, for short) on X, if it satisfies the following four properties:

(i) $\eta(\overline{conv}(Y)) = \eta(Y)$, for all bounded subsets $Y \subset X$,

(ii) Monotonocity: For any bounded subsets Y_1,Y_2 of X we have

$$Y_1 \subset Y_2 \Rightarrow \eta(Y_1) \leq \eta(Y_2).$$

(iii) Non-singularity: $\eta(Y \cup \{a\}) = \eta(Y)$ for all $a \in X$, Y bounded set of X.

(iv) $\eta(Y) = 0$ if and only if Y is relatively weakly compact in X.

The MWNC η is said to be positive homogeneous provided

$$\eta(\lambda Y) = \lambda \eta(Y), \text{ for all } \lambda > 0 \text{ and } Y \in P_{bd}(X).$$

The MWNC η is said to be sub-additive, if

$$\eta(Y_1 + Y_2) \leq \eta(Y_1) + \eta(Y_2), \text{ for all } Y_1, Y_2 \in P_{bd}(X).$$

The above notion is a generalization of the De Blasi measure of weak non-compactness ω (see [**17**]) defined on each bounded set Y of X by

$$\omega(Y) = \inf\{\varepsilon > 0, \text{ there exists a weakly compact set } D \text{ such that } Y \subset D + B_\varepsilon(\theta)\}.$$

It is well known that ω is homogeneous, sub-additive and satisfies the set additivity property

$$\omega(Y_1 \cup Y_2) = \max\{\omega(Y_1), \omega(Y_2)\}, \ Y_1, Y_2 \in P_{bd}(X).$$

Note that ω is the counterpart for the weak topology of the classical Hausdorff measure of non-compactness. For more examples and properties of measures of weak non-compactness, we refer the reader to [**3**, **5**, **6**, **26**, **27**].

Let Ω be a subset of a Banach space X and ω be a measure of weak non-compactness, and $k \in [0, 1)$. An operator $T : \Omega \to X$ is called

1. $k-$Lipschitzian, if $\|T(x) - T(y)\| \leq k\|x - y\|$ with $k \geq 0$ and $(x, y) \in \Omega^2$. If $k = 1$, T is called nonexpansive and if $k \in [0, 1)$, T is called a contraction.
2. $\mathcal{D}$-Lipschitz if there exists a continuous nondecreasing function Φ_T : $\mathbb{R}_+ \to \mathbb{R}_+$ satisfying
$$\|Tx - Ty\| \leq \Phi_T(\|x - y\|)$$
for all $x, y \in X$ with $\Phi_T(0) = 0$;
3. $\omega-$condensing, if $\omega(T(A)) < \omega(A)$ for any bounded set $A \subset \Omega$ with $\omega(A) > 0$;

Recall that a mapping mapping $T : \Omega_1 \cup \Omega_2 \to \Omega_1 \cup \Omega_1$ where (Ω_1, Ω_2) is a pair of subset of a Banach space X is called *cyclic* if $T(\Omega_1) \subset \Omega_2$ and $T(\Omega_2) \subset \Omega_1$. In the sequel, we shall consider the following condition: Let $T : \Omega \to X$ be a mapping.

($\mathcal{H}$) If $(x_n)_{n\geq 0}$ is a weakly convergent sequence in Ω, then $(Tx_n)_{n\geq 0}$ has a weakly convergent subsequence in X.

Theorem 2.3 ([**2**]). *Let T be a $\mathcal{D}$-Lipschitz mapping defined on a Banach space X with a D- function φ. If, in addition, T satisfies ($\mathcal{H}$), then, for each bounded subset M of X we have $\omega(TM) \leq \varphi(\omega(M))$, where ω stands for the De Blasi measure of weak noncompactness.*

One of the advantages of the weak topology of a Banach space X is the fact that if a set Ω is weakly compact, then every sequentially weakly continuous mapping $F : \Omega \to X$ is weakly continuous. This is an immediate consequence of the Eberlein-Smulian theorem (see [**21**, Theorem 8.12.4, p. 549]).

Theorem 2.4 ([**24**]). *Let K be a convex closed set of a Banach space X and let $T : K \to K$ be a single-valued mapping with $T(K)$ is bounded. If T is weakly sequentially continuous and weakly condensing, then T has a fixed point.*

Lemma 2.5 ([**8**]). *Let X be a Banach algebra with a condition $(\mathcal{P})$. Then for any bounded subset D of X and weakly compact subset K of X, we have $\omega(D.K) \leq \|K\|\omega(D)$, where $\|K\| = \sup\{\|x\|\ ,\ x \in K\}$.*

To deduce our main result we will need to recall the notion of proximal normal structure. Throughout this paper, (Ω_1^o, Ω_2^o) denote the proximal pair obtained from (Ω_1, Ω_2) upon setting

$$\Omega_1^o = \{x \in \Omega_1 : \|x - y\| = \text{dist}(\Omega_1, \Omega_2) \text{ for some } y \in \Omega_2\}$$

$$\Omega_2^o = \{y \in \Omega_2 : \|x - y\| = \text{dist}(\Omega_1, \Omega_2) \text{ for some } x \in \Omega_1\}.$$

A pair (Ω_1, Ω_2) in a space X is said to satisfy a property if both Ω_1 and Ω_2 satisfy that property. For instance, (Ω_1, Ω_2) is closed (resp. convex, bounded) if and only if Ω_1 and Ω_2 are closed (resp. convex, bounded). The pair (Ω_1, Ω_2) is not reduced to one point means that Ω_1 and Ω_2 are not singletons.

Definition 2.6. A pair (K_1, K_2) of subsets (Ω_1, Ω_2) of a normed linear space is said to be a proximal pair if for each $(x, y) \in K_1 \times K_2$ there exists $(x', y') \in K_1 \times K_2$ such that

$$\|x - y'\| = \|x' - y\| = \text{dist}(\Omega_1, \Omega_2).$$

Definition 2.7 (Proximal normal structure [**22**]). A convex pair (Ω_1, Ω_2) in a Banach space is said to have proximal normal structure if for any closed, bounded, convex proximal pair $(K_1, K_2) \subset (\Omega_1, \Omega_2)$ for which $\text{dist}(K_1, K_2) = \text{dist}(\Omega_1, \Omega_2)$ and $\delta(K_1, K_2) > \text{dist}(K_1, K_2)$, there exists $(x_1, x_2) \in K_1 \times K_2$ such that $\delta(x_1, K_2) < \delta(K_1, K_2)$, $\delta(x_2, K_1) < \delta(K_1, K_2)$, where $\delta(K_1, K_2) = \sup\{\|k_1 - k_2\| : k_1 \in K_1\ ,\ k_2 \in K_2\}$.

Proposition 2.8 ([**22**, Proposition 2.2]). *Every compact convex pair (Ω_1, Ω_2) in a Banach space has proximal normal structure.*

Using the concept of proximal normal structure, Eldred et. al [**22**] proved the existence of best proximity points for relatively nonexpansive mappings.

Theorem 2.9 ([**22**, Theorem 1.2]). *Let* (Ω_1, Ω_2) *be a nonempty, weakly compact convex pair in a Banach space* $(X, \|.\|)$, *and suppose* (Ω_1, Ω_2) *has proximal normal structure. Let* $T : \Omega_1 \cup \Omega_2 \to \Omega_1 \cup \Omega_2$ *be a cyclic relatively nonexpansive mapping. Then, there exists* $(x, y) \in \Omega_1 \times \Omega_2$ *such that* $\|x - Tx\| = \|Ty - y\| = \operatorname{dist}(\Omega_1, \Omega_2)$.

For the sake of completeness we state the next lemma which is, Lemma 3.1 in [**11**].

Lemma 2.10 ([**11**]). *Let* Ω *be a nonempty closed convex subset of a Banach space* X, ω *is a MWNC on* X *and let* $T : \Omega \to \Omega$ *be a* $\omega-$*condensing mapping with bounded range. Assume* T *maps weakly compact sets into relatively weakly compact sets, then there is a convex relatively weakly compact subset* K *of* Ω *such that* $T(K) \subset K$.

3 - Main results

We start this section by generalizing the Lemma 2.10 above (see [**11**, Lemma 3.1]) where T maps weakly compact sets into relatively weakly compact sets and it is cyclic $\omega-$condensing.

Theorem 3.1. *Let* (Ω_1, Ω_2) *be a nonempty closed convex pair of a Banach space* X, ω *is a MWNC on* X *and let* $T : \Omega_1 \cup \Omega_2 \to \Omega_1 \cup \Omega_2$ *be a cyclic and* $\omega-$*condensing mapping with bounded range. We assume* T *maps weakly compact sets into relatively weakly compact sets and* Ω_1^o *is nonempty. Then, there is a convex relatively weakly compact pair* $(H, K) \subset (\Omega_1, \Omega_2)$ *such that* $T(H) \subset K$, $T(K) \subset H$ *and* $\operatorname{dist}(H, K) = \operatorname{dist}(\Omega_1, \Omega_2)$.

Proof. As Ω_1^o is nonempty, there exists $(x_o, y_o) \in \Omega_1 \times \Omega_2$ such that $\operatorname{dist}(\Omega_1, \Omega_2) = \|x_o - y_o\|$. We consider the family $\mathcal{F}$ of all bounded convex pair $(E, F) \subset (\Omega_1, \Omega_2)$ such that $(x_o, y_o) \in E \times F$ and $T(E) \subset F,\ T(F) \subset E$.

$\mathcal{F}$ is nonempty, indeed $conv(T(\Omega_1) \cup \{y_o\}) \subset \Omega_2$, so $T\left(conv(T(\Omega_1) \cup \{y_o\})\right) \subset T(\Omega_2) \cup \{x_o\}$. Therefore,

$$T\left(conv(T(\Omega_1) \cup \{y_o\})\right) \subset conv\left(T(conv(T(\Omega_1) \cup \{y_o\}))\right) \subset conv(T(\Omega_2) \cup \{x_o\}).$$

Similarly, we have $T\left(conv(T(\Omega_2) \cup \{x_o\})\right) \subset conv(T(\Omega_1) \cup \{y_o\})$. Hence,

$$(conv(T(\Omega_2) \cup \{x_o\}), conv(T(\Omega_1) \cup \{y_o\})) \in \mathcal{F}.$$

Let $H = \bigcap_{E \in \mathcal{F}_1} E$, $K = \bigcap_{F \in \mathcal{F}_2} F$, where

$$\mathcal{F}_1 = \{E \subset \Omega_1 : \text{ there exists } F \subset \Omega_2 \text{ such that } (E, F) \in \mathcal{F}\}$$

and

$$\mathcal{F}_2 = \{F \subset \Omega_2 : \text{ there exists } E \subset \Omega_1 \text{ such that } (E, F) \in \mathcal{F}\}.$$

Note that the pair (H, K) is convex and $(x_o, y_o) \in H \times K$. We have $T(H) = T(\bigcap_{E\in\mathcal{F}_1} E) \subset \bigcap_{E\in\mathcal{F}_1}(T(E)) \subset \bigcap_{F\in\mathcal{F}_2} F = K$. Similarly, $T(K) \subset H$. Hence, $(H, K) \in \mathcal{F}$.

We now show (H, K) is relatively weakly compact. Let $H_* = conv(T(K) \cup \{x_o\})$ and $K_* = conv(T(H) \cup \{y_o\})$. We have $(H_*, K_*) \subset (H, K)$, so $(T(H_*), T(K_*)) \subset (T(H), T(K)) \subset (K_*, H_*)$, therefore $(H_*, K_*) \in \mathcal{F}$ and $(H_*, K_*) = (H, K)$. Hence,

$$H = conv(T(K) \cup \{x_o\}) \text{ and } K = conv(T(H) \cup \{y_o\}).$$

We put $a = \sup\{\omega(C) : C \text{ is subset of } H\}$ and $b = \sup\{\omega(D) : D \text{ is subset of } K\}$. There exist two sequences $(C_n)_{n\geq1}$ and $(D_n)_{n\geq1}$ of subsets of H and K respectively such that $\lim\limits_{n\to+\infty} \omega(C_n) = a$ and $\lim\limits_{n\to+\infty} \omega(D_n) = b$. Let $C = \cup_{n\geq1}C_n$ and $D = \cup_{n\geq1}D_n$. We have $\omega(C_n) \leq \omega(C) \leq a$ and $\omega(D_n) \leq \omega(D) \leq b$, for all $n \in \mathbb{N} \setminus \{0\}$. Taking $n \to +\infty$, we obtain $\omega(C) = a$ and $\omega(D) = b$.

Assume that $b \leq a$. Let $x \in C$. There exists $p_x \in \mathbb{N} \setminus \{0\}$ and $z_1, \cdots, z_{p_x} \in T(K) \cup \{x_o\}$ such that $x = \sum\limits_{k=1}^{p_x} \lambda_k.z_k$, where $\lambda_k \geq 0$, for all $k \in \{1, \cdots, p_x\}$, and $\sum\limits_{k=1}^{p_x} \lambda_k = 1$. Let $I_x = \{i \in \{1, \cdots, p_x\} : z_i = x_o\}$. For every $k \in \{1, \cdots, p_x\} \setminus I_x$, there exists $u_k \in K$ such that $z_k = T(u_k)$. We put $P_x = \{u_k : k \in \{1, \cdots, p_x\}\setminus I_x\}$, and $P = \cup_{x\in C}P_x$. Since $x = (\sum\limits_{i\in I_x} \lambda_i).x_o + \sum\limits_{k\in\{1,\cdots,p_x\}\setminus I_x} \lambda_k.T(u_k) \in conv(T(P) \cup \{x_o\})$, so $C \subset conv(T(P) \cup \{x_o\})$. We have, $T(\Omega_1 \cup \Omega_2)$ is bounded, so also are H, K, C and P. If $\omega(P) > 0$, then $\omega(C) \leq \omega(T(P)) < \omega(P) \leq b$, because T is $\omega-$condensing. We obtain $a < b$, a contradiction.

Hence, $\omega(P) = 0$, so $\overline{P}^{\omega}$ is weakly compact. By hypothesis, $T(\overline{P}^{\omega})$ is relatively weakly compact. Therefore,

$$\omega(C) \leq \omega(T(P)) \leq \omega(T(\overline{P}^{\omega})) = 0.$$

Thus, $\omega(C) = 0$ i.e. $a = 0 = b$.

Let $(x_n)_{n\geq1}$ be a sequence of H, so $\omega(\{x_n : n \in \mathbb{N}\setminus\{0\}) \leq a = 0$. Thus, H is relatively weakly sequentially compact and now the Eberlein-Smulian theorem guarantees that H is relatively weakly compact. Similarly, $\omega(D) = 0$, so K is relatively weakly compact. As $(x_o, y_o) \in H \times K$ and $\|x_o - y_o\| = \text{dist}(\Omega_1, \Omega_2)$, then

$$\|x_o - y_o\| = \text{dist}(H, K).$$

□

Definition 3.2. Let $(\alpha, \beta) \in (\mathbb{R}_+)^2$ and $\phi, \psi : \mathbb{R}_+ \to \mathbb{R}_+$ two mappings. We say that the pair (ϕ, ψ) has the property (α, β)-monotone if

(i) $\phi(0) = 0 = \psi(0)$,

(ii) $I - \alpha.\phi - \beta.\psi$ is nondecreasing on $\mathbb{R}_+$ and $\lim_{r \to +\infty} (r - \alpha\phi(r) - \beta\psi(r)) = +\infty$ where I stands for the identity mapping.

Remark 3.3. If (ϕ, ψ) has the property (α, β)-monotone and ϕ, ψ are continuous, then the mapping $I - \alpha.\phi - \beta.\psi : \mathbb{R}_+ \to \mathbb{R}_+$ is invertible.

Example 3.4. *Let $\phi, \psi : \mathbb{R}_+ \to \mathbb{R}_+$ be the mappings defined by :*

$$\phi(r) = \log(r + 1) \text{ and } \psi(r) = \frac{r}{3}, \quad \text{for all } r \in \mathbb{R}_+.$$

So, the mapping $I - \frac{1}{2}\phi - \psi : \mathbb{R}_+ \to \mathbb{R}_+$ has the property $(\frac{1}{2}, 1)$-monotone.

Recall that an operator A from a Banach algebra X is said to be regular on X if A maps X into the set of all invertible elements of X.

Theorem 3.5. *Let (Ω_1, Ω_2) be a nonempty closed convex bounded pair of a Banach algebra X satisfying condition $(\mathcal{P})$. Suppose (Ω_1, Ω_2) has proximal normal structure and Ω_1^o is nonempty set. Let $A, C : X \longrightarrow X$ and $B : \Omega_1 \cup \Omega_2 \longrightarrow X$ be three weakly sequentially continuous operators which satisfy the following conditions:*

(i) *A is regular on X, $A(\Omega_1 \cup \Omega_2)$ is relatively weakly compact and $\|A\| < 1$.*

(ii) *A and C are $\mathcal{D}$-Lipschitz with the $\mathcal{D}$-functions Φ_A and Φ_C respectively, $B(\Omega_1 \cup \Omega_2)$ is bounded with bound M, $M\Phi_A(r) + \Phi_C(r) \leq (1 - \|A\|)r$, for all $r > 0$, and $I - M\Phi_A - \Phi_C$, is nondecreasing,*

(iii) *B is cyclic relatively nonexpansive and $\omega-$condensing on $\Omega_1 \cup \Omega_2$,*

(iv) *$x = A(x).B(y) + C(x), y \in \Omega_i \Rightarrow x \in \Omega_j$, $\forall i, j \in \{1, 2\}$ with $i \neq j$.*

Then, there exists $(x, y) \in \Omega_1 \times \Omega_2$ such that

$$\left\| \frac{x - A(x).B(x) - C(x)}{A(x)} \right\| = \text{dist}(\Omega_1, \Omega_2) = \left\| \frac{y - A(y).B(y) - C(y)}{A(y)} \right\|.$$

Proof.

- Let y be fixed in $\Omega_1 \cup \Omega_2$ and let us define the mapping F_y on X by
$$F_y(x) = A(x).B(y) + C(x), \quad \text{for all } x \in X.$$
Let $x_1, x_2 \in X$. The use of assumption (ii) leads to
$$\begin{aligned}\|F_y(x_1) - F_y(x_2)\| &\le \|A(x_1).B(y) - A(x_2).B(y)\| + \|C(x_1) - C(x_2)\|\\ &\le \|A(x_1) - A(x_2)\|\,\|B(y)\| + \|C(x_1) - C(x_2)\|\\ &\le M\Phi_A(\|x_1 - x_2\|) + \Phi_C(\|x_1 - x_2\|).\end{aligned}$$
Now an application of Banach contraction leads to the existence of a unique point $x_y \in X$ such that $F_y(x_y) = x_y$. Hence, the operator $T := \left(\frac{I-C}{A}\right)^{-1} B : \Omega_1 \cup \Omega_2 \to X$ is well defined.

- Moreover, assumption (iv) implies that $T(\Omega_1) \subset \Omega_2$ and $T(\Omega_2) \subset \Omega_1$. Indeed, let $x \in \Omega_1$ and $y \in X$ such that $y = A(y).B(x) + C(y)$, so $T(x) = \left(\frac{I-C}{A}\right)^{-1} B(x) = y \in \Omega_2$. Similarly, for all $y \in \Omega_2$, $T(y) \in \Omega_1$. Hence, T is cyclic on $\Omega_1 \cup \Omega_2$.

- The operator $T : \Omega_1 \cup \Omega_2 \to \Omega_1 \cup \Omega_2$ is weakly sequentially continuous and ω-condensing w.r.t. the measure of weak noncompactness of De Blasi. Indeed, consider $(x_n)_{n\ge 0}$ as a sequence in $\Omega_1 \cup \Omega_2$ that is weakly convergent to x. In this case, the set $\{x_n : n \in \mathbb{N}\}$ is relatively weakly compact, and since B is weakly sequentially continuous, then $\{B(x_n) : n \in \mathbb{N}\}$ is also relatively weakly compact. Using the following equality
$$T(x) = A(T(x)).B(x) + C(T(x)) \quad \text{for all } x \in \Omega_1 \cup \Omega_2, \tag{3.1}$$
combined with the fact that $A(\Omega_1 \cup \Omega_2)$ is relatively weakly compact, C is $\mathcal{D}$-Lipschitz and $\Phi_C(r) < r$, for all $r > 0$, we obtain if $\omega(\{T(x_n) : n \in \mathbb{N}\}) > 0$,
$$\begin{aligned}\omega(\{T(x_n) : n \in \mathbb{N}\}) &\le \omega(\{A(T(x_n)).B(x_n) : n \in \mathbb{N}\})\\ &\qquad + \omega(\{C(T(x_n)) : n \in \mathbb{N}\})\\ &\le \|A\|\omega(\{B(x_n) : n \in \mathbb{N}\}) + \Phi_C(\{\omega(T(x_n)) : n \in \mathbb{N}\})\\ &\le \Phi_C(\{\omega(T(x_n)) : n \in \mathbb{N}\})\\ &< \omega(\{T(x_n) : n \in \mathbb{N}\}),\end{aligned}$$

which is in contradiction. Hence, $\{T(x_n) : n \in \mathbb{N}\}$ is relatively weakly compact. Consequently, there exists a subsequence $(x_{n_k})_{k\geq 0}$ of $(x_n)_{n\geq 0}$ such that $(T(x_{n_k})_{k\geq 0}$ is weakly convergent to $y \in \Omega_1 \cup \Omega_2$. We put $y_{n_k} = \left(\frac{I-C}{A}\right)^{-1} B(x_{n_k})$, for all $k \in \mathbb{N}$. However the subsequence $(y_{n_k})_{k\geq 0}$ satisfies

$$y_{n_k} - C(y_{n_k}) = A(y_{n_k}).B(x_{n_k}).$$

Therefore, from assumption (iii) and in view of condition $(\mathcal{P})$, we deduce that y satisfies

$$y - C(y) = A(y).B(x),$$

or, equivalently

$$y = \left(\frac{I-C}{A}\right)^{-1} B(x) = T(x).$$

Next we claim that the whole sequence $(x_n)_{n\geq 0}$ satisfies

$$T(x_n) = \left(\frac{I-C}{A}\right)^{-1} B(x_n) \rightharpoonup T(x).$$

Assume that there exists a subsequence $(x_{\sigma(n)})_{n\geq 0}$ of $(x_n)_{n\geq 0}$ and a weak neighborhood V^w of $T(x)$ such that $T(x_{\sigma(n)}) \notin V^w$, for all $n \in \mathbb{N}$. Since $(x_{\sigma(n)})_{n\geq 0}$ converge weakly to x, we may extract a subsequence $(x_{\sigma\circ\psi(n)})_{n\geq 0}$ of $(x_{\sigma(n)})_{n\geq 0}$ such that $T(x_{\sigma\circ\psi(n)}) \rightharpoonup Tx$ and $T(x_{\sigma\circ\psi(n)}) \notin V^w$, a contradiction. Hence, $T(x_n) \rightharpoonup T(x)$; it follows that T is weakly sequentially continuous.

T is $\omega-$condensing. Indeed, let N be a subset of $\Omega_1 \cup \Omega_2$ with $\omega(N) > 0$. We have $T(N) \subset A(T(N)).B(N) + C(T(N))$, so

$$\begin{aligned}\omega(T(N)) &\leq \omega(A(T(N)).B(N)) + \omega(C(T(N))) \\ &\leq \|A\|\ \omega(B(N)) + \Phi_C(\omega(T(N)))\end{aligned}$$

$$(I - \Phi_C)(\omega(T(N))) \leq \|A\|\omega(B(N)).$$

By hypothesis (ii), $I - \Phi_C$ is nondecreasing (since $r \mapsto (I - M\Phi_A - \Phi_C)(r)$ is nondecreasing), continuous, $(I - \phi_C)(0) = 0$ and $\lim_{r\to+\infty}(I - \Phi_C)(r) = +\infty$, then $I - \Phi_C : \mathbb{R}_+ \to \mathbb{R}_+$ is invertible. We have also

$$\Phi_C(r) \leq (1 - \|A\|)r \Leftrightarrow \|A\|r \leq (I - \Phi_C)(r) \Leftrightarrow (I - \Phi_C)^{-1}(\|A\|r) \leq r,$$

for all $r \geq 0$, as B is ω-condensing, we get

$$\omega(T(N)) \leq (I - \Phi_C)^{-1}(\|A\|\omega(B(N))) \leq \omega(B(N)) < \omega(N).$$

Thus, T is ω-condensing.

- By Theorem 3.1, there is a convex relatively weakly compact pair $(H,K) \subset (\Omega_1,\Omega_2)$ such that $T(H) \subset K$, $T(K) \subset H$, and $\text{dist}(H,K) = \text{dist}(\Omega_1,\Omega_2)$.

  Let $(H',K') \subset (H,K)$ be a closed, bounded, convex proximal pair for which $\text{dist}(H',K') = \text{dist}(H,K)$ and $\delta(H',K') > \text{dist}(H',K')$. As, $\text{dist}(H',K') = \text{dist}(H,K) = \text{dist}(\Omega_1,\Omega_2)$ and (Ω_1,Ω_2) have proximal normal structure, there exists $(x,y) \in H' \times K'$ such that $\delta(x,K') < \delta(H',K')$, $\delta(y,H') < \delta(H',K')$, where $\delta(H',K') = \sup\{\|a-b\| : a \in H' \ , \ b \in K'\}$. Thus, (H,K) have proximal normal structure.

  The mapping $T : H \cup K \to H \cup K$ is cyclic and relatively nonexpansive. Indeed, let $(x,y) \in H \times K$, The use of assumptions (ii) and (iii) leads to

$$\begin{aligned}\|T(x)-T(y)\| &\le \|A(T(x)).B(x) - A(T(y)).B(y)\| + \|C(T(x) - C(T(y)\| \\ &\le \|A(T(x)) - A(T(y))\| \, \|B\| + \|B(x) - B(y)\| \, \|A\| \\ &\qquad + \|C(T(x) - C(T(y)\| \\ &\le \|A\|\|x-y\| + M\Phi_A(\|T(x)-T(y)\|) \\ &\qquad + \Phi_C(\|T(x)-T(y)\|).\end{aligned}$$

  Since, (Φ_A,Φ_C) has the property $(M,1)$-monotone and Φ_A,Φ_C are continuous, then $I - M\phi_A - \Phi_C$ is invertible, so

$$\|T(x)-T(y)\| \le (I - M\Phi_A - \Phi_C)^{-1}(\|A\|\|x-y\|).$$

  We have also $M\Phi_A(r)+\Phi_C(r) \le (1-\|A\|)r$, so $(I-M\Phi_A-\Phi_C)^{-1}(\|A\|r) \le r$, $\forall r > 0$. Hence,

$$\|T(x)-T(y)\| \le \|x-y\|.$$

  Thus, by Theorem 2.9 there exists $(x,y) \in H \times K$ such that $\|x-T(x)\| = \text{dist}(H,K) = \|y-T(y)\|$. And since $\text{dist}(H,K) = \text{dist}(\Omega_1,\Omega_2)$, then $\|x - \left(\frac{I-C}{A}\right)^{-1} B(x)\| = \text{dist}(\Omega_1,\Omega_2) = \|y - \left(\frac{I-C}{A}\right)^{-1} B(y)\|$.
  Let $z = \left(\frac{I-C}{A}\right)^{-1} B(x)$. So $\left(\frac{I-C}{A}\right)(z) = B(x) \in \Omega_2$ and $z \in \Omega_2$. By (iii), B is cyclic relatively nonexpansive on $\Omega_1 \cup \Omega_2$, so

$$\begin{aligned}\text{dist}(\Omega_1,\Omega_2) &\le \left\|\frac{z - A(z).B(z) - C(z)}{A(z)}\right\| = \left\|\left(\frac{I-C}{A}\right)(z) - B(z)\right\| \\ &\le \|B(x) - B(z)\| \\ &\le \|x - z\| \\ &\le \left\|x - \left(\frac{I-C}{A}\right)^{-1}(B(x))\right\| \\ &= \text{dist}(\Omega_1,\Omega_2).\end{aligned}$$

Similarly we obtain

$$\left\| \frac{y - A(y).B(y) - C(y)}{A(y)} \right\| = \mathrm{dist}(\Omega_1, \Omega_2).$$

□

Remark 3.6. Note that the hypothesis A, B and C are weakly sequentially continuous in Theorem 3.5 can be replaced by one of the following conditions:

1. B and $\left(\frac{I-C}{A}\right)$ are weakly sequentially continuous, without Banach algebra X satisfying the condition $(\mathcal{P})$.
2. The operators A, B and C have to satisfy the condition $(\mathcal{H})$.

Remark 3.7. Recall that, every Lipschitz mapping is $\mathcal{D}$-Lipschitz, so if we replace condition (ii) in Theorem 3.5 by

(ii') the operators A and C are Lipschitzian mappings with constants k_A and k_C, respectively, $B(\Omega_1 \cup \Omega_2)$ is bounded with bound M, and $Mk_A + k_C \leq 1 - \|A\|$,

we get the same conclusion as Theorem 3.5.

Remark 3.8. We can drop the assumptions $A(\Omega_1 \cup \Omega_2)$ is relatively weakly compact and the operators A, B and C are weakly sequentially continuous from Theorem 3.5 and replace them by the pair (Ω_1, Ω_2) is weakly compact. On the other hand, since every closed convex subset of a reflexive Banach space is weakly compact we can get the same result when the Banach algebra is uniformly convex.

Example 3.9. *Consider the Banach algebra $X = \mathcal{C}(J, \mathbb{R})$ of all continuous real-valued functions on $J = [0, 1]$, endowed with the sup-norm $\|.\|_\infty$, defined by*

$$\|x\|_\infty = \sup\{|x(t)| \ ; \ t \in J\},$$

for each $x \in X$. Let (Ω_1, Ω_2) be the pair of the nonempty sets of X defined by:

$$\Omega_1 = \{\tilde{x} : t \to \int_0^t x(s)\, ds \ : \ x \in X \ \text{and}\ \forall t \in J,\ \frac{t}{6} \leq x(t) \leq \frac{t}{4}\}$$

and

$$\Omega_2 = \{\tilde{y} : t \to \int_0^t y(s)\, ds \ : \ y \in X \ \text{and}\ \forall t \in J,\ \frac{3t}{8} \leq y(t) \leq t\}.$$

- *For each* $\tilde{x} \in \Omega_1$ *and* $t \in J$,

$$|\tilde{x}(t)| \leq \int_0^t |x(s)|\, ds \leq \frac{t^2}{8},$$

so $\|\tilde{x}\|_\infty \leq \frac{1}{8}$. *Similarly, for all* $\tilde{y} \in \Omega_2$, $\|\tilde{y}\|_\infty \leq \frac{1}{2}$. *Thus,* (Ω_1, Ω_1) *is bounded.*

- *Let* $\tilde{x_1}, \tilde{x_2} \in \Omega_1$ *and* $\lambda \in J$, *we have, for all* $t \in J$,

$$(\lambda\tilde{x_1} + (1-\lambda)\tilde{x_2})(t) = \int_0^t (\lambda x_1 + (1-\lambda)x_2)(s)\, ds,$$

then

$$\frac{t}{6} \leq (\lambda x_1 + (1-\lambda)x_2)(t) \leq \frac{t}{4}.$$

Thus, $\lambda\tilde{x_1} + (1-\lambda)\tilde{x_2} \in \Omega_1$. *Similarly,* Ω_2 *is convex.*

- *Let* $\tilde{x} \in \Omega_1$ *and* $t, t' \in J$ *such that* $t < t'$,

$$|\tilde{x}(t) - \tilde{x}(t')| \leq \int_t^{t'} |x(s)|\, ds \leq \frac{1}{4}|t - t'|.$$

Hence, Ω_1 *is a family of equicontinuous functions and as it is is uniformly bounded, by Arzela-Ascoli's theorem,* Ω_1 *lies in a compact subset of* X, *and since it is closed it is compact. Similarly,* Ω_2 *is compact in* X. *Consequently,* (Ω_1, Ω_2) *has proximal normal structure and* Ω_1^o *is nonempty set (see Proposition* 2.8*).*

- *For each* $(\tilde{x}, \tilde{y}) \in \Omega_1 \times \Omega_2$ *and* $t \in J$,

$$|\tilde{x}(t) - \tilde{y}(t)| = \int_0^t (y(s) - x(s))\, ds \geq \int_0^t (\frac{3s}{8} - \frac{s}{4})\, ds = \frac{t^2}{16},$$

so

$$\|\tilde{x} - \tilde{y}\|_\infty = \sup_{t \in J} |\tilde{x}(t) - \tilde{y}(t)| \geq \sup_{t \in J} \left(\int_0^t (\frac{3s}{8} - \frac{s}{4})\, ds \right) = \frac{1}{16}.$$

Thus, $\mathrm{dist}(\Omega_1, \Omega_2) = \frac{1}{16}$.

- *Let* B *be the function defined on* $\Omega_1 \cup \Omega_2$ *by*

$$B(\tilde{x}) = \begin{cases} \tilde{x}_1 & \textit{if } \tilde{x} \in \Omega_1 \\ \tilde{y}_1 & \textit{if } \tilde{x} \in \Omega_2 \end{cases},$$

where $\tilde{x}_1 : t \to \int_0^t \frac{3s}{8} ds$ *and* $\tilde{y}_1 : t \to \int_0^t \frac{s}{4} ds$. *Let* $(\tilde{x}, \tilde{y}) \in \Omega_1 \times \Omega_2$, *we have*

$$\|\tilde{x} - \tilde{y}\|_\infty \geq \frac{1}{16},$$

so

$$\|B(\tilde{x}) - B(\tilde{y})\|_\infty = \|\tilde{x}_1 - \tilde{y}_1\|_\infty = \frac{1}{16} \leq \|\tilde{x} - \tilde{y}\|_\infty.$$

Thus, B *is cyclic relatively nonexpansive on* $\Omega_1 \cup \Omega_2$ *and* $M = \sup\{\|B(x)\| : x \in \Omega_1 \cup \Omega_2\} = \|\tilde{y}_1\|_\infty$. *Let* N *be a subset of* $\Omega_1 \cup \Omega_2$ *such that* $\omega(N) > 0$.

$$\begin{aligned}\omega(B(N)) &= \omega(\{B(\tilde{x}) \ : \ \tilde{x} \in N\}) \\ &\leq \omega(\{\tilde{x}_1, \tilde{y}_1\}) = 0, \ \textit{since } \{\tilde{x}_1, \tilde{y}_1\} \textit{ is weakly relatively compact.}\end{aligned}$$

Thus, B *is* $\omega-$*condensing on* $\Omega_1 \cup \Omega_2$.

- *Let* A *be the function defined on* X *by* $A(x) = a$, *where* $a(t) = \frac{1}{6}$ *for all* $t \in J$. *The function* A *is* $\mathcal{D}$*-Lipschitz with the* $\mathcal{D}$*-function* $\Phi_A = 0$, *and* $\|A\| = \sup_{x \in E} \|A(x)\|_\infty = \frac{1}{6}$.
- *Let* C *be the function defined on* X *by* $C(x) = \frac{5}{6}x$. *For each* $(x, y) \in E^2$,

$$\|C(x) - C(y)\|_\infty \leq \frac{5}{6}\|x - y\|_\infty.$$

Then C *is* $\mathcal{D}$*-Lipschitz with the* $\mathcal{D}$*-function* $\Phi_C(r) = \frac{5}{6}.r$ *for all* $r \geq 0$. *In addition,*

$$(I - M\Phi_A - \Phi_C)(r) = (I - \Phi_C)(r) = \frac{1}{6}r$$

so the function $I - M\Phi_A - \Phi_C$ *is nondecreasing, and for all* $r > 0$,

$$(M\Phi_A + \Phi_C)(r) = \frac{5}{6}r \leq (1 - \|A\|)r.$$

- *Let* $x \in X$ *and* $\tilde{y} \in \Omega_2$. *Suppose* $x = A(x).B(\tilde{y}) + C(x)$. *For each* $t \in J$,

$$x(t) = Ax(t).B\tilde{y}(t) + Cx(t) = (\frac{1}{6}).(\frac{t^2}{8}) + \frac{5x(t)}{6} = \frac{t^2}{48} + \frac{5x(t)}{6},$$

so $x(t) = \frac{t^2}{8} = \int_0^t \frac{s}{4} ds$. *Thus,* $x \in \Omega_1$.

Let $y \in X$ *and* $\tilde{x} \in \Omega_2$. *Suppose* $y = A(y).B(\tilde{x}) + C(y)$. *For each* $t \in J$,

$$y(t) = Ay(t).B\tilde{x}(t) + Cy(t) = (\frac{1}{6}).(\frac{3t^2}{16}) + \frac{5y(t)}{6} = \frac{t^2}{32} + \frac{5y(t)}{6},$$

so $y(t) = \frac{3t^2}{16} = \int_0^t \frac{3s}{8} ds$. *Thus,* $y \in \Omega_2$.

- – *The mapping A is constant on X, so A is weakly sequentially continuous on X.*
  – *The mapping B is weakly sequentially continuous on Ω_1 and Ω_2 since it is constant on each part, as the pair (Ω_1, Ω_2) is closed and $\Omega_1 \cap \Omega_2 = \emptyset$ so B is weakly sequentially continuous on $\Omega_1 \cup \Omega_2$.*
  – *Now, we show that C is weakly sequentially continuous on X; for this, let $(x_n)_{n\geq 0}$ in X such that $x_n \rightharpoonup x \in X$, then $(x_n)_{n\geq 0}$ is bounded on X; from Dobrakov's theorem (see Theorem 2.2), we get for all $t \in J$, $x_n(t) \rightharpoonup x(t)$.*

    Put $Cx(t) = K(t, x(t))$ for each $t \in J$ and $x \in X$. For all $t \in J$, the function $K(t, .) : \mathbb{R}_+ \to \mathbb{R}_+$ defined by $K(t, u) = \frac{5u}{6}$ is continuous, so is weakly sequentially continuous, so for all $t \in J$, we have $Cx_n(t) \rightharpoonup Cx(t)$. Again, from Dobrakov's theorem, we deduce that $C(x_n) \rightharpoonup C(x)$, then C is weakly sequentially continuous on X.

Hence by Theorem 3.5, there exists $(x, y) \in \Omega_1 \times \Omega_2$ such that

$$\left\| \frac{x - A(x).B(x) - C(x)}{A(x)} \right\|_\infty = \operatorname{dist}(\Omega_1, \Omega_2) = \left\| \frac{y - A(y).B(y) - C(y)}{A(y)} \right\|_\infty .$$

That is

$$\sup_{t\in J} |x(t) - \tilde{x}_1(t)| = \frac{1}{16} = \sup |y(t) - \tilde{y}_1(t)|.$$

Hence, $(x, y) = (\tilde{y}_1, \tilde{x}_1)$.

Proposition 3.10. *Let (Ω_1, Ω_2) be a nonempty weakly compact convex pair in a Banach algebra X. Suppose (Ω_1, Ω_2) has proximal normal structure. Let $A, C : X \longrightarrow X$ and $B : \Omega_1 \cup \Omega_2 \longrightarrow X$ be three operators which satisfy the following conditions:*

(i) *A is regular on X and $\|A\| < 1$,*

(ii) *A and C are $\mathcal{D}$-Lipschitz with the $\mathcal{D}$-functions Φ_A and Φ_C respectively, $B(\Omega_1 \cup \Omega_2)$ is bounded with bound M, and $M\Phi_A(r) + \Phi_C(r) \leq (1 - \|A\|)r$ for all $r > 0$, and $I - M\Phi_A - \Phi_C$, is nondecreasing,*

(iii) *B is cyclic relatively nonexpansive and $\omega-$condensing on $\Omega_1 \cup \Omega_2$,*

(iv) *$x = A(x).B(y) + C(x), y \in \Omega_i \Rightarrow x \in \Omega_j$, $\forall i, j \in \{1, 2\}$ with $i \neq j$.*

Then, there exists $(x, y) \in \Omega_1 \times \Omega_2$ such that

$$\left\| \frac{x - A(x).B(x) - C(x)}{A(x)} \right\| = \operatorname{dist}(\Omega_1, \Omega_2) = \left\| \frac{y - A(y).B(y) - C(y)}{A(y)} \right\| .$$

Proof. As the proof of the previous theorem; by (i) and (ii), we show that $T := (\frac{I-C}{A})^{-1} \circ B : \Omega_1 \cup \Omega_2 \to X$ is well defined. Moreover, the use of assumption (iv), T is cyclic on $\Omega_1 \cup \Omega_2$.

By assumptions (ii), (iii) the mapping $T : \Omega_1 \cup \Omega_2 \to \Omega_1 \cup \Omega_2$ is relatively nonexpansive.

Thus, by Theorem 2.9 there exists $(x, y) \in \Omega_1 \times \Omega_2$ such that

$$\left\| \frac{x - A(x).B(x) - C(x)}{A(x)} \right\| = \mathrm{dist}(\Omega_1, \Omega_2) = \left\| \frac{y - A(y).B(y) - C(y)}{A(y)} \right\|.$$

□

In the same vein of the above theorem for two operators we have the next result, the proof follows the same steps as in the proof of Theorem 3.5. For sake of completeness we give the complete proof. Note that the Banach algebra does not need to satisfy condition $(\mathcal{P})$.

Theorem 3.11. *Let (Ω_1, Ω_2) be a nonempty closed convex bounded pair of a Banach algebra X. Suppose (Ω_1, Ω_2) has proximal normal structure and Ω_1^o is nonempty set. Let $C : X \longrightarrow X$ and $B : \Omega_1 \cup \Omega_2 \longrightarrow X$ be two operators which satisfy the following conditions:*

(i) *C is $\mathcal{D}$-Lipschitz with the $\mathcal{D}$-function Φ_C, $\Phi_C(r) < r$, for all $r > 0$,*

(ii) *C is weakly sequentially continuous on X and B is weakly sequentially continuous on $\Omega_1 \cup \Omega_2$,*

(iii) *B is cyclic relatively nonexpansive and $\omega-$condensing on $\Omega_1 \cup \Omega_2$,*

(iv) *$(I - C)^{-1}$ is nonexpansive on $\Omega_1 \cup \Omega_2$,*

(v) *$x = B(y) + C(x), y \in \Omega_i \Rightarrow x \in \Omega_j$, $\forall i, j \in \{1, 2\}$ with $i \neq j$.*

Then, there exists $(x, y) \in \Omega_1 \times \Omega_2$ such that

$$\|x - B(x) - C(x)\| = \mathrm{dist}(\Omega_1, \Omega_2) = \|y - B(y) - C(y)\|.$$

Proof.

- Let y be fixed in $\Omega_1 \cup \Omega_2$ and let define the mapping G_y on X by

$$G_y(x) = B(y) + C(x), \quad \text{for all } x \in X.$$

Let $x_1, x_2 \in X$. The use of assumption (ii) leads to

$$\begin{aligned}\|G_y(x_1) - G_y(x_2)\| &\leq \|C(x_1) - C(x_2)\| \\ &\leq \|C(x_1) - C(x_2)\| \\ &\leq \Phi_C(\|x_1 - x_2\|).\end{aligned}$$

Now, an application of Boyd and Wong's fixed point theorem [**13**, Theorem 1] leads to the existence of a unique point $x_y \in X$ such that $G_y(x_y) = x_y$. Hence, the operator $T := (I - C)^{-1} \circ B : \Omega_1 \cup \Omega_2 \to X$ is well defined.

- By assumption (v) we have $T(\Omega_1) \subset \Omega_2$ and $T(\Omega_2) \subset \Omega_1$. Indeed, let $x \in \Omega_1$ and $y \in X$ such that $y = B(x) + C(y)$, so $Tx = (I - C)^{-1}B(x) = y \in \Omega_2$. Similarly, for all $y \in \Omega_2$, $T(y) \in \Omega_1$. Hence, T is cyclic on $\Omega_1 \cup \Omega_2$.
- The operator $T : \Omega_1 \cup \Omega_2 \to \Omega_1 \cup \Omega_2$ is weakly sequentially continuous. Indeed, consider $(x_n)_{n\geq 0}$ as a sequence in $\Omega_1 \cup \Omega_2$ that is weakly convergent to x. In this case, the set $\{x_n : n \in \mathbb{N}\}$ is relatively weakly compact, and since B is weakly sequentially continuous, then $\{B(x_n) : n \in \mathbb{N}\}$ is also relatively weakly compact. Using the following equality

$$T(x) = B(x) + C(T(x)) \text{ for all } x \in \Omega_1 \cup \Omega_2, \tag{3.2}$$

combined with the fact that C is $\mathcal{D}$-Lipschitz and $\Phi_C(r) < r$, for all $r > 0$, we obtain if $\omega(\{T(x_n) : n \in \mathbb{N}\}) > 0$,

$$\begin{aligned}\omega(\{T(x_n) : n \in \mathbb{N}\}) &\leq \omega(\{B(x_n) : n \in \mathbb{N}\}) + \omega(\{C(T(x_n)) : n \in \mathbb{N}\}) \\ &\leq \omega(\{B(x_n) : n \in \mathbb{N}\}) + \Phi_C(\{\omega(T(x_n)) : n \in \mathbb{N}\}) \\ &\leq \Phi_C(\{\omega(T(x_n)) : n \in \mathbb{N}\}) \\ &< \omega(\{T(x_n) : n \in \mathbb{N}\}),\end{aligned}$$

which is in contradiction. Hence, $\{T(x_n) : n \in \mathbb{N}\}$ is relatively weakly compact. Consequently, there exists a subsequence $(x_{n_k})_{k\geq 0}$ of $(x_n)_{n\geq 0}$ such that $(T(x_{n_k})_{k\geq 0}$ is weakly convergent to $y \in \Omega_1 \cup \Omega_2$. We put $y_{n_k} = (I - C)^{-1}B(x_{n_k})$, for all $k \in \mathbb{N}$. However the subsequence $(y_{n_k})_{k\geq 0}$ satisfies

$$y_{n_k} - C(y_{n_k}) = B(x_{n_k}).$$

Therefore, from assumption (iii), we deduce that y satisfies

$$y - C(y) = B(x),$$

or, equivalently
$$y = (I - C)^{-1} B(x) = T(x).$$
Next we claim that the whole sequence $(x_n)_{n\geq 0}$ satisfies
$$T(x_n) = (I - C)^{-1} B(x_n) \rightharpoonup T(x).$$
As the proof of Theorem 3.5, we obtain $T(x_n) \rightharpoonup T(x)$; it follows that T is weakly sequentially continuous.

- T is $\omega-$condensing. Indeed, let N be a subset of $\Omega_1 \cup \Omega_2$ with $\omega(N) > 0$. As $(I - C)^{-1}$ is $1-$Lipschitzian and B is $\omega-$condensing on $\Omega_1 \cup \Omega_2$, we have,
$$\omega(T(N)) = \omega((I - C)^{-1} B(N)) \leq \omega(B(N)) < \omega(N).$$
Thus, T is ω-condensing.

- By Theorem 3.1, there is a convex relatively weakly compact pair $(H, K) \subset (\Omega_1, \Omega_2)$ such that $T(H) \subset K$, $T(K) \subset H$, and $\operatorname{dist}(H, K) = \operatorname{dist}(\Omega_1, \Omega_2)$. Moreover, (H, K) have proximal normal structure. The mapping $T_{H,K} : H \cup K \to H \cup K$ defined by $T_{H,K}(x) = T(x)$, is relatively nonexpansive. Indeed, let $(x, y) \in H \times K$, The use of assumptions (ii) and (iv) leads to
$$\begin{aligned}\|T(x) - T(y)\| &= \|(I - C)^{-1} B(x) - (I - C)^{-1} B(y)\| \leq \|B(x) - B(y)\| \\ &\leq \|x - y\|.\end{aligned}$$
Thus, By Theorem 2.9, there exists $(x, y) \in H \times K$ such that $\|x - T(x)\| = \operatorname{dist}(H, K) = \|y - T(y)\|$. And since $\operatorname{dist}(H, K) = \operatorname{dist}(\Omega_1, \Omega_2)$, then $\|x - B(x) - C(x)\| = \operatorname{dist}(\Omega_1, \Omega_2) = \|y - B(y) - C(y)\|$.

□

The following theorem deals with the classical result for one operator, i.e. when does it exists an $x \in X$ such that $\|x - T(x)\| = \operatorname{dist}(\Omega_1, \Omega_2)$. Note that in our framework, the Banach algebra is not necessarily reflexive, so the normal structure assumption is not sufficient to get best proximity point, that's why we need additional conditions.

Theorem 3.12. *Let (Ω_1, Ω_2) be a nonempty closed convex bounded pair of a Banach algebra X. Suppose (Ω_1, Ω_2) has proximal normal structure and Ω_1^o is nonempty set. Let $B : \Omega_1 \cup \Omega_2 \longrightarrow X$ be an operator such that:*

(i) *B is weakly sequentially continuous on $\Omega_1 \cup \Omega_2$,*

(ii) *B is ω-condensing,*

(iii) *B is cyclic relatively nonexpansive on $\Omega_1 \cup \Omega_2$.*

Then, there exists $(x, y) \in \Omega_1 \times \Omega_2$ such that

$$\|x - B(x)\| = \text{dist}(\Omega_1, \Omega_2) = \|y - B(y)\|.$$

Proof. Using Theorem 3.1 and taking $T := B$, the proof follows same steps as the above proofs. □

We conclude this paper with a fixed point result, note that this result needs neither the normal structure nor the nonexpansiveness of the mapping B.

Theorem 3.13. *Let (Ω_1, Ω_2) be a nonempty closed convex bounded pair of a Banach algebra X satisfying condition $(\mathcal{P})$. Let $A, C : X \longrightarrow X$ and $B : \Omega_1 \cup \Omega_2 \longrightarrow X$ be three operators which satisfy the following conditions:*

(i) *A is regular on X, $A(\Omega_1 \cup \Omega_2)$ is relatively weakly compact and $\|A\| < 1$.*

(ii) *A and C are D-Lipschitz with the D-functions Φ_A and Φ_C respectively, $B(\Omega_1 \cup \Omega_2)$ is bounded with bound M, and $M\Phi_A(r) + \Phi_C(r) \leq (1 - \|A\|)r$ for all $r > 0$, and $I - \Phi_C$, is nondecreasing,*

(iii) *B is weakly sequentially continuous on $\Omega_1 \cup \Omega_2$ and A, C are weakly sequentially continuous on X,*

(iv) *B is $\omega-$condensing on $\Omega_1 \cup \Omega_2$,*

(v) *$x = A(x).B(y) + C(x), y \in \Omega_i \Rightarrow x \in \Omega_j$, $\forall i, j \in \{1, 2\}$ with $i \neq j$.*

Then, if $\Omega_1 \cap \Omega_2$ is nonempty, there exists $x \in \Omega_1 \cap \Omega_2$ such that $x = A(x).B(x) + C(x)$.

Proof. By (ii), we show that $T = (\frac{I-C}{A})^{-1} \circ B : \Omega_1 \cup \Omega_2 \to X$ is well defined. Moreover, the use of assumption (v), T is cyclic on $\Omega_1 \cup \Omega_2$.

The set $\Omega_1 \cap \Omega_2$ is empty set convex bounded pair of a Banach algebra X. Furthermore, $T(\Omega_1 \cap \Omega_2) \subset \Omega_1 \cap \Omega_2$. We consider the operator $T_{\Omega_1 \cap \Omega_2} : \Omega_1 \cap \Omega_2 \to \Omega_1 \cap \Omega_2$ defined by, for all $x \in \Omega_1 \cap \Omega_2$, $T_{\Omega_1 \cap \Omega_2}(x) = T(x)$.

The use of assumption (i), (ii), (iii) and (iv) leads to, the map. $T_{\Omega_1 \cap \Omega_2}$ is weakly sequentially continuous and $\omega-$condensing. Since $T_{\Omega_1 \cap \Omega_2}(\Omega_1 \cap \Omega_2)$ is bounded, according to Theorem 2.4, $T_{\Omega_1 \cap \Omega_2}$ has a fixed point. There exists $x \in \Omega_1 \cap \Omega_2$ such that $x = A(x).B(x) + C(x)$. □

Acknowledgments. The authors would like to thank the anonymous reviewer for their positive comment and careful review, which helped improve the manuscript.

References

[**1**] A. Abkar and M. Gabeleh, *Best proximity points for asymptotic cyclic contraction mappings*, Nonlinear Anal. **74** (2011), 7261–7268.

[**2**] R. P. Agarwal, N. Hussain and M.-A. Taoudi, *Fixed point theorems in ordered Banach spaces and applications to nonlinear integral equations*, Abstr. Appl. Anal. **2012**, Art. ID 245872, 15 pp.

[**3**] C. Angosto and B. Cascales, *Measures of weak noncompactness in Banach spaces*, Topology Appl. **156** (2009), 1412–1421.

[**4**] J. Banas and M. Lecko, *Fixed points of the product of operators in Banach algebra*, PanAmer. Math. J. **12** (2002), 101–109.

[**5**] J. Banaś and L. Olszowy, *On a class of measures of noncompactness in Banach algebras and their application to nonlinear integral equations*, Z. Anal. Anwend. **28** (2009), 475–498.

[**6**] J. Banaś and J. Rivero, *On measures of weak noncompactness*, Ann. Mat. Pura Appl. **151** (1988), 213–224.

[**7**] J. Banaś and K. Sadarangani, *Solutions of some functional-integral equations in Banach algebra*, Math. Comput. Modelling **38** (2003), 245–250.

[**8**] J. Banaś and M.-A. Taoudi, *Fixed points and solutions of operator equations for the weak topology in Banach algebras*, Taiwanese J. Math. **18** (2014), 871–893.

[**9**] S. Sadiq Basha, N. Shahzad and R. Jeyaraj, *Best proximity points: approximation and optimization*, Optim. Lett. **7** (2013), 145–155.

[**10**] A. Ben Amar, S. Chouayekh and A. Jeribi, *New fixed point theorems in Banach algebras under weak topology features and applications to nonlinear integral equations*, J. Funct. Anal. **259** (2010), 2215–2237.

[**11**] A. Ben Amar, S. Derbel, D. O'Regan and T. Xiang, *Fixed point theory for countably weakly condensing maps and multimaps in non-separable Banach spaces*, J. Fixed Point Theory Appl. **21** (2019), Paper No. 8, 25 pp.

[**12**] A. Ben Amar, A. Jeribi and R. Moalla, *Leray-Schauder alternatives in Banach algebra involving three operators with applications*, Fixed Point Theory **15** (2014), 359–372.

[**13**] D. W. Boyd and J. S. W. Wong, *On nonlinear contractions*, Proc. Amer. Math. Soc. **20** (1969), 458–464.

[**14**] M. S. Brodskii and D. P. Mil'man, *On the center of a convex set*, (Russian), Doklady Akad. Nauk SSSR (N.S.) **59** (1948), 837–840.

[**15**] T. A. Burton, *A fixed-point theorem of Krasnoselskii*, Appl. Math. Lett. **11** (1998), 85–88.

[**16**] K. Chaira and S. Lazaiz, *Best proximity point theorems for cyclic relatively ρ-nonexpansive mappings in modular spaces*, Abstr. Appl. Anal. **2018**, Art. ID 8084712, 8 pp.

[**17**] F. S. De Blasi, *On a property of the unit sphere in a Banach space*, Bull. Math. Soc. Sci. Math. R. S. Roumanie (N.S.) **21(69)** (1977), 259–262.

[**18**] B. C. Dhage, *A fixed point theorem in Banach algebras involving three operators with applications*, Kyungpook Math. J. **44** (2004), 145–155.

[**19**] B. C. Dhage, *On a fixed point theorem in Banach algebras with applications*, Appl. Math. Lett. **18** (2005), 273–280.

[**20**] I. Dobrakov, *On representation of linear operators on* $\mathcal{C}_0(T, X)$, Czechoslovak Math. J. **21(96)** (1971), 13–30.

[**21**] R. E. Edwards, *Functional Analysis, Theory and Applications*, Holt, Rinehart and Winston, New York, 1965.

[**22**] A. A. Eldred, W. A. Kirk and P. Veeramani, *Proximal normal structure and relatively nonexpansive mappings*, Studia Math. **171** (2005), 283–293.

[**23**] A. A. Eldred and P. Veeramani, *Existence and convergence of best proximity points*, J. Math. Anal. Appl. **323** (2006), 1001–1006.

[**24**] G. Emmanuele, *Measure of weak noncompactness and fixed point theorems*, Bull. Math. Soc. Sci. Math. R. S. Roumanie (N.S.) **25(73)** (1981), 353–358.

[**25**] M. A. Krasnosel'skii, *Two remarks on the method of successive approximations*, (Russian), Uspekhi Mat. Nauk **10** (1955), 123–127.

[**26**] A. Kryczka and S. Prus, *Measure of weak noncompactness under complex interpolation*, Studia Math. **147** (2001), 89–102.

[**27**] A. Kryczka, S. Prus and M. Szczepanik, *Measure of weak noncompactness and real interpolation of operators*, Bull. Austral. Math. Soc. **62** (2000), 389–401.

[**28**] B. N. Sadovskii, *A fixed-point principle*, Funct. Anal. Its Appl. **1** (1967), 151–153.

[**29**] V. Sankar Raj and P. Veeramani, *Best proximity pair theorems for relatively nonexpansive mappings*, Appl. Gen. Topol. **10** (2009), 21–28.

Karim Chaira
L3A Laboratory
Department of Mathematics and Computer Sciences
Faculty of Sciences Ben M'sik
University of Hassan II Casablanca, Morocco
e-mail: chaira_karim@yahoo.fr

Soumia Chaira
Laboratory of Mathematics and Applications
Faculty of Sciences and Technologies Mohammedia
University Hassan II Casablanca, Morocco
e-mail: soumia_chaira@yahoo.fr

Samih Lazaiz
LaSMA Laboratory,
Department of Mathematics
Faculty of Sciences Dhar El Mahraz
University Sidi Mohamed Ben Abdellah, Fes, Morocco
e-mail: samih.lazaiz@usmba.ac.ma