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Some results on warped and twisted products

Abstract. We give a result for a pseudo-Riemannian manifold to be a warped product. We obtain a necessary and sufficient condition for a twisted product to be a warped product.

Keywords. Warped product, twisted product, pseudo-Riemannian manifold, Lie derivative.

Mathematics Subject Classification: 53C05, 53C25, 53C50.

1 - Introduction

The notion of warped product was introduced by Bishop and O' Neill in [**1**] in order to construct a large class of complete manifolds of negative curvature. In fact, this notion appeared in the literature before [**1**] under the name of semi-reducible spaces [**7**]. Also, this notion is a natural and fruitful generalization of the notion of direct (or Riemannian) product. One of the reasons why warped products have been studied actively is that they play very important roles in physics as well as in differential geometry, especially in the theory of relativity. In fact, the standard space-time models such as Robertson-Walker, Schwarschild, static and Kruskal are warped products. Moreover, the simplest models of neighborhoods of stars and black holes are warped products [**8**]. Hiepko characterized the warped products in [**6**] as follows:

Hiepko's Theorem. *Let M be a pseudo-Riemannian manifold with pseudo-Riemann metric g and call $(\mathcal{D}_1, \mathcal{D}_2)$ the canonical foliations. Suppose that $\mathcal{D}_1$ and $\mathcal{D}_2$ are orthogonal with respect to g. Then (M, g) is a warped product $M_1 \times_f M_2$ if and only if $\mathcal{D}_1$ is totally geodesic and $\mathcal{D}_2$ is spherical, where M_1 (resp. M_2) is the integral manifold of $\mathcal{D}_1$ (resp. $\mathcal{D}_2$).*

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On the other hand, twisted product manifolds [**3**] are natural extensions of warped product manifolds. In this paper, we will prove a result for both warped and twisted product manifolds.

2 - Preliminaries

Let (M_1, g_1) and (M_2, g_2) be any Riemannian manifolds, and let f is a positive smooth function defined on M_1. Also, π_1 and π_2 are canonical projections of $M_1 \times M_2$ onto M_1 and M_2, respectively. Then the *warped product manifold* [**1**] $M_1 \times_f M_2$ is the product manifold $\bar{M} = M_1 \times M_2$ equipped with metric g defined by

$$g = \pi_1^*(g_1) + (f \circ \pi_1)^2 \pi_2^*(g_2)$$

where $\pi_i^*(g_i)$ is the pullback of g_i via π_i for $i = 1, 2$. The function f is called a *warping function* of the warped product $M_1 \times_f M_2$. If f is constant then the manifold is a *direct product* [**4**].

Let $(M_1 \times_f M_2, g)$ be a warped product manifold with the Levi-Civita connection ∇ and ∇^i denote the Levi-Civita connection of M_i for $i \in \{1, 2\}$. By usual convenience, we denote the set of lifts of vector fields on M_i by $\mathfrak{L}(M_i)$ and use the same notation for a vector field and for its lift. On the other hand, π_1 is an isometry and π_2 is a (positive) homothety, so they preserve the Levi-Civita connection. Thus, there is no confusion using the same notation for a connection on M_i and its pullback via π_i. Then, we have

$$\nabla_X Y = \nabla^1_X Y, \tag{2.1}$$

$$\nabla_X V = \nabla_V X = X(\ln(f \circ \pi_1))V, \tag{2.2}$$

$$\nabla_U V = \nabla^2_U V - g(U, V)\nabla(\ln(f \circ \pi_1)) \tag{2.3}$$

for $X, Y \in \mathfrak{L}(M_1)$ and $U, V \in \mathfrak{L}(M_2)$. The manifold $p \times M_2$ is called a *fiber* of the warped product and the manifold $M_1 \times q$ is called a *base manifold* of $M_1 \times_f M_2$, where $p \in M_1$ and $q \in M_2$. It is well known that the base manifold is totally geodesic and the fiber is spherical in $M_1 \times_f M_2$. For more details on warped products, we refer to the book [**4**].

3 - Main Results

Theorem 3.1. *Let (M, g) be a pseudo-Riemannian manifold and $\mathcal{D}_1$ and $\mathcal{D}_2$ be canonical foliations on M. Suppose that $\mathcal{D}_1$ and $\mathcal{D}_2$ intersect everywhere* *orthogonally. Then g is the metric of a warped product $M_1 \times_f M_2$ if and only if for any $W \in \mathfrak{L}(M_2)$*

$$\mathcal{L}_W g = 0 \quad on \quad M_1 \tag{3.1}$$

and there exists a smooth function μ on M_1 such that for any $Z \in \mathfrak{L}(M_1)$, we have

$$\mathcal{L}_Z g = 2Z[\mu]g \quad on \quad M_2, \tag{3.2}$$

where $\mathcal{L}_W$ is the Lie derivative with respect to W and M_1 (resp. M_2) is the integral manifold of $\mathcal{D}_1$ (resp. $\mathcal{D}_2$).

P r o o f. Let $M_1 \times_f M_2$ be a warped product with the metric g. Then using the Lie derivative formula, for any $X, Y, Z \in \mathfrak{L}(M_1)$ and $U, V, W \in \mathfrak{L}(M_2)$, we have

$$(\mathcal{L}_W g)(X, Y) = -2g(h_1(X, Y), W) \tag{3.3}$$

and

$$(\mathcal{L}_Z g)(U, V) = -2g(h_2(U, V), Z), \tag{3.4}$$

where h_1 (resp. h_2) denotes the second fundamental form of $\mathcal{D}_1$ (resp. $\mathcal{D}_2$), (e.g. see [2, p. 195]). Hence, using (2.1), we get

$$(\mathcal{L}_W g)(X, Y) = 0 \tag{3.5}$$

from (3.3). Which gives (3.1). Next, using (2.3), we get

$$(\mathcal{L}_Z g)(U, V) = -2g\Big(- g(U, V)\nabla(\ln(f \circ \pi_1)),\ Z\Big) \tag{3.6}$$

from (3.4). By direct calculation, we obtain,

$$(\mathcal{L}_Z g)(U, V) = 2Z[\ln(f \circ \pi_1)]g(U, V) \tag{3.7}$$

from (3.6). Thus, we get (3.2) for $\mu = \ln(f \circ \pi_1)$.

Conversely, suppose that the conditions (3.1) and (3.2) hold. Then for any $X, Y \in \mathfrak{L}(M_1)$ and $W \in \mathfrak{L}(M_2)$ using (3.1) and (3.3), we get $g(h_1(X, Y), W) = 0$. It follows that $h_1(X, Y) = 0$ for all $X, Y \in \mathfrak{L}(M_1)$. And this tells us that $\mathcal{D}_1$ is totally geodesic. On the other hand, for any $Z \in \mathfrak{L}(M_1)$ and $U, V \in \mathfrak{L}(M_2)$, using (3.2) and (3.4), we have

$$-2g(h_2(U, V), Z) = 2Z[\mu]g(U, V).$$

After a straightforward computation, we get

$$g(h_2(U,V),Z) = g\Big(-g(U,V)\nabla\mu,\, Z\Big).$$

It follows that $h_2(U,V) = -g(U,V)\nabla\mu$ for all $U,V \in \mathfrak{L}(M_2)$. Which means that $\mathcal{D}_2$ is totally umbilical with mean curvature vector field $-\nabla\mu$. Next, we show that $-\nabla\mu$ is parallel. For any $Y \in \mathfrak{L}(M_1)$ and $U \in \mathfrak{L}(M_2)$, by direct calculation, we have

$$\begin{aligned} g(\nabla_U\nabla\mu, Y) &= Ug(\nabla\mu, Y) - g(\nabla\mu, \nabla_U Y) \\ &= U[Y[\mu]] - [U,Y][\mu] - g(\nabla\mu, \nabla_Y U) \\ &= [U,Y][\mu] + Y[U[\mu]] - [U,Y][\mu] - g(\nabla\mu, \nabla_Y U) \\ &= Y[U[\mu]] - g(\nabla\mu, \nabla_Y U). \end{aligned}$$

Since $U[\mu] = 0$, we obtain $g(\nabla_U\nabla\mu, Y) = -g(\nabla\mu, \nabla_Y U)$. On the other hand, since $\nabla\mu$ is tangent to M_1 and M_1 is totally geodesic in M, we have $g(\nabla_Y\nabla\mu, U) = 0$. Thus, it follows that $g(\nabla_U\nabla\mu, Y) = -g(\nabla\mu, \nabla_Y U) = g(\nabla_Y\nabla\mu, U) = 0$. Namely, $-\nabla\mu$ is parallel. Hence, $\mathcal{D}_2$ is spherical, since it is totally umbilical with the parallel mean curvature vector filed $-\nabla\mu$. Thus, by Hiepko's Theorem, g is the metric of a warped product $M_1 \times_f M_2$. □

As we mentioned earlier, twisted product manifolds are natural extensions of warped product manifolds. Namely, the warping function f of a warped product $M_1 \times_f M_2$ were replaced by a twisting function, i.e. f is a positive smooth function on $M_1 \times M_2$. In this case, the covariant derivatives formula (2.1) remains same while the covariant derivatives formulas (2.2) and (2.3) change as

$$\nabla_X V = \nabla_V X = X(\ln f)V \tag{3.8}$$

and

$$\nabla_U V = \nabla^2_U V + U(\ln f)V + V(\ln f)U - g(U,V)\nabla(\ln f). \tag{3.9}$$

Next, motivated by Lemma 2.3 of [**5**], we give the second result of this paper.

Theorem 3.2. *Let $M_1 \times_f M_2$ be a twisted product. Then it is a warped product if and only if the mean curvature vector field of the canonical foliation $\mathcal{D}_2$ is closed.*

Proof. Let $M_1 \times_f M_2$ be a twisted product. If it is a warped product, then it follows the mean curvature vector field H_2 of the canonical distribution of $\mathcal{D}_2$ is $-\nabla(\ln f \circ \pi_1)$ from (2.3). This vector field is closed, since its dual 1-form, i.e. $-d(f \circ \pi_1)$ is closed.

Conversely, let the mean curvature vector field $H_2 = -P_1\nabla(\ln f)$ of the canonical foliation $\mathcal{D}_2$ of the twisted product $M_1 \times_f M_2$ be closed, where $P_1 : \mathcal{L}(M_1 \times M_2) \to \mathcal{L}(M_1)$ is canonical projection. We denote by μ the dual 1-form of H_2. Hence μ is closed, i.e. $d\mu = 0$, then for any $X \in \mathcal{L}(M_1)$ and $U \in \mathcal{L}(M_2)$, using the exterior differentiation formula (see, [9, p.17])), we have

$$0 = d\mu(U, X) = U\mu(X) - X\mu(U) - \mu([U, X]).$$

Since $X\mu(U) = 0$ and $[U, X] = 0$, we obtain

$$U\mu(X) = 0.$$

But,

$$\begin{aligned} U\mu(X) &= Ug(X, H_2) \\ &= Ug(X, -P_1\nabla(\ln f)) = -Ug(X, \nabla(\ln f)) = -UX(\ln f). \end{aligned}$$

Thus, we get

$$UX(\ln f) = 0. \tag{3.10}$$

On the other hand, we have

$$UX(\ln f) = -Ug(X, H_2) = -g(\nabla_U X, H_2) - g(X, \nabla_U H_2).$$

Using (3.8), we obtain

$$UX(\ln f) = -g(\nabla_X U, H_2) - g(X, \nabla_U H_2) = g(\nabla_X H_2, U) - g(X, \nabla_U H_2).$$

By (2.1), we get

$$UX(\ln f) = -g(X, \nabla_U H_2). \tag{3.11}$$

It follows that $g(X, \nabla_U H_2) = 0$ from (3.10) and (3.11). Which means that H_2 is parallel. Thus $\mathcal{D}_2$ is spherical. We already know that the other canonical distribution $\mathcal{D}_1$ is totally geodesic, since $M_1 \times_f M_2$ is a twisted product. Therefore, the assertion follows from Hiepko's Theorem. □

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References

[**1**] R. L. Bishop and B. O'Neill, *Manifolds of negative curvature*, Trans. Amer. Math. Soc. **145** (1969), 1–49.

[**2**] R. A. Blumenthal and J. J. Hebda, *An analogue of the holonomy bundle for a foliated manifold*, Tôhoku Math. J. **40** (1988), 189–197.

[**3**] B.-Y. Chen, *Geometry of submanifolds and its applications*, Science University of Tokyo, Tokyo, 1981.

[**4**] B.-Y. Chen, *Differential geometry of warped product manifolds and submanifolds*, World Scientific Publishing, Hackensack, NJ, 2017.

[**5**] M. Gutiérrez and B. Olea, *Semi-Riemannian manifolds with a doubly warped structure*, Rev. Mat. Iberoam. **28** (2012), 1–24.

[**6**] S. Hiepko, *Eine innere Kennzeichnung der verzerrten Produkte*, (German), Math. Ann. **241** (1979), 209–215.

[**7**] G. I. Kruchkovich, *On semireducible Riemannian spaces*, (Russian), Dokl. Akad. Nauk SSSR **115** (1957), 862–865.

[**8**] B. O'Neill, *Semi-Riemannian geometry With applications to relativity*, Pure Appl. Math., **103**, Academic Press, New York, 1983.

[**9**] K. Yano and M. Kon, *Structures on manifolds*, Ser. Pure Math., **3**, World Scientific Publishing, Singapore, 1984.

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