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Characterizing a surface by invariants

Abstract. Canonical principal parameters are introduced for surfaces in $\mathbb{R}^3$ without umbilical points. It is proved that in these parameters, the surface is determined (up to position in space) by a pair of invariants satisfying a partial differential equation equivalent to the Gauss equation. The principal curvatures or the Gauss and the mean curvature may be used as such a pair of invariants.

Keywords. Surfaces, equations of Gauss and Codazzi, canonical principal parameters.

Mathematics Subject Classification: 53A05.

1 - Introduction

An important problem in differential geometry is to characterize a geometric object by its invariants. For example, it is well known that any curve in $\mathbb{R}^3$ is determined (up to position in space) by its curvature and torsion as functions of its natural parameter.

Until now, no similar theorem for surfaces in $\mathbb{R}^3$ has been known. A standard characterization of a surface is given by Bonnet's classical theorem, according to which the surface is determined (up to position in space) by six functions – the coefficients of the first and the second fundamental forms satisfying the equations of Gauss and Codazzi. However, the coefficients of the fundamental forms are not invariant functions, unlike the curvature and torsion of a curve, although these coefficients remain unchanged in motions. Nevertheless, as we shall see, the above mentioned Bonnet's theorem can help us study the determination of a surface by invariants.

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Investigations of surfaces by their invariants have a long history. The determination of surfaces by their first fundamental form and principal curvatures was first studied by Bonnet [**1**]. There are several subsequent works in this direction, see e.g. Cartan [**4**] and S.-S. Chern [**5**]. Finikoff and Gambier [**8**] and recently Bryant [**2**] have studied surfaces in view of their curvature lines and principal curvatures, or their shape operator. In all of the above studies, **entire families** of surfaces with the considered properties appear. So we have to specify the conditions to obtain a characterization of surfaces analogous to the above mentioned characterization of curves.

Note that some differential equations between the invariants of the surfaces arise in a natural way as a result of the Gauss and Codazzi equations. The so-called Lund-Regge problem here is to find the minimum possible invariants and relations between them that characterize a surface, see [**12**], [**13**]. When trying to reduce the number of invariants and the compatibility conditions involved, it is common to search for special parameters, just as in the case of curves and their natural parameters.

New results in the above directions have been obtained in [**10**] – a work that has actually inspired the present paper. More precisely, in [**10**] it is proved that a strongly regular surface is determined (up to position in space) by **four** invariants – the principal curvatures ν_1, ν_2, and the geodesic curvatures γ_1, γ_2 of the principal lines. These invariants satisfy **three** partial differential equations equivalent to the Gauss and Codazzi equations. In particular, for the class of Weingarten surfaces, the authors use special parameters that they call *geometric* and they prove that in these parameters the surface is determined by three functions, one of which is invariant and the two other determine the Weingarten nature of the surface. These three functions are closely related to the principal curvatures and are subject to a single partial differential equation equivalent to the Gauss equation.

In the present paper, we introduce canonical parameters for an arbitrary surface in $\mathbb{R}^3$ without umbilical points and we prove that in these parameters the surface is locally determined up to position in space by just **two** invariant functions related by just **one** partial differential equation equivalent to the Gauss equation. In Theorem 3.1 these two invariants are the principal curvatures and in Theorem 3.2 – the Gauss curvature and the mean curvature. It is clear that the surface cannot be determined by just one of these invariant functions – for example there exist many surfaces with the same constant Gauss or mean curvature. For example, Bonnet [**1**] proved that a surface of constant mean curvature can be isometrically deformed, preserving the mean curvature. So it appears that our results solve the Lund-Regge problem for surfaces without umbilical points.

In the last section, some particular cases are considered and relations with some results in [**6**], [**10**] and [**16**] are shown.

For similar investigations about surfaces in upper dimensional spaces of constant curvature c, we refer the reader e.g. to [**14**], where some special isothermal parameters are used in the case of **minimal** non-superconformal surfaces in $\mathbb{Q}^4_c$ and it is proved that the surface is determined by its Gauss and normal curvatures, which satisfy a system of two partial differential equations; see also [**9**].

2 - Preliminaries

Let a regular surface in $\mathbb{R}^3$ be given by the parametric equation $S : x = x(u,v)$. We denote by E, F, G, resp. L, M, N the coefficients of the first, resp. the second fundamental form. A point of S is called *umbilical* if the two fundamental forms are proportional at that point. The Gauss curvature K and the mean curvature H of S, which are the most important invariants of the surface, are expressed with these coefficients respectively by

$$K = \frac{LN - M^2}{EG - F^2} \qquad\qquad H = \frac{EN - 2FM + GL}{2(EG - F^2)} \ .$$

Moreover, the coefficients of the two fundamental forms satisfy Gauss's equation

$$K = -\frac{1}{2W}\left\{\left(\frac{E_v - F_u}{W}\right)_v + \left(\frac{G_u - F_v}{W}\right)_u\right\} - \frac{1}{4W^4}\begin{vmatrix} E & F & G \\ E_u & F_u & G_u \\ E_v & F_v & G_v \end{vmatrix}$$

and Codazzi's equations

$$2W^2(L_v - M_u) = (EN - 2FM + GL)(E_v - F_u) + \begin{vmatrix} E & F & G \\ L & M & N \\ E_u & F_u & G_u \end{vmatrix} ,$$

$$2W^2(M_v - N_u) = (EN - 2FM + GL)(F_v - G_u) + \begin{vmatrix} E & F & G \\ L & M & N \\ E_v & F_v & G_v \end{vmatrix} ,$$

where $W = \sqrt{EG - F^2}$. The classical theorem of Bonnet [**1**] states that conversely, given six functions E, F, G, L, M, N ($E > 0$, $EG - F^2 > 0$) satisfying these three equations, then locally there exists a unique (up to position in space) surface, having E, F, G as coefficients of its first fundamental form and L, M, N as coefficients of its second fundamental form; see also e.g. [**3**], p. 236.

Suppose that a curve c on S is defined by

$$c\ :\quad u = u(s)\ ,\quad v = v(s)\ ,$$

where s is the natural parameter of c. Then the Frenet formulas are

$$\begin{aligned} t' &= \gamma p + \nu l \\ p' &= -\gamma t + \alpha l \\ l' &= -\nu t - \alpha p \end{aligned}$$

where t is the unit tangent vector field of c, l is the unit normal vector field of S, and $p = l \times t$. The functions γ, ν, α are the geodesic curvature, the normal curvature and the geodesic torsion of c on S, respectively. The normal curvature of c is given by

$$\nu = \frac{L\dot{u}^2 + 2M\dot{u}\,\dot{v} + N\dot{v}^2}{E\dot{u}^2 + 2F\dot{u}\,\dot{v} + G\dot{v}^2}\ .$$

Actually, at each point of c the normal curvature ν depends not on the curve itself, but on the direction of its tangent vector at that point, so we speak about a normal curvature of a direction at any point. The maximal and the minimal values of the normal curvatures at a point are called *principal curvatures* and the corresponding directions and vectors – *principal directions* and *principal vectors*. A curve on S is called *principal* if its tangent vector is principal at any point. When the surface has no umbilical points, the parameters (u, v) can be chosen (at least locally) such that the parametric lines are principal. Then the parameters (u, v) of S are called *principal*. In terms of the coefficients of the fundamental forms, this means that $F = M = 0$ on S. In this case, the geodesic torsions of the parametric lines vanish identically and the geodesic curvatures of the parametric lines are

$$\gamma_1 = -\frac{E_v}{2E\sqrt{G}}\ ,\qquad \gamma_2 = \frac{G_u}{2G\sqrt{E}}\ . \tag{2.1}$$

Let ν_1 and ν_2 be the principal curvatures of S. Then the classical definition of the Gauss curvature and the mean curvature becomes

$$K = \nu_1\nu_2\ ,\qquad H = \frac{1}{2}(\nu_1 + \nu_2)\ . \tag{2.2}$$

3 - Determining non-umbilical surfaces

Suppose that S has no umbilical points and the parametric lines are principal, i.e. $F = M = 0$ on S. Then Gauss's equation is

$$K = \nu_1\nu_2 = -\frac{1}{2\sqrt{EG}}\left\{\left(\frac{E_v}{\sqrt{EG}}\right)_v + \left(\frac{G_u}{\sqrt{EG}}\right)_u\right\} \tag{3.1}$$

and Codazzi's equations take the form

$$2EGL_v = (EN + GL)E_v\ , \qquad 2EGN_u = (EN + GL)G_u\ .$$

On the other hand, the principal curvatures ν_1, ν_2 are given by

$$\nu_1 = \frac{L}{E}\ , \qquad \nu_2 = \frac{N}{G}\ . \tag{3.2}$$

Since the surface has no umbilical points, the difference $\nu_1 - \nu_2$ cannot vanish. Hence it is easy to see that Codazzi's equations may be written as

$$\frac{E_v}{2E} = -\frac{(\nu_1)_v}{\nu_1 - \nu_2}\ , \qquad \frac{G_u}{2G} = \frac{(\nu_2)_u}{\nu_1 - \nu_2}\ . \tag{3.3}$$

Let us fix a point (u_0, v_0). The last equations imply that there exist two functions $\varphi_1(u)$ and $\varphi_2(v)$, such that

$$\sqrt{E} = \varphi_1(u) e^{-\int_{v_0}^{v} \frac{(\nu_1)_v}{\nu_1 - \nu_2} dv}\ , \qquad \sqrt{G} = \varphi_2(v) e^{\int_{u_0}^{u} \frac{(\nu_2)_u}{\nu_1 - \nu_2} du}\ .$$

In other words, for any functions $\phi_1(u)$, $\phi_2(v)$, the function

$$\phi_1(u)\sqrt{E}\, e^{\int_{v_0}^{v} \frac{(\nu_1)_v}{\nu_1 - \nu_2} dv}$$

does not depend on v and the function

$$\phi_2(v)\sqrt{G}\, e^{-\int_{u_0}^{u} \frac{(\nu_2)_u}{\nu_1 - \nu_2} du}$$

does not depend on u. Now we introduce new parameters $(\bar{u}, \bar{v})$ by the formulas

$$\overline{u} = \frac{1}{\sqrt{E(u_0, v_0)}} \int_{u_0}^{u} \sqrt{E}\, e^{\int_{v_0}^{v} \frac{(\nu_1)_v}{\nu_1 - \nu_2} dv + \int_{u_0}^{u} \frac{(\nu_1)_u}{\nu_1 - \nu_2}(u, v_0) du} du + \overline{u}_0$$

$$\overline{v} = \frac{1}{\sqrt{G(u_0, v_0)}} \int_{v_0}^{v} \sqrt{G}\, e^{-\int_{u_0}^{u} \frac{(\nu_2)_u}{\nu_1 - \nu_2} du - \int_{v_0}^{v} \frac{(\nu_2)_v}{\nu_1 - \nu_2}(u_0, v) dv} dv + \overline{v}_0$$

for some constants $\overline{u}_0$, $\overline{v}_0$. The parameters $(\overline{u}, \overline{v})$ are principal too, since $\bar{u} = \bar{u}(u)$, $\bar{v} = \bar{v}(v)$. Moreover we have

$$\begin{aligned} &\frac{\sqrt{\overline{E}}}{\sqrt{\overline{E}(\overline{u}_0, \overline{v}_0)}}\, e^{\int_{\overline{v}_0}^{\overline{v}} \frac{(\overline{\nu}_1)_{\overline{v}}}{\overline{\nu}_1 - \overline{\nu}_2} d\overline{v} + \int_{\overline{u}_0}^{\overline{u}} \frac{(\overline{\nu}_1)_{\overline{u}}}{\overline{\nu}_1 - \overline{\nu}_2}(\overline{u}, \overline{v}_0) d\overline{u}} = 1 \\ &\frac{\sqrt{\overline{G}}}{\sqrt{\overline{G}(\overline{u}_0, \overline{v}_0)}}\, e^{-\int_{\overline{u}_0}^{\overline{u}} \frac{(\overline{\nu}_2)_{\overline{u}}}{\overline{\nu}_1 - \overline{\nu}_2} d\overline{u} - \int_{\overline{v}_0}^{\overline{v}} \frac{(\overline{\nu}_2)_{\overline{v}}}{\overline{\nu}_1 - \overline{\nu}_2}(\overline{u}_0, \overline{v}) d\overline{v}} = 1\,. \end{aligned} \tag{3.4}$$

We shall call *canonical principal parameters* any principal parameters $(\overline{u},\overline{v})$ satisfying (3.4) for certain constants $(\overline{u}_0,\overline{v}_0)$. For Weingarten surfaces, these parameters were found by Weingarten [**15**], see also [**7**], p. 292, and are called in [**10**] *geometric principal parameters.*

We can easily see that if (u,v) are also canonical principal parameters, then

$$\begin{aligned}\overline{u}&=\lambda\,u+c_1\\ \overline{v}&=\mu\,v+c_2\end{aligned}\qquad\text{or}\qquad\begin{aligned}\overline{u}&=\lambda\,v+c_1\\ \overline{v}&=\mu\,u+c_2\end{aligned}$$

for some constants λ, μ, c_1, c_2 ($\lambda\neq 0$, $\mu\neq 0$). More precisely, if for example $\overline{u}_0=\overline{u}(u_0)$, $\overline{v}_0=\overline{v}(v_0)$, then

$$\begin{aligned}(\overline{u}-\overline{u}_0)\sqrt{\overline{E}(\overline{u}_0,\overline{v}_0)}&=\pm(u-u_0)\sqrt{E(u_0,v_0)}\\ (\overline{v}-\overline{v}_0)\sqrt{\overline{G}(\overline{u}_0,\overline{v}_0)}&=\pm(v-v_0)\sqrt{G(u_0,v_0)}\,.\end{aligned}$$

In the following, we assume that the surface is parametrized with canonical principal parameters (u,v). Then the coefficients E and G of the first fundamental form satisfy

$$\begin{aligned}E&=a\,e^{-2\int_{v_0}^{v}\frac{(\nu_1)_v}{\nu_1-\nu_2}dv-2\int_{u_0}^{u}\frac{(\nu_1)_u}{\nu_1-\nu_2}(u,v_0)du}\\ G&=b\,e^{2\int_{u_0}^{u}\frac{(\nu_2)_u}{\nu_1-\nu_2}du+2\int_{v_0}^{v}\frac{(\nu_2)_v}{\nu_1-\nu_2}(u_0,v)dv}\end{aligned}\tag{3.5}$$

where $a=E(u_0,v_0)$, $b=G(u_0,v_0)$. In this case the Gauss equation (3.1) can be written in the following equivalent form

$$\nu_1\nu_2\Psi_1\Psi_2=\frac{1}{b}\left(\frac{(\nu_1)_v}{\nu_1-\nu_2}\,\frac{\Psi_1}{\Psi_2}\right)_v-\frac{1}{a}\left(\frac{(\nu_2)_u}{\nu_1-\nu_2}\,\frac{\Psi_2}{\Psi_1}\right)_u\tag{3.6}$$

where the functions Ψ_1 and Ψ_2 are defined by

$$\begin{aligned}\Psi_1&=e^{-\int_{v_0}^{v}\frac{(\nu_1)_v}{\nu_1-\nu_2}dv-\int_{u_0}^{u}\frac{(\nu_1)_u}{\nu_1-\nu_2}(u,v_0)du}\\ \Psi_2&=e^{\int_{u_0}^{u}\frac{(\nu_2)_u}{\nu_1-\nu_2}du+\int_{v_0}^{v}\frac{(\nu_2)_v}{\nu_1-\nu_2}(u_0,v)dv}\,.\end{aligned}\tag{3.7}$$

Conversely, consider two differentiable functions ν_1, ν_2 that satisfy the equation (3.6) for some positive constants a, b, the functions Ψ_i being defined by (3.7) (of course we suppose that the difference $\nu_1-\nu_2$ never vanishes). With these functions ν_1, ν_2 we define E and G by (3.5) and after that, L and N by (3.2). Note that as a consequence of the definition (3.5) the equations (3.3) (which are equivalent to the Codazzi equations) are satisfied. Then using Bonnet's theorem we obtain:

Theorem 3.1. *Let there be given two differentiable functions $\nu_1(u,v)$, $\nu_2(u,v)$, such that $\nu_1 - \nu_2$ never vanishes. Define Ψ_1, Ψ_2 by* (3.7) *and suppose that* (3.6) *is satisfied for some positive constants a and b. Then locally there exists a unique (up to position in space) surface, such that ν_1 and ν_2 are its principal curvatures in canonical principal parameters. For this surface $E(u_0,v_0) = a$, $G(u_0,v_0) = b$.*

Note that the integrability condition (3.6) (which is a form of the Gauss equation) is expressed only by the two invariants ν_1 and ν_2 – the principal curvature functions of the surface in canonical principal parameters.

Note also that the above theorem and the Gauss integrability equation (3.6) can be put in a different form in terms of the Gauss curvature and the normal curvature instead of the principal curvatures ν_1, ν_2. Indeed, according to (2.2) we have (supposing $\nu_1 > \nu_2$)

$$\nu_1 = H + \sqrt{H^2 - K}\ , \qquad \nu_2 = H - \sqrt{H^2 - K}\ .$$

In this case, the condition that $\nu_1 - \nu_2$ never vanishes is replaced by the condition that $H^2 - K$ never vanishes. As a result, the surface is determined up to position in space by its Gauss and mean curvature. More precisely, let as define

$$\Phi_1 = e^{-\displaystyle\int_{v_0}^{v} \frac{H_v}{2\sqrt{H^2-K}}dv - \int_{u_0}^{u} \frac{H_u}{2\sqrt{H^2-K}}(u,v_0)du} \tag{3.8}$$

$$\Phi_2 = e^{\displaystyle\int_{u_0}^{u} \frac{H_u}{2\sqrt{H^2-K}}du + \int_{v_0}^{v} \frac{H_v}{2\sqrt{H^2-K}}(u_0,v)dv}\ .$$

Then we can see that

$$\Psi_1 = \frac{\sqrt[4]{(H^2-K)(u_0,v_0)}}{\sqrt[4]{H^2-K}}\Phi_1\ , \qquad \Psi_2 = \frac{\sqrt[4]{(H^2-K)(u_0,v_0)}}{\sqrt[4]{H^2-K}}\Phi_2\ .$$

Substituting these in (3.6) and using Theorem 3.1 we obtain

Theorem 3.2. *Let $K(u,v)$, $H(u,v)$ be differentiable functions such that $H^2 - K$ never vanishes and define Φ_1, Φ_2 by* (3.8). *Suppose that the equation*

$$\frac{2K}{\sqrt{H^2-K}}\Phi_1\Phi_2 = \frac{1}{b}\left(\frac{\Phi_1}{\Phi_2}\frac{\left(H+\sqrt{H^2-K}\right)_v}{\sqrt{H^2-K}}\right)_v - \frac{1}{a}\left(\frac{\Phi_2}{\Phi_1}\frac{\left(H-\sqrt{H^2-K}\right)_u}{\sqrt{H^2-K}}\right)_u$$

is satisfied for some positive constants a, b. Then locally there exists a unique (up to position in space) surface, such that K and H are respectively its Gauss curvature and mean curvature in canonical principal parameters. For this surface $\big(E\sqrt{(H^2-K)}\big)(u_0,v_0)=a$, $\big(G\sqrt{(H^2-K)}\big)(u_0,v_0)=b$.

Having two functions ν_1, ν_2 satisfying the conditions of Theorem 3.1 (or, which is the same, two functions K, H satisfying the conditions of Theorem 3.2), we determine the coefficients E, G of the first fundamental form of the induced surface S by (3.5). Now we can find the geodesic curvatures γ_1, γ_2 of the principal lines of the surface using (2.1). A geometric method to construct the surface with invariants ν_1, ν_2, γ_1, γ_2 is obtained in [**10**].

4 - Particular cases

The surface $S\ :\ x=x(u,v),\ (u,v)\in D$ is called *strongly regular Weingarten surface* (see [**10**]) if there exist two differentiable functions $f(t)$, $g(t)$ defined on an interval I and a function $\nu(u,v)$ defined on D, such that

$$f(t)-g(t)>0\,,\qquad f'(t)g'(t)\neq 0\,,\quad t\in I\,, \tag{4.1}$$

$$\nu_u(u,v)\nu_v(u,v)\neq 0,\quad (u,v)\in D\,, \tag{4.2}$$

$$\nu_1=f(\nu)\,,\quad \nu_2=g(\nu)\,. \tag{4.3}$$

Theorem 3.1 implies that given three functions $f(t)$, $g(t)$, $\nu(u,v)$ with the properties (4.1), (4.2) and satisfying the equation

$$\begin{aligned} &A\left\{f'\nu_{vv}+\left(f''-\frac{2f'^2}{f-g}\right)\nu_v^2\right\}e^{2\int_{\nu_0}^{\nu}\frac{g'dt}{g-f}} \\ &-B\left\{g'\nu_{uu}+\left(g''+\frac{2g'^2}{f-g}\right)\nu_u^2\right\}e^{2\int_{\nu_0}^{\nu}\frac{f'dt}{f-g}} \quad = \quad fg(f-g) \end{aligned} \tag{4.4}$$

for two positive constants A, B and $\nu_0=\nu(u_0,v_0)$ for $(u_0,v_0)\in D$, then there exists a unique (up to position in space) Weingarten surface S with principal curvatures in canonical principal parameters given by (4.3). This is one of the main results in [**10**]. Note again that in this case our canonical principal parameters coincide with the geometric principal parameters defined in [**10**].

For the form of the Gauss equation (4.4) for some important subclasses of Weingarten surfaces, e.g. surfaces of constant mean curvature, see [**10**], where the principal curvatures are used.

It is more interesting to consider the surfaces of constant mean curvature H from another point of view. Namely, according to Theorem 3.2 such a surface is uniquely determined by its Gauss curvature. More precisely, Theorem 3.2 (with $a = b = 1$) implies that for a real number H and a differentiable function $K = K(x, y)$ satisfying $K < H^2$ and the differential equation

$$\Delta(\log(H^2 - K)) = \frac{4K}{\sqrt{H^2 - K}} \ , \tag{4.5}$$

where Δ is the Laplace operator, there exists a unique (up to position) surface with Gauss curvature K and constant mean curvature H. Conversely, the Gauss curvature $K(x, y)$ of any surface of constant mean curvature H gives a solution of the above partial differential equation. Note that the equation (4.5) is exactly equation (6.2) from [**16**]:

$$\Delta\lambda = 2(e^{-\lambda} - H^2 e^{\lambda}) \ , \tag{4.6}$$

where $\lambda(x, y) = -\frac{1}{2}\log(H^2 - K(x, y))$. It is shown in [**16**] that (4.6), or, which is the same, (4.5), is satisfied for a family of surfaces X_t with coefficients of the first and the second fundamental form given by

$$E_t = G_t = e^{\lambda} \qquad F_t = 0 \ ,$$

$$L_t = He^{\lambda} + \cos 2t \qquad M_t = \sin 2t \qquad N_t = He^{\lambda} - \cos 2t \ .$$

Moreover, it is proved in [**16**] that any surface of Gauss curvature $K(u, v)$ and constant mean curvature H satisfies locally the above. Of course for $t \neq k\pi$ (k – integer) the surface X_t is not in principal parameters. More precisely, it can be seen that in canonical principal parameters the Gauss curvature of the surface X_t is

$$K(u\cos t - v\sin t, u\sin t + v\cos t)$$

and this function also satisfies (4.5). So these functions give a family of solutions of (4.5) (and (4.6)).

In particular, for minimal surfaces ($H = 0$) (4.5) reduces to

$$\Delta\big(\log\sqrt{-K}\big) + 2\sqrt{-K} = 0$$

or, if $\nu = \sqrt{-K}$ is the positive principal curvature,

$$\Delta(\log\nu) + 2\nu = 0\,. \tag{4.7}$$

It follows from (3.5) that in this case $E = G$. Hence, since $F = 0$, the canonical principal parameters (u, v) are isothermal. When we consider minimal surfaces, it is very common to use complex parameters; in real coordinates this gives isothermal parameters. A method to obtain canonical principal parameters for a minimal surface from arbitrary isothermal ones is found in [**11**]. In [**10**], a different form of (4.7) is called *natural partial differential equation of minimal surfaces*.

Flat surfaces, i.e. surfaces with vanishing Gauss curvature K, are well studied – they are in some sense (locally) general cylinders, general cones and tangent developable surfaces. When a surface has no umbilical points (for example for a tangent developable surface the torsion of the directrix must not vanish), the mean curvature H cannot vanish. It follows from Theorem 3.2 that these surfaces are characterized by

$$\left(\frac{1}{H}\right)_{vv} = 0 \qquad \text{or} \qquad H = \frac{1}{f(u)v + g(u)}$$

in canonical principal parameters for some functions $f(u)$, $g(u)$. In particular, up to Euclidean motion for any nonzero real number H there exists only one flat surface with mean curvature H (rather than a nontrivial family of isometric surfaces, as stated in [**6**]).

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