Riv. Mat. Univ. Parma, Vol. **12** (2021), 201-219

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Some fixed point results in ordered bicomplex-valued metric spaces

Abstract. Recently, fixed point results on bicomplex valued metric spaces have had many applications in functional analysis, graph theory, probability theory and other areas. Very recently, Fuli He et al. (J. Funct. Spaces, 2020, Art. ID 4070324) introduced fixed point theorems for Mizoguchi-Takahashi type contraction in bicomplex-valued metric spaces and applications. In this direction of research, we demonstrate some fixed point theorems in ordered bicomplex valued metric spaces for type contraction mappings with illustrative examples. The reported results here along with those stated in earlier papers were also specified.

Keywords. Bicomplex numbers, Partial order, Fixed point, Contraction mapping, Compatible mapping.

Mathematics Subject Classification: 47H10, 54D99, 54E99.

1 - Introduction

In recent decades, fixed point theories are very important tools in the different areas of mathematical analysis, applied mathematics and other science, which have engaged many researchers (see, e.g., [**1**, **2**, **4**, **5**, **6**, **7**, **11**]). Nowadays, there have been a number of generalizations of metric spaces and some fixed point results. In particular, Azam et al. [**9**] proved some fixed point results in complex-valued metric spaces. The extension of this work established recently in [**17**].

Later on, results on bicomplex functional analysis and their applications have been presented and discussed by many authors (see, for example, [**3**, **13**, **14**]).

Received: March 6, 2020; accepted in revised form: February 11, 2021.
This research was supported by Deanship of Scientific Research at King Khalid University.

In [**10**], Choi et al. introduced the notion of bicomplex-valued metric space which is a generalization of complex-valued metric space and proved some various fixed point results. Also, see, [**15**].

Very recently, some new fixed point theorems for contractive maps that satisfied Mizoguchi-Takahashi type condition in the setting of bicomplex-valued metric spaces are studied by [**12**].

Motivated essentially by the above-mentioned results. In this manuscript, we demonstrate a unique common fixed point theorem for contractive mappings satisfying the notion of weak compatibility. To substantiate the authenticity of our results, some illustrative examples are also outlined. Furthermore, some special cases and consequences of our main results are presented.

2 - Preliminaries

In this section, we recall some definitions and terminologies which will be used to prove the main results.

2.1 - *Bicomplex numbers*

The set of bicomplex numbers denoted by $\mathbb{BC}$ is the first setting in an infinite sequence of multicomplex sets which are generalizations of the set of complex numbers $\mathbb{C}$. Here we recall the set of bicomplex numbers $\mathbb{BC}$ (see, for example, [**13, 16**]):

$$\mathbb{BC} = \{w = x_0 + i_1x_1 + i_2x_2 + i_1i_2x_3;\;\; x_k \in \mathbb{R},\ (k = 0, 1, 2, 3)\}.$$

Since each element w in $\mathbb{BC}$ can be written as $w = x_0 + i_1x_1 + i_2(x_2 + i_1x_3)$ or $w = z_1 + i_2z_2$; $(z_1, z_2 \in \mathbb{C})$ we can also express $\mathbb{BC}$ as

$$\mathbb{BC} := \{w = z_1 + i_2z_2,\ z_1, z_2 \in \mathbb{C}\}, \tag{2.1}$$

where $z_1 = x_0+i_1x_1$, $z_2 = x_2+i_1x_3$ and i_1, i_2 are independent imaginary units such that $i_1^2 = -1 = i_2^2$. The product of i_1 and i_2 defines a hyperbolic unit j such that $j^2 = 1$. The product of units is commutative and is defined as

$$i_1i_2 = j, \quad i_1j = -i_2, \quad i_2j = -i_1.$$

With the addition and multiplication of two bicomplex numbers defined in the obvious way, the set $\mathbb{BC}$ makes up a commutative ring. Three important subsets of $\mathbb{BC}$ can be specified as

$$\mathbb{C}(i_k) := \{x + yi_k;\ x, y \in \mathbb{R}\}, \quad \text{for } k = 1, 2. \tag{2.2}$$

$$\mathbb{D} := \{x + yj;\ x, y \in \mathbb{R}\}. \tag{2.3}$$

Each of the sets $\mathbb{C}(i_k)$ is isomorphic to the field of complex numbers, while $\mathbb{D}$ is the set of the so-called hyperbolic numbers.

2.2 - *Conjugation and moduli*

Three kinds of conjugation can be defined on bicomplex numbers. With w specified as in (2.1) and the bar $\bar{\cdot}$ denoting complex conjugation in $\mathbb{C}$, we define

$$w^{\dagger_1} := \bar{z}_1 + \bar{z}_2\, i_2,\ \ w^{\dagger_2} := z_1 - z_2\, i_2,\ \ w^{\dagger_3} := \bar{z}_1 - \bar{z}_2\, i_2. \tag{2.4}$$

It is easy to check that each conjugation has the following properties

$$(u + v)^{\dagger_k} = u^{\dagger_k} + v^{\dagger_k},\ \ (u^{\dagger_k})^{\dagger_k} = u,\ \ (u.v)^{\dagger_k} = u^{\dagger_k}.v^{\dagger_k}. \tag{2.5}$$

here $u, v \in \mathbb{BC}$ and $k = 1, 2, 3$. With each kind of conjugation, one can define a specific bicomplex modulus as

$$\begin{aligned} |w|^2_{i_1} :=& w.w^{\dagger_2} = z_1^2 + z_2^2 \in \mathbb{C}(i_1),\\ |w|^2_{i_2} :=& w.w^{\dagger_1} = (|z_1|^2 - |z_2|^2) + 2\mathrm{Re}(z_1\bar{z}_2)i_2 \in \mathbb{C}(i_2),\\ |w|^2_{j} :=& w.w^{\dagger_3} = (|z_1|^2 + |z_2|^2) - 2\mathrm{Im}(z_1\bar{z}_2)j \in \mathbb{D}. \end{aligned}$$

It can be shown that $|u.v|^2_s = |u|^2_s.|v|^2_s$, where $s = i_1, i_2$ or j.

A norm of a bicomplex number $w = z_1 + i_2 z_2$ denoted by $\|w\|$ is defined by

$$\|w\| = \|z_1 + i_2 z_2\| = (|z_1|^2 + |z_2|^2)^{\frac{1}{2}},$$

which, upon choosing $w = x_0 + i_1 x_1 + i_2 x_2 + i_1 i_2 x_3, (x_k \in \mathbb{R}, k = 0, 1, 2, 3)$, gives

$$\|w\| = (x_0^2 + x_1^2 + x_2^2 + x_3^2)^{\frac{1}{2}}.$$

For any two bicomplex numbers $u, v \in \mathbb{BC}$, one can easily verify that $\|u + v\| \leq \|u\| + \|v\|$, $\|\alpha u\| = \alpha\|u\|$, where α is nonnegative real number. Further, for any two bicomplex numbers $u, v \in \mathbb{BC}$, $\|uv\| \leq \sqrt{2}\|u\|\|v\|$ holds.

Next, we recall some necessary definitions and lemmas.

First, define a partial order relation $\prec_{i_1}$ on $\mathbb{C}$ as $u_1 \in \mathbb{C}$, $u_1 \prec_{i_1} v_1$ if and only if $\mathrm{Re}(u_1) < \mathrm{Re}(v_1)$ and $\mathrm{Im}(u_1) < \mathrm{Im}(v_1)$. Let $u = u_1 + i_2 u_2 \in \mathbb{BC}$ and $v = v_1 + i_2 v_2 \in \mathbb{BC}$, then we define a partial order relation $\preceq_{i_2}$ on $\mathbb{BC}$ as follows

$$u \preceq_{i_2} v \Longleftrightarrow u_1 \preceq_{i_1} v_1 \quad \text{and} \quad u_2 \preceq_{i_1} v_2,$$

where $\prec_{i_1}$ is a partial order relation on $\mathbb{C}$ and $\preceq_{i_1}$ is the reflexive closure of $\prec_i$.

It is easy to verify $0 \prec_{i_2} u \prec_{i_2} v \Longrightarrow \|u\| \leq \|v\|$.

The definition of the complex metric space is introduced in [**9**], we extend this definition to Bicomplex analysis as follows.

Definition 2.1. Let X be a nonempty set. A function $d_{\mathbb{BC}} : X \times X \longrightarrow \mathbb{BC}$ is called a bicomplex-valued metric on X if for $x, y, z \in X$ the following conditions are satisfied:

(m_1) $0 \preceq_{i_2} d_{\mathbb{BC}}(x, y)$,

(m_2) $d_{\mathbb{BC}}(x, y) = 0$ if and only if $x = y$,

(m_3) $d_{\mathbb{BC}}(x, y) = d_{\mathbb{BC}}(y, x)$,

(m_4) $d_{\mathbb{BC}}(x, y) \preceq_{i_2} d_{\mathbb{BC}}(x, z) + d_{\mathbb{BC}}(z, y)$.

Then $(X, d_{\mathbb{BC}})$ is called a bicomplex-valued metric space.

Some known examples of bicomplex-valued metric, which show that a bicomplex-valued metric, are the following.

Example 2.2. *On set of real numbers together consider the functionals*

$$d^1_{\mathbb{BC}}(x, y) = (1 + i_1 + i_2 + i_1 i_2)|x - y|$$

$$d^2_{\mathbb{BC}}(x, y) = i_1 i_2 |x - y|$$

for all $x, y \in \mathbb{R}$ where $|.|$ is the usual real modulus. One can easily check that $(X, d^1_{\mathbb{BC}})$, $(X, d^2_{\mathbb{BC}})$ are a bicomplex-valued metric spaces.

A bicomplex-valued metric space $(X, d_{\mathbb{BC}})$ together with a partial order relation $\precsim_{i_2}$ on X is called partially ordered bicomplex-valued metric space.

Definition 2.3. [**12**] Let (x_n) be a sequence in a bicomplex-valued metric space $(X, d_{\mathbb{BC}})$. The sequence (x_n) is said to converge to $x \in X$ if and only if for any $0 \prec_{i_2} \varepsilon \in \mathbb{BC}$, there exists $N \in \mathbb{N}$ depending on ε such that $d_{\mathbb{BC}}(x_n, x) \prec_{i_2} \varepsilon$ for all $n > N$. It is denoted by $x_n \to x$ as $n \to +\infty$ or $\lim\limits_{n \to +\infty} x_n = x$.

Definition 2.4. [**12**] A sequence (x_n) in a bicomplex-valued metric space $(X, d_{\mathbb{BC}})$ is said to be a Cauchy sequence if and only if for any $0 \prec_{i_2} \varepsilon \in \mathbb{BC}$, there exists $N \in \mathbb{N}$ depending on ε such that $d_{\mathbb{BC}}(x_n, x_m) \prec_{i_2} \varepsilon$ for all $n, m > N$

Definition 2.5. [**12**] A bicomplex-valued metric space $(X, d_{\mathbb{BC}})$ is said to be complete if and only if every Cauchy sequence in X converges in X.

L e m m a 2.6 ([**10**]). *Let* $(X, d_{\mathbb{BC}})$ *be a bicomplex-valued metric space and* (x_n) *be a sequence in* X. *Then* (x_n) *converges to* $x \in X$ *if and only if* $\|d_{\mathbb{BC}}(x_n, x)\| \to 0$ *as* $n \to +\infty$.

L e m m a 2.7 ([**10**]). *Let* $(X, d_{\mathbb{BC}})$ *be a bicomplex-valued metric space and* (x_n) *be a sequence in* X *such that* $\lim_{n\to+\infty} x_n = x$. *Then for any* $a \in X$, $\lim_{n\to+\infty} \|d_{\mathbb{BC}}(x_n, a)\| = \|d_{\mathbb{BC}}(x, a)\|$.

Let (X, d) be a metric space and $T, S : X \to X$ be two mappings. A point $x \in X$ is said to be a coincidence point of T and S if and only if $Sx = Tx$ and a point $y \in X$ is said to be a common fixed point of T and S if and only if $Sx = Tx = x$.

D e f i n i t i o n 2.8 ([**1**, **15**]). The mappings $T, S : X \to X$ are called commuting if $TSx = STx$ for all $x \in X$, compatible if $\lim_{n\to+\infty} d(TSx_n, STx_n) = 0$ whenever (x_n) is a sequence such that $\lim_{n\to+\infty} Tx_n = \lim_{n\to+\infty} Sx_n = t$ for some t in X, weakly compatible if they commute at their coincidence points, that is, if $STx = TSx$ whenever $Sx = Tx$.

R e m a r k 2.9. For the partially order set $(X, \preceq)$, we say that S is non-decreasing if for $x, y \in X$, $x \preceq y$, we have $Sx \preceq Sy$. Similarly, we say that S is non-increasing if for $x, y \in X$, $x \preceq y$, we have $Sx \succeq Sy$ and we say that T is $S-$non-decreasing if for $x, y \in X$, $Sx \preceq Sy$, we have $Tx \preceq Ty$. Note that if S is the identity mapping, then T is $S-$non-decreasing means T is monotone nondecreasing.

A subset Y of a partially ordered set X is said to be well-ordered if every two elements of Y are comparable.

3 - Statement of Results

In this section, we will present some fixed point theorems in ordered bicomplex-valued metric spaces for generalized rational type contraction mappings. Furthermore, we will give examples and applications to our main results. The first result in this work is the following theorem.

T h e o r e m 3.1. *Let* $(X, \preceq, d_{\mathbb{BC}})$ *be a partially ordered complete bicomplex-valued metric space. Suppose that* T *and* S *are continuous self mappings on* X,

$T(X) \subseteq S(X)$, T is a $S-$ non-decreasing mapping and

$$\begin{aligned} d_{\mathbb{BC}}(Tx,Ty) \quad \preceq_{i_2} \quad & \alpha\Big(\frac{d_{\mathbb{BC}}(Sx,Tx)d_{\mathbb{BC}}(Sy,Ty)}{d_{\mathbb{BC}}(Sx,Sy)}\Big) + \beta d(Sx,Sy) \\ & + \gamma\big[d_{\mathbb{BC}}(Sx,Tx) + d_{\mathbb{BC}}(Sy,Ty)\big] \\ & + \delta\big[d_{\mathbb{BC}}(Sx,Ty) + d_{\mathbb{BC}}(Sy,Tx)\big], \end{aligned}$$

for all $x, y \in X$ with $Sx \succeq Sy$, $Sx \neq Sy$ and for some $\alpha, \beta, \gamma, \delta \in [0,1)$ with $\alpha + 2\beta + 2\gamma + 2\delta < 1$. If there exists $x_0 \in X$ such that $Sx_0 \preceq Tx_0$, T and S are compatible, then T and S have a coincidence point.

Proof. By the condition of the theorem there exists $x_0 \in X$ such that $Sx_0 \preceq Tx_0$. Since $T(X) \subseteq S(X)$, we can find a point $x_1 \in X$ such that $Sx_1 = Tx_0$, then $Sx_0 \preceq Tx_0 = Sx_1$. Since T is $S-$non-decreasing, we have $Sx_1 = Tx_0 \preceq Tx_1$. In this way, we construct the sequence (x_n) recursively as

$$Tx_n = Sx_{n+1} \text{ for all } n \geq 1,$$

for which

$$\begin{aligned} & Sx_0 \preceq Tx_0 = Sx_1 \preceq Tx_1 = Sx_2 \preceq Tx_2 \preceq ... \\ & \qquad\qquad \preceq Tx_{n-1} = Sx_n \preceq Tx_n = Sx_{n+1} \preceq \dots . \end{aligned}$$

We suppose that $d_{\mathbb{BC}}(Tx_n, Tx_{n+1}) \succ_{i_2} 0$ for all n. If not, then $Tx_{n+1} = Tx_n$ for some n, $Tx_{n+1} = Sx_{n+1}$, i.e., T and S have a coincidence point x_{n+1}, and thus, we have the result. Consider

$$\begin{aligned} & d_{\mathbb{BC}}(Tx_{n+1}, Tx_n) \\ \preceq_{i_2} & \alpha\Big(\frac{d_{\mathbb{BC}}(Sx_{n+1}, Tx_{n+1})d_{\mathbb{BC}}(Sx_n, Tx_n)}{d_{\mathbb{BC}}(Sx_{n+1}, fx_n)}\Big) + \beta d_{\mathbb{BC}}(Sx_{n+1}, Sx_n) \\ & + \gamma\big[d_{\mathbb{BC}}(Sx_{n+1}, Tx_{n+1}) + d_{\mathbb{BC}}(Sx_n, Tx_n)\big] \\ & + \delta\big[d_{\mathbb{BC}}(Sx_{n+1}, Tx_n) + d_{\mathbb{BC}}(Sx_n, Tx_{n+1})\big] \\ = & \alpha\Big(\frac{d_{\mathbb{BC}}(Sx_{n+1}, Sx_{n+2})d_{\mathbb{BC}}(Sx_n, Sx_{n+1})}{d_{\mathbb{BC}}(Sx_{n+1}, Sx_n)}\Big) + \beta d_{\mathbb{BC}}(Sx_{n+1}, Sx_n) \\ & + \gamma\big[d_{\mathbb{BC}}(Sx_{n+1}, Sx_{n+2}) + d_{\mathbb{BC}}(Sx_n, Sx_{n+1})\big] \\ & + \delta\big[d_{\mathbb{BC}}(Sx_{n+1}, Sx_{n+1}) + d_{\mathbb{BC}}(Sx_n, Sx_{n+2})\big] \\ \preceq_{i_2} & \alpha d_{\mathbb{BC}}(Sx_{n+1}, Sx_{n+2}) + \beta d_{\mathbb{BC}}(Sx_{n+1}, Sx_n) \\ & + \gamma\big[d_{\mathbb{BC}}(Sx_{n+1}, Sx_{n+2}) + d_{\mathbb{BC}}(Sx_n, Sx_{n+1})\big] \\ & + \delta\big[d_{\mathbb{BC}}(Sx_n, Sx_{n+1}) + d_{\mathbb{BC}}(Sx_{n+1}, Sx_{n+2})\big] \end{aligned}$$

$$
\begin{aligned}
&=(\alpha+\gamma+\delta)d_{\mathbb{BC}}(Sx_{n+1},Sx_{n+2})+(\beta+\gamma+\delta)d_{\mathbb{BC}}(Sx_n,Sx_{n+1})\\
&=(\alpha+\gamma+\delta)d_{\mathbb{BC}}(Tx_n,Tx_{n+1})+(\beta+\gamma+\delta)d_{\mathbb{BC}}(Tx_{n-1},Tx_n),
\end{aligned}
$$

which implies that

$$d_{\mathbb{BC}}(Tx_{n+1},Tx_n)\preceq_{i_2}\frac{(\beta+\gamma+\delta)}{1-(\alpha+\gamma+\delta)}d_{\mathbb{BC}}(Tx_n,Tx_{n-1}).$$

Using mathematical induction we have

$$d_{\mathbb{BC}}(Tx_{n+1},Tx_n)\preceq_{i_2}\Big(\frac{\beta+\gamma+\delta}{1-(\alpha+\gamma+\delta)}\Big)^n d_{\mathbb{BC}}(Tx_1,Tx_0).$$

Put $h=\frac{(\beta+\gamma+\delta)}{1-(\alpha+\gamma+\delta)}$, since $\alpha+\beta+2\gamma+2\delta<1$, then $0\le h<1$ and consequently for all $n>0$,

$$
\begin{aligned}
d_{\mathbb{BC}}(Tx_{n+1},Tx_{n+2})&\preceq_{i_2}hd_{\mathbb{BC}}(Tx_n,Tx_{n+1})\preceq_{i_2}h^2d_{\mathbb{BC}}(Tx_{n-1},Tx_n)\\
&\preceq_{i_2}\dots\preceq_{i_2}h^{n+1}d_{\mathbb{BC}}(Tx_0,Tx_1).
\end{aligned}
$$

Now, for $m>n$, we have

$$
\begin{aligned}
&d_{\mathbb{BC}}(Tx_n,Tx_m)\\
&\quad\preceq_{i_2}d_{\mathbb{BC}}(Tx_n,Tx_{n+1})+d_{\mathbb{BC}}(Tx_{n+1},Tx_{n+2})+\dots+d_{\mathbb{BC}}(Tx_{m-1},Tx_m)\\
&\quad\preceq_{i_2}(h^n+h^{n+1}+h^{n+2}+\dots+h^{m-1})d_{\mathbb{BC}}(Tx_0,Tx_1)\\
&\quad\preceq_{i_2}\frac{h^n}{1-h}d_{\mathbb{BC}}(Tx_0,Tx_1).
\end{aligned}
$$

That is,

$$\| d_{\mathbb{BC}}(Tx_n,Tx_m)\|\le\frac{h^n}{1-h}\| d_{\mathbb{BC}}(Tx_0,Tx_1)\|.$$

Taking limit as $n\to+\infty$, we obtain $\| d_{\mathbb{BC}}(Tx_n,Tx_m)\|\to 0$, that is $d_{\mathbb{BC}}(Tx_n,Tx_m)\to 0$ as $n\to+\infty$. Therefore, (Tx_n) is a Cauchy sequence in the complete bicomplex-valued metric space. Then, there exists $u\in X$ such that

$$\lim_{n\to+\infty}d_{\mathbb{BC}}(Tx_n,u)=0.$$

By the continuity of T, we have

$$\lim_{n\to+\infty}d_{\mathbb{BC}}(T(Tx_n),Tu)=0.$$

Since $Sx_{n+1}=Tx_n\to u$ and the pair (T,S) is compatible, we have

$$\lim_{n\to+\infty}d_{\mathbb{BC}}(S(Tx_n),T(Sx_n))=0.$$

Then, since $Tx_n \to u$ as $n \to +\infty$ in $(X, d_{\mathbb{BC}})$, and S is continuous, we get $STx_n \to Su$ as $n \to +\infty$ in $(X, d_{\mathbb{BC}})$. Therefore, we get,

$$\begin{aligned} d_{\mathbb{BC}}(Tu, Su) &\preceq_{i_2} d_{\mathbb{BC}}(Tu, T(Sx_n)) + d_{\mathbb{BC}}(T(Sx_n), fu) \\ &\preceq_{i_2} d_{\mathbb{BC}}(Tu, T(Sx_n)) + d_{\mathbb{BC}}(TSx_n, STx_n) + d_{\mathbb{BC}}(STx_n, Su), \end{aligned}$$

that is,

$$\begin{aligned} \| d_{\mathbb{BC}}(Tu, Su) \| &\preceq_{i_2} \| d_{\mathbb{BC}}(Tu, T(Sx_n)) \| + \| d_{\mathbb{BC}}(T(Sx_n), Su) \| \\ &\preceq_{i_2} \| d_{\mathbb{BC}}(Tu, T(Sx_n)) \| + \| d_{\mathbb{BC}}(TSx_n, STx_n) \| \\ &\qquad + \| d_{\mathbb{BC}}(STx_n, Su) \| . \end{aligned}$$

Letting $n \to +\infty$ in the above inequality, we get that $\| d_{\mathbb{BC}}(Tu, Su) \| = 0$. i.e, $Tu = Su$ and u is a coincidence point of T and S. □

Example 3.2. *Let $X = \{1, 2, 3\}$ reendowed with the partial order $\preceq$ is defined as $\{(1, 1), (2, 2), (3, 3), (3, 1)\}$. Let a mapping $d_{\mathbb{BC}} : X \times X \to \mathbb{BC}$ be defined by $d_{\mathbb{BC}}(x, y) = (1+i_1+i_2+i_1i_2)|x-y|, (\forall x, y \in X)$, where $|.|$ is the usual real modulus. One can easily check that $(X, d_{\mathbb{BC}})$ is a complete bicomplex-valued metric on $\mathbb{C}$.*

Define $T(1) = 1$, $T(2) = 2$ and $T(3) = 1$ and $S(1) = 1$, $S(2) = 2$ and $S(3) = 3$. Then, for $x = 3$, $y = 1$, we have $d(Tx, Ty) = 0$ and

$$\begin{aligned} d_{\mathbb{BC}}(T(3), T(1)) = 0 \quad &\preceq_{i_2} \alpha\Big(\frac{d_{\mathbb{BC}}(S3, T3)d_{\mathbb{BC}}(S1, T1)}{d_{\mathbb{BC}}(S3, S1)}\Big) + \beta d_{\mathbb{BC}}(S3, S1) \\ &\quad + \gamma\big[d_{\mathbb{BC}}(S3, T3) + d_{\mathbb{BC}}(S1, T1)\big] \\ &\quad + \delta\big[d_{\mathbb{BC}}(S3, T1) + d_{\mathbb{BC}}(S1, T3)\big] \\ &= 2(\beta + \gamma + \delta)(1 + i_1 + i_2 + i_1i_2). \end{aligned}$$

Hence the inequality holds. On the other hand, it is obvious that T is $S-$non decreasing mapping and there exists $x_0 = 3$ such that $Sx_0 \preceq Tx_0$ and 1 is a unique common fixed point of S and T.

Example 3.3. *Let $X = \{0, 1, 3\}$ reendowed with the partial order $\preceq$ is defined as $\{(0, 0), (1, 1), (3, 3), (0, 1), (0, 3), (3, 1)\}$. Let a mapping $d : X \times X \to \mathbb{BC}$ be defined by $d_{\mathbb{BC}}(x, y) = (i_1i_2)|x - y|, (\forall x, y \in X)$, where $|.|$ is the usual real modulus.*

Define $Tx = 1$, for all $x \in X$ and $S(0) = 3$, $S(1) = 1$ and $S(3) = 1$. Then, *for $x=0$, $y=1$, we have* $d_{\mathbb{BC}}(Tx,Ty)=0$ *and*

$$\begin{aligned} d_{\mathbb{BC}}(T0,T1)=0 &\preceq_{i_2} \alpha\Big(\frac{d_{\mathbb{BC}}(S0,T0)d_{\mathbb{BC}}(S1,T1)}{d_{\mathbb{BC}}(S0,S1)}\Big) \\ &\quad + \beta d_{\mathbb{BC}}(S0,S1)+\gamma\big[d_{\mathbb{BC}}(S0,T0)+d_{\mathbb{BC}}(S1,T1)\big] \\ &\quad + \delta\big[d_{\mathbb{BC}}(S0,T1)+d_{\mathbb{BC}}(S1,T0)\big] \\ &=_{i_2} (\beta+\gamma+\delta)d(3,1) \\ &=_{i_2} 2(\beta+\gamma+\delta)(i_1 i_2). \end{aligned}$$

In the same way, for $x=0$, $y=3$, we have $d_{\mathbb{BC}}(Tx,Ty)=0$ and

$$\begin{aligned} d_{\mathbb{BC}}(T0,T3)=0 &\preceq_{i_2} \alpha\Big(\frac{d_{\mathbb{BC}}(S0,T0)d_{\mathbb{BC}}(S3,T3)}{d_{\mathbb{BC}}(S0,S3)}\Big) \\ &\quad + \beta d_{\mathbb{BC}}(S0,S3)+\gamma\big[d_{\mathbb{BC}}(S0,T0)+d_{\mathbb{BC}}(S3,T3)\big] \\ &\quad + \delta\big[d_{\mathbb{BC}}(S0,T3)+d_{\mathbb{BC}}(S3,T0)\big] \\ &=_{i_2} (\beta+\gamma+\delta)d_{\mathbb{BC}}(3,1) \\ &=_{i_2} 2(\beta+\gamma+\delta)(i_1 i_2). \end{aligned}$$

Hence the inequality holds. On the other hand, it is obvious that T is $S-$non-decreasing mapping with respect to $\preceq$ and there exists $x_0 = 3$ such that $Sx_0 \preceq Tx_0$, 1 is a common fixed point of S and T and 3 is a coincidence fixed point of S and T.

As a consequence of Theorem 3.1, we have the following corollaries.

If $S = I$ (the identity mapping) in Theorem 3.1, we have the following result:

Corollary 3.4. *Let $(X,\preceq,d_{\mathbb{BC}})$ be a partially ordered complete bicomplex-valued metric space. Suppose that T is continuous self mapping on X, T is a monotone non-decreasing mapping and*

$$\begin{aligned} d_{\mathbb{BC}}(Tx,Ty) &\preceq_{i_2} \alpha\Big(\frac{d_{\mathbb{BC}}(x,Tx)d(y,Ty)}{d(x,y)}\Big) \\ &\quad + \beta d_{\mathbb{BC}}(x,y)+\gamma\big[d_{\mathbb{BC}}(x,Tx)+d_{\mathbb{BC}}(y,Ty)\big] \\ &\quad + \delta\big[d_{\mathbb{BC}}(x,Ty)+d_{\mathbb{BC}}(y,Tx)\big], \end{aligned}$$

for all $x,y\in X$ with $x\succeq y$, $x\neq y$ and for some $\alpha,\beta,\gamma,\delta\in[0,1)$ with $\alpha+\beta+2\gamma+2\delta<1$. If there exists $x_0\in X$ such that $x_0\preceq Tx_0$, then T has a fixed point.

If $\gamma = \delta = 0$ in Theorem 3.1, we have the following result:

Corollary 3.5. *Let $(X, \preceq, d_{\mathbb{BC}})$ be a partially ordered complete bicomplex-valued metric space. T and S are continuous self mappings on X, $T(X) \subseteq S(X)$, T is a f non-decreasing mapping and*

$$d_{\mathbb{BC}}(Tx, Ty) \quad \preceq_{i_2} \quad \alpha\Big(\frac{d_{\mathbb{BC}}(Sx, Tx) d_{\mathbb{BC}}(Sy, Ty)}{d_{\mathbb{BC}}(fx, fy)}\Big) + \beta d_{\mathbb{BC}}(Sx, Sy),$$

for all $x, y \in X$ with $Sx \succeq Sy$, $Sx \neq Sy$ and for some $\alpha, \beta \in [0, 1)$ with $\alpha + \beta < 1$. If there exists $x_0 \in X$ such that $Sx_0 \preceq Tx_0$, T and S are compatible, then T and S have a coincidence point.

Remark 3.6. Corollary 3.5 extends Theorem 2.1 of [**8**] from metric spaces to bicomplex-valued metric spaces.

If $\beta = 0$ in Corollary 3.5 we have the following result:

Corollary 3.7. *Let $(X, \preceq, d_{\mathbb{BC}})$ be a partially ordered complete bicomplex-valued metric. T and S are continuous self mappings on X, $T(X) \subseteq S(X)$, T is a S non-decreasing mapping and*

$$d_{\mathbb{BC}}(Tx, Ty) \quad \preceq_{i_2} \quad \alpha\Big(\frac{d(Sx, Tx) d_{\mathbb{BC}}(Sy, Ty)}{d_{\mathbb{BC}}(Sx, Sy)}\Big),$$

for all $x, y \in X$ with $Sx \succeq Sy$, $Sx \neq Sy$ and for some $\alpha \in [0, 1)$ with $\alpha < 1$. If there exists $x_0 \in X$ such that $Sx_0 \preceq Tx_0$, T and S are compatible, then T and S have a coincidence point.

Remark 3.8. Corollary 3.7 extends Corollary 2.2 of [**8**] from metric spaces to bicomplex-valued metric spaces.

In what follows, we prove that Theorem 3.1 is still valid for T not necessarily continuous, assuming the following hypothesis in X :

$$\text{if } (x_n) \text{ is a nondecreasing sequence in } X \text{ such that} \\ x_n \to x, \text{ then } x_n \preceq x \text{ for all } n \in \mathbb{N},$$

Theorem 3.9. *Let $(X, \preceq, d_{\mathbb{BC}})$ be a partially ordered complete bicomplex-valued. Suppose that T and S are self mappings on X, $T(X) \subseteq S(X)$, T is a S-monotone non-decreasing mapping and*

$$\begin{aligned} d_{\mathbb{BC}}(Tx, Ty) \preceq_{i_2} \alpha\Big(\frac{d_{\mathbb{BC}}(Sx, Tx) d_{\mathbb{BC}}(Sy, Ty)}{d_{\mathbb{BC}}(Sx, Sy)}\Big) + \beta d_{\mathbb{BC}}(Sx, Sy) \\ + \gamma\big[d_{\mathbb{BC}}(Sx, Tx) + d(Sy, Ty)\big] \\ + \delta\big[d_{\mathbb{BC}}(Sx, Ty) + d_{\mathbb{BC}}(Sy, Tx)\big], \end{aligned}$$

for all $x, y \in X$ with $Sx \succeq Sy$, $Sx \neq Sy$ and for some $\alpha, \beta, \gamma, \delta \in [0,1)$ with $\alpha + \beta + 2\gamma + 2\delta < 1$.

Also assume that $S(X)$ is closed and for any non-decreasing sequence (x_n) in X which converges to x we have $x_n \preceq x$ for all n. If there exists $x_0 \in X$ such that $Sx_0 \preceq Tx_0$, then T and f have a coincidence point.

Further, if T and S are weakly compatible, then T and S have a common fixed point. Moreover, the set of common fixed points of T and S is well ordered if and only if T and S have one and only one common fixed point.

Proof. Following the proof of Theorem 3.1, we have (Tx_n) is a Cauchy sequence and so is (Sx_n). Since $S(X)$ is closed and X is complete,

$$\lim_{n\to+\infty} Tx_n = \lim_{n\to+\infty} Sx_n = Su, \quad \text{for some } u \in X.$$

Notice that the sequences (Tx_n) and (Sx_n) are non-decreasing, then from our assumptions we have

$$Tx_n \preceq Su, \quad \text{and} \quad Sx_n \preceq Su, \quad \forall n \in \mathbb{N}.$$

Keeping in mind that T is S-monotone non-decreasing we get $Tx_n \preceq Tu$ for all n, let $n \to +\infty$, we obtain $Su \preceq Tu$. Suppose $Su \prec Tu$ (otherwise we are done), construct a sequence (u_n) as $u_0 = u$ and $Su_{n+1} = Tu_n$ for all n, a similar argument as in the proof of Theorem 3.1 yields (Su_n) is a non-decreasing sequence and

$$\lim_{n\to+\infty} Tu_n = \lim_{n\to+\infty} Su_n = Sv, \quad \text{for some } v \in X.$$

From our assumptions it follows that

$$\sup Su_n \preceq Sv \quad \text{and} \quad \sup Tu_n \preceq Sv.$$

Notice that

$$Sx_n \preceq Su \preceq Su_1 \preceq \cdots \preceq Su_n \preceq \cdots \preceq Sv,$$

we distinguish two cases:

Case 1: Suppose there is $n_0 \geq 1$ with $Sx_{n_0} = Su_{n_0}$, then $Sx_{n_0} = Su = Su_{n_0} = Su_1 = Tu$, we are done.

Case 2: Suppose $Sx_n \neq Su_n$ for all $n \geq 1$, then from the assumption we obtain

$$\begin{aligned}
d_{\mathbb{BC}}(Sx_{n+1}, Su_{n+1}) &= d_{\mathbb{BC}}(Tx_n, Tu_n) \\
&\preceq_{i_2} \alpha\Big(\frac{d(Sx_n, Tx_n) d_{\mathbb{BC}}(Su_n, Tu_n)}{d_{\mathbb{BC}}(Sx_n, Su_n)}\Big) + \beta d_{\mathbb{BC}}(Sx_n, Su_n) \\
&\quad + \gamma\big[d_{\mathbb{BC}}(Sx_n, Tx_n) + d_{\mathbb{BC}}(Su_n, Tu_n)\big] \\
&\quad + \delta\big[d_{\mathbb{BC}}(Sx_n, Tu_n) + d_{\mathbb{BC}}(Su_n, Tx_n)\big].
\end{aligned}$$

Let $n \to +\infty$, we get $d_{\mathbb{BC}}(Su, Sv) \preceq_{i_2} (\beta + 2\delta) d_{\mathbb{BC}}(Su, Sv)$, which implies that $Su = Sv$ since $\beta + 2\delta < 1$. Hence $Su = Sv = Su_1 = Tu$.

Now suppose that T and f are weakly compatible. Let $w = Tu = Su$. Then $Tw = TSu = STu = Sw$. Consider

$$\begin{aligned} d_{\mathbb{BC}}(T(u), T(w)) \preceq_{i_2} & \alpha \frac{d_{\mathbb{BC}}(Su, Tu) d_{\mathbb{BC}}(Sw, Tw)}{d_{\mathbb{BC}}(Su, Sw)} + \beta d_{\mathbb{BC}}(Su, Sw) \\ & + \gamma [d_{\mathbb{BC}}(Su, Tu) + d_{\mathbb{BC}}(Sw, Tw)] \\ & + \delta [d_{\mathbb{BC}}(Su, Tw) + d_{\mathbb{BC}}(Sw, Tu)] \\ \preceq_{i_2} & \beta d_{\mathbb{BC}}(Tu, Tw) + \delta [d_{\mathbb{BC}}(Tu, Tw) + d_{\mathbb{BC}}(Tw, Tu)] \\ \preceq_{i_2} & (\beta + 2\delta) d_{\mathbb{BC}}(Tu, Tw). \end{aligned}$$

This implies that $d_{\mathbb{BC}}(Tu, Tw) = 0$, as $\beta + 2\delta < 1$. Therefore, $Tw = Sw = w$. Now suppose that the set of common fixed points of T and S is well ordered. We claim that the common fixed point of T and S is unique. Assume on the contrary that $Tu = fSu = u$ and $Tv = Sv = v$ but $u \neq v$. Consider

$$\begin{aligned} d_{\mathbb{BC}}(u, v) &= d_{\mathbb{BC}}(Tu, Tv) \\ &\preceq_{i_2} \alpha \frac{d_{\mathbb{BC}}(Su, Tu) d_{\mathbb{BC}}(Sv, Tv)}{d_{\mathbb{BC}}(Su, Sv)} + \beta d_{\mathbb{BC}}(Su, Sv) \\ &\quad + \gamma [d_{\mathbb{BC}}(Su, Tu) + d_{\mathbb{BC}}(Sv, Tv)] + \delta [d_{\mathbb{BC}}(Su, Tv) + d_{\mathbb{BC}}(Sv, Tu)] \\ &\preceq_{i_2} \beta d_{\mathbb{BC}}(Tu, Tv) + \delta [d_{\mathbb{BC}}(Tu, Tv) + d_{\mathbb{BC}}(Tv, Tu)] \\ &\preceq_{i_2} (\beta + 2\delta) d_{\mathbb{BC}}(Tu, Tv). \end{aligned}$$

This implies that $d_{\mathbb{BC}}(u, v) = 0$, as $\beta + 2\delta < 1$. Hence $u = v$. Conversely, if T and S have only one common fixed point then the set of common fixed points of S and T being a singleton is well ordered, the proof is complete. □

According to Theorem 3.2, we have the following corollaries.
If we take $\gamma = \delta = 0$ in Theorem 3.9 we get the following result:

Corollary 3.10. *Let $(X, \preceq, d_{\mathbb{BC}})$ be a partially ordered complete bicomplex-valued metric space. Suppose that T and S are self mappings on X, $T(X) \subseteq S(X)$, T is a S-monotone non-decreasing mapping and*

$$d_{\mathbb{BC}}(Tx, Ty) \quad \preceq_{i_2} \quad \alpha \Big(\frac{d_{\mathbb{BC}}(Sx, Tx) d_{\mathbb{BC}}(Sy, Ty)}{d_{\mathbb{BC}}(Sx, Sy)} \Big) + \beta d_{\mathbb{BC}}(Sx, Sy),$$

for all $x, y \in X$ with $Sx \succeq Sy$, $Sx \neq Sy$ and for some $\alpha, \beta \in [0, 1)$ with $\alpha + \beta < 1$.

Also assume that $S(X)$ is closed and for any non-decreasing sequence (x_n) in X which converges to x we have $x_n \preceq x$ for all n. If there exists $x_0 \in X$ such that $Sx_0 \preceq Tx_0$, then T and f have a coincidence point.

Further, if T and S are weakly compatible, then T and S have a common fixed point. Moreover, the set of common fixed points of T and S is well ordered if and only if T and S have one and only one common fixed point.

Remark 3.11. Corollary 3.10 extends Theorem 2.3 [**8**] from metric spaces to bicomplex-valued metric spaces.

Corollary 3.12. *Let $(X, \preceq, d_{\mathbb{BC}})$ be a partially ordered complete bicomplex-valued metric space. Suppose that T and S are self mappings on X, $T(X) \subseteq S(X)$, T is a S-monotone non-decreasing mapping and*

$$d_{\mathbb{BC}}(Tx, Ty) \quad \preceq_{i_2} \quad \alpha\Big(\frac{d_{\mathbb{BC}}(Sx, Tx)d_{\mathbb{BC}}(Sy, Ty)}{d_{\mathbb{BC}}(Sx, Sy)}\Big),$$

for all $x, y \in X$ with $Sx \succeq Sy$, $Sx \neq Sy$ and for some $\alpha \in [0, 1)$. Also assume that $S(X)$ is closed and for any non-decreasing sequence (x_n) in X which converges to x we have $x_n \preceq x$ for all n. If there exists $x_0 \in X$ such that $Sx_0 \preceq Tx_0$, then T and f have a coincidence point.

Further, if T and S are weakly compatible, then T and S have a common fixed point. Moreover, the set of common fixed points of T and S is well ordered if and only if T and S have one and only one common fixed point.

Remark 3.13. Corollary 3.12 extends Corollary 2.4 of [**8**] from metric spaces to bicomplex-valued metric spaces.

Theorem 3.14. *Let $(X, \preceq, d_{\mathbb{BC}})$ be a partially ordered complete bicomplex-valued metric space. Let $T : X \to X$ be a continuous and nondecreasing mapping. Suppose there exist nonnegative real numbers α, β and γ with $\alpha+2\beta+2\gamma < 1$ such that for all $x, y \in X$ with $x \preceq y$,*

$$\begin{aligned} d_{\mathbb{BC}}(Tx, Ty) \quad \preceq_{i_2} \quad & \alpha\Big(\frac{d_{\mathbb{BC}}(y, Ty)[1 + d_{\mathbb{BC}}(x, Tx)]}{1 + d_{\mathbb{BC}}(x, y)}\Big) \\ & + \beta\big[d_{\mathbb{BC}}(x, Tx) + d_{\mathbb{BC}}(y, Ty)\big] \\ & + \gamma\big[d_{\mathbb{BC}}(y, Tx) + d_{\mathbb{BC}}(x, Ty)\big]. \end{aligned} \tag{3.1}$$

If there exists $x_0 \in X$ with $x_0 \preceq Tx_0$, then T has a fixed point.

Proof. If $x_0 = Tx_0$, then we have the result. Suppose that $x_0 \prec Tx_0$. Then we construct a sequence (x_n) in X such that

$$x_{n+1} = Tx_n, \quad \text{for every } \; n \geq 0. \tag{3.2}$$

Since T is a nondecreasing mapping, we obtain by induction that

$$x_0 \prec Tx_0 = x_1 \preceq Tx_1 = x_2 \preceq \ldots \preceq Tx_{n-1} = x_n \preceq Tx_n = x_{n+1} \preceq \ldots . \tag{3.3}$$

If there exists some $n_0 \geq 1$ such that $x_{n_0+1} = x_{n_0}$, then from (3.2), $x_{n_0+1} = Tx_{n_0} = x_{n_0}$, that is, x_{n_0} is a fixed point of T and the proof is finished. Then we can suppose that $x_{n+1} \neq x_n$, for all $n \geq 1$, since $x_n \prec x_{n+1}$, for all $n \geq 1$, applying (3.1) we have

$$\begin{aligned} d_{\mathbb{BC}}(Tx_n, Tx_{n+1}) \;\; &\preceq_{i_2} \;\; \alpha\Big(\frac{d_{\mathbb{BC}}(x_{n+1}, Tx_{n+1})[1 + d_{\mathbb{BC}}(x_n, Tx_n)]}{1 + d_{\mathbb{BC}}(x_n, x_{n+1})}\Big) \\ &\quad + \beta\big[d_{\mathbb{BC}}(x_n, Tx_n) + d_{\mathbb{BC}}(x_{n+1}, Tx_{n+1})\big] \\ &\quad + \gamma\big[d_{\mathbb{BC}}(x_{n+1}, Tx_n) + d_{\mathbb{BC}}(x_n, Tx_{n+1})\big], \end{aligned}$$

that is,

$$\begin{aligned} d_{\mathbb{BC}}(x_{n+1}, x_{n+2}) \;\; &\preceq_{i_2} \;\; \alpha\Big(\frac{d_{\mathbb{BC}}(x_{n+1}, x_{n+2})[1 + d_{\mathbb{BC}}(x_n, x_{n+1})]}{1 + d_{\mathbb{BC}}(x_n, x_{n+1})}\Big) \\ &\quad + \beta\big[d_{\mathbb{BC}}(x_n, x_{n+1}) + d_{\mathbb{BC}}(x_{n+1}, x_{n+2})\big] \\ &\quad + \gamma\big[d_{\mathbb{BC}}(x_{n+1}, x_{n+1}) + d_{\mathbb{BC}}(x_n, x_{n+2})\big], \end{aligned}$$

which implies that

$$\begin{aligned} d_{\mathbb{BC}}(x_{n+1}, x_{n+2}) \preceq_{i_2} \; &\alpha\, d_{\mathbb{BC}}(x_{n+1}, x_{n+2}) + \beta\big[d_{\mathbb{BC}}(x_n, x_{n+1}) + d_{\mathbb{BC}}(x_{n+1}, x_{n+2})\big] \\ &+ \gamma\big[d_{\mathbb{BC}}(x_n, x_{n+1}) + d_{\mathbb{BC}}(x_{n+1}, x_{n+2})\big], \end{aligned}$$

that is

$$d_{\mathbb{BC}}(x_{n+1}, x_{n+2}) \preceq_{i_2} \frac{\beta + \gamma}{1 - \alpha - \beta - \gamma} d_{\mathbb{BC}}(x_n, x_{n+1}). \tag{3.4}$$

Now, $\alpha + 2\beta + 2\gamma < 1$ implies that $\frac{\beta+\gamma}{1-\alpha-\beta-\gamma} < 1$, put $\frac{\beta+\gamma}{1-\alpha-\beta-\gamma} = h$, then by repeated application (3.4), we have

$$\begin{aligned} d_{\mathbb{BC}}(x_{n+1}, x_{n+2}) \preceq_{i_2} & \; h d_{\mathbb{BC}}(x_n, x_{n+1}) \\ &\preceq h^2 d_{\mathbb{BC}}(x_{n-1}, x_n) \preceq_{i_2} \ldots \preceq h^{n+1} d_{\mathbb{BC}}(x_0, x_1). \end{aligned} \tag{3.5}$$

For any $m > n$,

$$\begin{aligned} d_{\mathbb{BC}}(x_m, x_n) \;\; &\preceq_{i_2} \;\; d_{\mathbb{BC}}(x_n, x_{n+1}) + d_{\mathbb{BC}}(x_{n+1}, x_{n+2}) \preceq_{i_2} + \ldots + d_{\mathbb{BC}}(x_{m-1}, x_m) \\ &\preceq_{i_2} \;\; [h^n + h^{n+1} + h^{n+2} + \ldots + h^{m-1}] d_{\mathbb{BC}}(x_0, x_1) \\ &\preceq_{i_2} \;\; \tfrac{h^n}{1-h} d_{\mathbb{BC}}(x_0, x_1) \end{aligned} ,$$

that is

$$\| d_{\mathbb{BC}}(x_m, x_n) \| \leq \frac{h^n}{1-h} \| d_{\mathbb{BC}}(x_0, x_1) \| .$$

Taking limit as $n \to +\infty$, we obtain $\| d_{\mathbb{BC}}(x_n, x_m) \| \to 0$, that is $d_{\mathbb{BC}}(x_n, x_m) \to 0$ as $n \to +\infty$, which implies that, (x_n) is a Cauchy sequence, from the completeness of X, there exists $w \in X$ such that

$$x_n \to w \text{ as } n \to +\infty.$$

The continuity of T implies that $Tw = \lim_{n \to +\infty} Tx_n = \lim_{n \to +\infty} x_{n+1} = w$, that is, w is a fixed point of T. □

In our next theorem we relax the continuity assumption of the mapping T in Theorem 3.14 by imposing the following order condition of the complex valued metric space X: if (x_n) is a non-decreasing sequence in X such that $x_n \to x$, then $x_n \preceq x$, for all $n \in \mathbb{N}$.

Theorem 3.15. *Let $(X, \preceq, d_{\mathbb{BC}})$ be a partially ordered complete Bicomplex valued metric space. Assume that if (x_n) is a nondecreasing sequence in X such that $x_n \to x$, then $x_n \preceq x$, for all $n \in \mathbb{N}$. Let $T : X \to X$ be a nondecreasing mapping such that for all $x, y \in X$ with $x \preceq y$,* (3.1) *is satisfied, where the condition on α, β and γ are same as in Theorem* 3.14*. If there exists $x_0 \in X$ with $x_0 \preceq Tx_0$, then T has a fixed point.*

Proof. We take the same sequence (x_n) as in the proof of Theorem 3.14. Arguing like in the proof of Theorem 3.14, we prove that (x_n) is a nondecreasing sequence which satisfies $x_n \to w$ as $n \to +\infty$. Then, by the conditions of the Theorem $x_n \preceq w$, for all $n \in \mathbb{N}$. Applying (3.1), we have

$$\begin{aligned} d_{\mathbb{BC}}(x_{n+1}, Tw) &= d_{\mathbb{BC}}(Tx_n, Tw) \\ &\preceq_{i_2} \alpha\Big(\frac{d_{\mathbb{BC}}(w, Tw)[1 + d_{\mathbb{BC}}(x_n, x_{n+1})]}{1 + d_{\mathbb{BC}}(x_n, w)}\Big) \\ &\quad + \beta\big[d_{\mathbb{BC}}(x_n, x_{n+1}) + d_{\mathbb{BC}}(w, Tw)\big] \\ &\quad + \gamma\big[d_{\mathbb{BC}}(w, x_{n+1}) + d_{\mathbb{BC}}(x_n, Tw)\big]. \end{aligned}$$

Taking the limit as $n \to +\infty$ in the above inequality, we have

$$d(w, Tw) \preceq_{i_2} (\alpha + \beta + \gamma) d(w, Tw),$$

that is

$$\| d(w, Tw) \| \leq (\alpha + \beta + \gamma) \| d(w, Tw) \| .$$

Since $\alpha+\beta+\gamma<1$, it is a contradiction unless $d(w,Tw)=0$, that is, $Tw=w$, that is, w is a fixed point of T. □

Now, we shall prove the uniqueness of the fixed point.

Theorem 3.16. *In addition to the hypotheses of Theorem* 3.14 *(or Theorem* 3.15*), suppose that for every $x,y\in X$, there exists $z\in X$ such that $z\preceq x$ and $z\preceq y$, then T has a unique fixed point.*

Proof. It follows from the Theorem 3.14 (or Theorem 3.15) that the set of fixed points of T is non-empty. We shall show that if v and w are two fixed points of T, that is, if $v=Tv$ and $w=Tw$, then $v=w$. By the assumption, there exists $z_0\in X$ such that $z_0\preceq v$ and $z_0\preceq w$. Then, similarly as in the proof of Theorem 3.14, we define the sequence (z_n) such that

$$z_{n+1}=Tz_n=T^{n+1}z_0,\ n=0,1,2,\dots \tag{3.6}$$

monotonicity of T implies that $T^nz_0=z_n\preceq v=T^nv$ and $T^nz_0=z_n\preceq w=T^nw$, if there exists a positive integer m such that $v=z_m$, then $v=Tv=Tz_n=z_{n+1}$, for all $n\geq m$. Then $z_n\to v$ as $n\to+\infty$. Now we suppose that $v\neq z_n$, for all $n\geq 0$, so $z_n\prec v$, for all $n\geq 0$, applying (3.1), we have

$$\begin{aligned}
d_{\mathbb{BC}}(z_{n+1},v) &= d_{\mathbb{BC}}(Tz_n,Tv)\\
&\preceq_{i_2} \alpha\Big(\frac{d_{\mathbb{BC}}(v,Tv)[1+d_{\mathbb{BC}}(z_n,z_{n+1})]}{1+d_{\mathbb{BC}}(z_n,v)}\Big)\\
&\quad +\beta\big[d_{\mathbb{BC}}(z_n,z_{n+1})+d_{\mathbb{BC}}(v,Tv)\big]\\
&\quad +\gamma\big[d_{\mathbb{BC}}(v,z_{n+1})+d_{\mathbb{BC}}(z_n,Tv)\big]\\
&= \beta d_{\mathbb{BC}}(z_n,z_{n+1})+\gamma\big[d_{\mathbb{BC}}(v,z_{n+1})+d_{\mathbb{BC}}(z_n,v)\big]\\
&\preceq_{i_2} \beta\big[d_{\mathbb{BC}}(z_n,v)+d_{\mathbb{BC}}(v,z_{n+1})\big]+\gamma\big[d_{\mathbb{BC}}(v,z_{n+1})+d_{\mathbb{BC}}(z_n,v)\big],
\end{aligned}$$

which implies that

$$d_{\mathbb{BC}}(z_{n+1},v)\preceq_{i_2}\frac{\beta+\gamma}{1-\beta-\gamma}d_{\mathbb{BC}}(z_n,v).$$

Put $\frac{\beta+\gamma}{1-\beta-\gamma}=k(<1)$. Then it follows that

$$d_{\mathbb{BC}}(z_{n+1},v)\preceq_{i_2} kd_{\mathbb{BC}}(z_n,v)\preceq_{i_2} k^2 d_{\mathbb{BC}}(z_{n-1},v)\preceq_{i_2}\dots\preceq_{i_2} k^{n+1}d(z_0,v),$$

then

$$\parallel d_{\mathbb{BC}}(z_{n+1},v)\parallel\preceq_{i_2} k^{n+1}\parallel d(z_0,v)\parallel .$$

Taking limit as $n \to +\infty$, we obtain $d_{\mathbb{BC}}(z_n, v) \to 0$ as $n \to +\infty$; that is, $z_n \to v$ as $n \to +\infty$. Using a similar argument, we can prove that $z_n \to w$ as $n \to +\infty$ Finally, the uniqueness of the limit implies that $v = w$. Hence T has a unique fixed point. $\square$

Example 3.17. *Let* $X = \{0, \frac{1}{2}, 2\}$. *Partial order* $\preceq$ *is defined as* $x \preceq y$ *iff* $x \geq y$. *Let the Bicomplex valued metric* d *be given as*

$$d_{\mathbb{BC}}(x, y) = \mid x - y \mid i_1 i_2, \quad \forall x, y \in X.$$

Let $T : X \to X$ *be defined as follows*

$$T(0) = 0, \quad T(\frac{1}{2}) = 0, \quad T(2) = \frac{1}{2}.$$

Let $\alpha = \gamma = \frac{1}{8}$ *and* $\beta = \frac{1}{4}$.

Here all the conditions of Theorems 3.1 *and* 3.14 *are satisfied. Additionally, the conditions of Theorem* 3.16 *are also satisfied and it is seen that* 0 *is the unique fixed point of* T.

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