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Some remarks on twists and Euler products in the Selberg class

Abstract. We present a new variant of the method of J. Kaczorowski and A. Perelli for obtaining the converse theorem via Euler products in the Selberg class. A standard method requires proving a weak zero-density estimate for functions from the Selberg class in the whole half-plane $\sigma > 1/2$. Such an estimate is not known in general, even for the classical L-functions of high degrees, to hold. Our modification, while providing the same result, does not need as an ingredient a weak zero-density estimates for functions in the Selberg class.

Keywords. Selberg class, Converse theorem, Twists, Euler products.

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1 - Introduction

In [**3**, Theorem 4] J. Kaczorowski and A. Perelli proved a converse theorem stating that the square of the Riemann zeta is the only function $F \in \mathrm{S}$ of degree two and conductor one with a pole at 1. This converse statement was obtained, among others, by means of [**3**, Theorem 1] giving polynomial shape of Euler factors for function $F \in \mathrm{S}$. To prove [**3**, Theorem 1] one requires a weak-zero density estimate for a function F to hold in the entire half-plane $\sigma > 1/2$. Such a condition is not known to be satisfied in general, even for classical L-functions of high degrees. In this paper we prove Theorem 1 which does not have any assumptions on zero distribution for F, but gives polynomial shape of almost all Euler factors for function F. Then we show how to change the strategy of J. Kaczorowski and A. Perelli to reprove the converse theorem.

We start by recalling definitions of the Selberg class S and the extended Selberg class $\mathrm{S}^{\#}$. We say that a not identically vanishing function F is an element of $\mathrm{S}^{\#}$ iff

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1. F is an absolutely convergent Dirichlet series $F(s) := \sum_{n=1}^{\infty} \frac{a_F(n)}{n^s}$, where $s = \sigma + it$ and $\sigma > 1$;
2. $(s-1)^{m_F} F(s)$ is an entire function of finite order for some integer $m_F \geq 0$;
3. F satisfies a functional equation of type $\Phi_F(s) = \omega \overline{\Phi_F(1-\overline{s})}$, where $|\omega| = 1$ and
$$\Phi_F(s) = \mathrm{Q}^s \prod_{j=1}^{r} \Gamma\left(\lambda_j s + \mu_j\right) F(s),$$
where $\mathrm{Q} > 0$, $r \geq 0$, $\lambda_j > 0$ and $\Re \mu_j \geq 0$.

Moreover, F is an element of S iff F belongs to $\mathrm{S}^{\#}$ and

4. (Ramanujan condition) the Dirichlet coefficients of F satisfy $a_F(n) \ll_{\varepsilon} n^{\varepsilon}$ for every $\varepsilon > 0$;
5. (Euler product) F has the following Euler product expansion
$$F(s) = \prod_{p-\mathrm{prime}} F_p(s), \quad \text{for } \sigma > 1,$$
where for each prime p we have
$$\log F_p(s) = \sum_{m=0}^{\infty} \frac{b_F\left(p^m\right)}{p^{ms}}, \quad \text{with } b_F\left(p^m\right) \ll p^{\theta m} \text{ for a certain } \theta < 1/2.$$

For a function $F \in \mathrm{S}^{\#}$ and a real number α we denote by $F(s,\alpha)$ the additive twist of F, i. e.
$$F\left(s,\alpha\right) := \sum_{n=1}^{\infty} \frac{a_F(n) e(-n\alpha)}{n^s}, \quad \sigma > 1,$$
where $e(x) = \mathrm{e}^{2\pi i x}$. Moreover, for a Dirichlet character χ, we denote by $F(s,\chi)$ the corresponding multiplicative twist,
$$F\left(s,\chi\right) := \sum_{n=1}^{\infty} \frac{a_F(n)\chi(n)}{n^s}, \quad \sigma > 1.$$

For the sake of clarity we recall the definition of the class $\mathrm{M}(d,h)$ from **[3]**. We say that $f \in \mathrm{M}(d,h)$ iff:

- $f(\sigma + it)$ is meromorphic over $\mathbb{C}$ and holomorphic for $\sigma < 1$;

- for every $A < B$ there exists a constant $C = C(A,B)$ such that

$$f(\sigma+it) \ll_{d,h,A,B,f} |\sigma|^{d|\sigma|} \left(\frac{h}{(2\pi\mathrm{e})^d}\right)^{|\sigma|} |\sigma|^C$$

  as $\sigma \to -\infty$ uniformly for $A \le t \le B$.

Let us recall that the degree and the conductor of $F \in \mathrm{S}^\#$ are defined by

$$d_F := \sum_{j=1}^{r} \lambda_j, \qquad q_F := (2\pi)^{d_F}\, \mathrm{Q}^2 \prod_{j=1}^{r} \lambda_j^{2\lambda_j},$$

respectively, and are well-known invariants of F (see [**2**]). Moreover, $\mathrm{S}_\mathrm{d}^\#$ and S_d denote subclasses of $\mathrm{S}^\#$ and S consisting of functions of degree d.

The following Theorem is a modified version of [**3**, Theorem 1].

Theorem 1. *Let $d > 0$ and $F \in \mathrm{S}_\mathrm{d}$. Moreover, let p be a prime number and $h > 0$. Then there exists a constant $\varkappa(F) > 0$ such that the following statements are equivalent*

1. *for every $a \bmod p$, $p \nmid a$ and every $p > \varkappa(F)$ the twist $F(s, a/p)$ belongs to $\mathrm{M}(d_F, h)$;*
2. *for every $\chi(\bmod p)$, $\chi \neq \chi_0$ and every $p > \varkappa(F)$ the twist $F(s,\chi)$ belongs to $\mathrm{M}(d_F, h)$ and moreover*

$$F_p(s) = \prod_{j=1}^{\partial_p} \left(1 - \frac{\alpha_j(p)}{p^s}\right)^{-1} \quad \text{with } |\alpha_j(p)| \le 1 \text{ and } \partial_p \le \left\lfloor \frac{\log(h/q_F)}{\log p} \right\rfloor.$$

Remark 1. For [**3**, Theorem 1] to hold, $F \in \mathrm{S}$ needs to satisfy a week zero-density estimate, namely $\mathrm{N}_F(\sigma,T) = \mathrm{o}(T)$ for every fixed $\sigma > 1/2$, where $\mathrm{N}_F(\sigma,T) := |\{F(\beta+i\gamma) = 0 \mid \beta > \sigma,\ |\gamma| \le T\}|$. Under this condition [**3**, Theorem 1] implies that for a function F and every prime number p, p-th Euler factor of F is of a polynomial type. Theorem 1 does not require any assumptions on the distribution of zeros of F, alas it guaranties that only for $p > \varkappa(F)$, p-th Euler of F is of a polynomial type.

Remark 2. Axiom (5) of S impose the restriction $\theta < 1/2$, yet, for Theorem 1 to hold, it is sufficient to have $\theta < 1$.

With the theorem above we reprove the following converse theorem.

Theorem 2 ([**3**, Theorem 4]). *Let $F \in \mathrm{S}_2$ with $q_F = 1$ has a pole at $s = 1$. Then $F(s) = \zeta^2(s)$.*

2 - Proofs of theorems

Following [**3**], for $F \in \mathrm{S}^\#$ we put

$$\tau_F := \max_{1\le j\le r} \left| \frac{\Im \mu_j}{\lambda_j} \right|.$$

Using criteria described in [**1**,**2**] one checks that the above number is an invariant of F.

In the proof of the Theorem 1 we use the following two lemmas.

Lemma 1 ([**3**], Lemma 2.1). *Let $F \in \mathrm{S}^\#_{\mathrm{d}}$ with $d_F > 0$; then $F \in \mathrm{M}(d_F, q_F)$. Moreover, if $[A,B] \cap [-\tau_F, \tau_F] = \emptyset$ and $\sigma > 1$ we also have*

$$F(-\sigma+it) \gg \sigma^{d\sigma}\left(\frac{q_F}{(2\pi \mathrm{e})^d}\right)\sigma^C$$

for some $C = C(A,B)$, uniformly as $\sigma \to \infty$.

Lemma 2 ([**3**], Lemma 2.2). *Let $F \in \mathrm{S}_{\mathrm{d}}$ with $d > 0$. Let p be a prime number, $\sigma > 1$ and $\tau(\chi)$ denote the Gauss sum of a Dirichlet character χ, $\chi \neq \chi_0$, we have*

$$F(s,\chi) = \frac{1}{\tau(\chi)}\sum_{a=1}^{p} \overline{\chi}(a) F\left(s, -{}^{a}\!/\!{}_{p}\right) \tag{2.1}$$

while for any $(a,p) = 1$ we have

$$F\left(s, -{}^{a}\!/\!{}_{p}\right) = \frac{1}{p-1}\sum_{\substack{\chi (\mathrm{mod} p)\\ \chi\neq\chi_0}} \chi(a)\tau(\overline{\chi})F(s,\chi) - \left(\frac{p}{p-1}\frac{1}{F_p(s)} - 1\right)F(s). \tag{2.2}$$

Proof of Theorem 1. In order to prove implication (2) ⇒ (1), one follows unmodified strategy of [**3**].

To prove (1) ⇒ (2) we observe first that by (2.1) and our assumptions we have, that $F(s,\chi)$ is meromorphic on $\mathbb{C}$ and holomorphic for $\sigma < 1$, provided $\chi \neq \chi_0$. Moreover, for such characters we have

$$F(\sigma+it,\chi) \ll \max_{a (\mathrm{mod}) p} \left|F\left(\sigma+it, -{}^{a}\!/\!{}_{p}\right)\right| \ll |\sigma|^{d_F|\sigma|}\left(\frac{h}{(2\pi\mathrm{e})^{d_F}}\right)^{|\sigma|}|\sigma|^C$$

uniformly for $A \le t \le B$ as $\sigma \to -\infty$, for some constant $C = C(A,B)$. Thus $F(s,\chi)$ belongs to $\mathrm{M}(d_F, h)$.

Since the Dirichlet series expansion of $\log F_p(s)$, provided by axiom (5) of the Selberg class, is absolutely convergent for $\sigma > \theta$, the function $1/F_p$ is holomorphic on a half-plane where $\sigma > \theta$. As remarked above (cf. *Remark* 2) we need only $\theta < 1$. By (2.2) we have

(2.3)
$$\frac{1}{F_p(s)} = \frac{p-1}{p}\left(1+\frac{1}{F(s)}\left(F\left(s,-a/p\right)-\frac{1}{p-1}\sum_{\substack{\chi(\mathrm{mod}p)\\ \chi\neq\chi_0}}\chi(a)\tau\left(\overline{\chi}\right)F(s,\chi)\right)\right)$$

which gives us a meromorphic continuation of $1/F_p$ to $\mathbb{C}$.

Since for $\sigma < 1$ both $F\left(s,-a/p\right)$ and $F(s,\chi)$ are holomorphic and moreover, for $\sigma < 0$ and $|t| > \tau_F$ the function F is non-vanishing, hence by (2.3) the function $1/F_p$ is also holomorphic in the region $\sigma < 0$, $|t| > \tau_F$. By $2\pi i/\log p$-periodicity of $1/F_p$ we conclude that $1/F_p$ is holomorphic in the entire half-plane $\sigma < 0$.

In the vertical strip $0 \le \sigma \le \theta < 1$ both $F\left(s,-a/p\right)$ and $F(s,\chi)$ are holomorphic, thus the singularities of $1/F_p$ may come only from the non-trivial zeros of F there. Let $\gamma_n(F) > 0$ denote the n^{th} ordinate of the non-trivial zero of F on the upper half-plane. Then we put

$$Z(F) := \sup_n \left|\gamma_{n+1}(F)-\gamma_n(F)\right|.$$

With this notation there exists rectangle $\mathbf{R}$ with vertices at the points iT_0, $1+iT_0$, $1+i(T_0+Z(F))$ and $i(T_0+Z(F))$, without non-trivial zeros of F and hence without singularities of $1/F_p$. We set

$$\varkappa(F) = \mathrm{e}^{\frac{2\pi}{Z(F)}}.$$

Then for $p > \varkappa(F)$ we have $2\pi/\log p < Z(F)$. Therefore for by $2\pi i/\log p$-periodicity of $1/F_p$ we infer that the function $1/F_p$ is holomorphic inside the vertical strip $0 \le \sigma \le \theta < 1$, and consequently it is an entire function.

We prove

$$F_p(s) = \prod_{j=1}^{\partial_p}\left(1-\frac{\alpha_j(p)}{p^s}\right)^{-1} \quad \text{with } |\alpha_j(p)| \le 1 \text{ and } \partial_p \le \left\lfloor \frac{\log\left(h/q_F\right)}{\log p}\right\rfloor$$

in the same way as in [**3**, Theorem 1] and the result follows. □

Remark 3. Observe that if

$$Z(F) > \frac{2\pi}{\log 2} \approx 9.06472028$$

then Theorem 1 holds for all primes p (cf. Remark 1).

Proof of Theorem 2. We need following results.

Lemma 3 ([**3**, Theorem 3]). *Let* $F \in \mathrm{S}_2^{\#}$ *with* $q_F = 1$. *Then for every* $q \geq 1$ *and* $1 \leq a \leq q$ *with* $(a, q) = 1$ *the linear twist* $F(s, a/q)$ *belongs to* $\mathrm{M}\left(2, q^2\right)$.

Following the strategy of [**3**], from Theorem 1 and Theorem 3 we immediately obtain the following

Corollary ([**3**, cf. Corollary]). *Let* $F \in \mathrm{S}_2$ *with* $q_F = 1$. *Then there exists a number* $\varkappa(F) > 0$ *such that for every prime* $p > \varkappa(F)$ *and every* $\chi(\mathrm{mod} p)$, $\chi \neq \chi_0$, *the twist* $F(s, \chi)$ *belongs to* $\mathrm{M}\left(2, p^2\right)$ *and*

$$F_p(s) = \prod_{j=1}^{\partial_p} \left(1 - \frac{\alpha_j(p)}{p^s}\right)^{-1} \quad \text{with } |\alpha_j(p)| \leq 1 \text{ and } \partial_p \leq 2.$$

We also need the following form of the strong multiplicity one property for the Selberg class.

Lemma 4 ([**4**]). *Let* $F, G \in \mathrm{S}$. *If* $F_p = G_p$ *for almost all primes* p, *then* $F = G$.

Essentially to prove Theorem 2 one follows the complete argument of [**3**] with a minor addendum at the end.

Since Corollary above holds only for $p > \varkappa(F)$, following the proof of [**3**, Theorem 4], under the assumptions of Theorem 2 we have

$$F_p(s) = \left(1 - \frac{1}{p^s}\right)^{-2}, \quad \text{for } p > \varkappa(F).$$

Hence the Euler factor F_p agrees with that of the square of the Riemann zeta function for almost all primes p. Thus by Lemma 4 we have that

$$F(s) = \zeta^2(s).$$

□

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