Riv. Mat. Univ. Parma, Vol. **11** (2020), 217-249

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Some combinatorial properties of the generalized derangement numbers

Abstract. In this paper, we give a simple description of the m-widened permutations (generalized m-permutations) and the m-widened derangements (generalized m-derangements) in terms of ordinary permutations and derangements with a suitable constraint. This approach allows us to give a natural combinatorial interpretation of the generalized derangement numbers and the generalized rencontres polynomials in terms of species of structures. Finally, we obtain some formulas relating the generalized derangement numbers with the r-Bell numbers. In particular, we give an extension of the *Clarke-Sved identity*.

Keywords. species, permutation, derangement, arrangement, enriched partition, enriched partition with no singleton block, rencontres polynomial, Stirling number, Bell number.

Mathematics Subject Classification: Primary 05A19, 05A15; Secondary 11B73, 11B83.

Contents

Received: April 9, 2019; accepted in revised form: August 27, 2020.

This research was partially supported by the NSF CCF-1526485 and DMS-1800443, the NIH R01GM109459-01, the Southeast Center for Mathematics and Biology (an NSF-Simons Research Center for Mathematics of Complex Biological Systems) under NSF DMS-1764406 and Simons Foundation 594594 (first author).

1 - Introduction

The *derangement number* d_n is the number of permutations of an n-set with no fixed points, [**61**, p. 65] (sequence A000166 in [**65**]), and (by a simple application of the principle of inclusion-exclusion) it can be expressed as

$$d_n = \sum_{k=0}^{n} \frac{n!}{k!} (-1)^k = \sum_{k=0}^{n} \binom{n}{k} (-1)^k (n-k)! \,. \tag{1}$$

Similarly, the *rencontres number*

$$D_{n,k} = \binom{n}{k} d_{n-k} \tag{2}$$

is the number of permutations of an n-set with exactly k fixed points [**61**, pp. 57, 58, 65] (sequence A008290 in [**65**]). The *rencontres polynomials*

$$D_n(x) = \sum_{k=0}^{n} D_{n,k} x^k = \sum_{k=0}^{n} \binom{n}{k} d_{n-k} x^k \tag{3}$$

are the polynomials associated with the rencontres numbers (first expression) or, equivalently, the *Appell polynomials* associated with the derangement numbers (second expression) [**54**, **61**].

The numbers d_n and the polynomials $D_n(x)$ go back to the *problème des rencontres* (*problem of coincidences*) [**29**,**30**,**52**,**69**]. They are very well known and widely studied in combinatorics and in probability theory, as witnessed by the large number of papers on this topic [**23**, p. 182] [**12**, Section 7.2, p. 202] [**38**, p. 194] [**47**, p. 102] [**61**, p. 65] [**63**, p. 23] [**1**, **2**, **4**, **5**, **6**, **13**, **17**, **24**, **36**, **37**, **39**, **40**, **41**, **48**, **57**, **60**, **64**, **68**, **70**, **73**, **74**]. Moreover, as often happens, they have been generalized in several ways [**3**,**9**,**10**,**16**,**18**,**19**,**20**,**25**,**26**,**31**,**34**,**35**,**42**,**44**,**45**,**46**, **50**, **51**, **53**, **56**, **71**, **72**, **75**, **76**, **77**]. Here, we are interested in the generalizations $d_n^{(m)}$ and $D_n^{(m)}(x)$ considered in [**7**,**14**,**15**,**33**,**55**] (see also [**21**,**27**,**28**,**32**,**58**,**59**] where the numbers $d_n^k = d_{n-k}^{(k)}$ are the entries of *Euler's difference table* [**65**, A068106]). In what follows, we briefly recall the combinatorial settings for these generalizations.

For any $m \in \mathbb{N}$, an *m-widened permutation* [**15**] is a bijection $f : X \cup U \to X \cup V$ where X is any n-set and U and V are two m-sets such that X, U and V are pairwise disjoint. For $m = 0$, we have the ordinary permutations on X, while, for $m = 1$, we have the case introduced and studied in [**7**].

If $U = \{u_1, \ldots, u_m\}$ and $V = \{v_1, \ldots, v_m\}$, then f is equivalent to an $(m+1)$-tuple $(\lambda_1, \ldots, \lambda_m, \sigma)$, where each λ_i is a linear order and σ is a permutation, defined as follows: $\lambda_i = [u_i, f(u_i), f^2(u_i), \ldots, f^h(u_i), v_j]$ where $h+1$ is the minimum positive integer for which there exists an index $j \in \{1, 2, \ldots, m\}$ such that $f^{h+1}(u_i) = v_j$, and σ is the remaining permutation on $X \backslash (X_{\lambda_1} \cup \cdots \cup X_{\lambda_m})$, where $X_{\lambda_i} = \{f(u_i), f^2(u_i), \ldots, f^h(u_i)\} \subseteq X$. For instance, the 5-widened permutation defined by

$$f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & u_1 & u_2 & u_3 & u_4 & u_5 \\ 7 & 1 & 9 & v_5 & 8 & v_4 & 2 & 6 & 3 & v_3 & v_1 & 5 & v_2 & 4 \end{pmatrix} \tag{4}$$

is equivalent to the sestuple $(\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \sigma)$, where $\lambda_1 = [u_1, v_3]$, $\lambda_2 = [u_2, v_1]$, $\lambda_3 = [u_3, 5, 8, 6, v_4]$, $\lambda_4 = [u_4, v_2]$, $\lambda_5 = [u_5, 4, v_5]$, and $\sigma = (172)(39)$.

An *m-widened derangement* is an m-widened permutation $f : X \cup U \to X \cup V$ with no fixed points, that is, with no points $x \in X$ such that $f(x) = x$. Notice that f is an m-widened derangement if and only if in the decomposition $(\lambda_1, \ldots, \lambda_m, \sigma)$ of f the permutation σ is a derangement. In the above example, f is a 5-widened derangement.

The m-widened permutations are equivalent to the *generalized m-permutations* [**15**] defined as the permutations of the symbols $1, 2, \ldots, n, v_1, v_2, \ldots, v_m$, where $v_i \notin \{1, 2, \ldots, n\}$ for every $i = 1, 2, \ldots, m$. In the above example, considering only the second line in the two-line representation (4) of f, we have the generalized 5-permutation $\tau = 7\,1\,9\,v_5\,8\,v_4\,2\,6\,3\,v_3\,v_1\,5\,v_2\,4$.

A fixed point of a generalized m-permutation is an integer $k \in \{1, 2, \ldots, n\}$ appearing in position k. A *generalized m-derangement* [**15**] is a generalized m-permutation with no fixed points (in the first n positions). The generalized 5-permutation τ of the above example is a generalized 5-derangement.

In this context [**15**], we have the *generalized derangement numbers*

$$d_n^{(m)} = \sum_{k=0}^{n} \binom{n}{k} (-1)^k (m+n-k)!\,, \tag{5}$$

counting the generalized m-derangements (m-widened derangements), the *generalized rencontres numbers*

$$D_{n,k}^{(m)} = \binom{n}{k} d_{n-k}^{(m)} \tag{6}$$

counting the generalized m-permutations (m-widened permutations) with exactly k fixed points, and the *generalized rencontres polynomials*

$$(7)\quad D_n^{(m)}(x) = \sum_{k=0}^{n} D_{n,k}^{(m)} x^k = \sum_{k=0}^{n} \binom{n}{k} d_{n-k}^{(m)} x^k = \sum_{k=0}^{n} \binom{n}{k} (m+k)!(x-1)^{n-k}$$

with exponential generating series

$$(8)\qquad D^{(m)}(x;t) = \sum_{n\geq 0} D_n^{(m)}(x)\,\frac{t^n}{n!} = \frac{m!\,\mathrm{e}^{(x-1)t}}{(1-t)^{m+1}}\,.$$

Clearly, formulas (5), (6) and (7) reduce to formulas (1), (2) and (3) when $m = 0$. On the other hand, the properties valid in the ordinary case ($m = 0$) can usually be extended to the general case. Many of these extensions have been obtained in a purely formal way [**14**, **15**, **33**]. In this paper, on the contrary, we use a more combinatorial approach, based on the *theory of species*. To do that, we simply observe that an m-widened permutation $f : X \cup U \to X \cup V$ is equivalent, in a natural way, to an ordinary permutation $\sigma : X \cup M \to X \cup M$, where M is any m-set (such that $X \cap M = \varnothing$), and an m-widened derangement is equivalent to a permutation $\sigma : X \cup M \to X \cup M$ with no fixed points in X. We call these permutations, just for simplicity, *m-permutations* and *m-derangements*[1], respectively. So, if $X = \{1,2,3,4,5,6,7,8,9\}$ and $M = \{m_1, m_2, m_3, m_4, m_5\}$, the 5-widened derangement (4) becomes the 5-derangement $\sigma = (172)(39)(4m_5)(586m_4m_2m_1m_3)$.

The paper is organized as follows. In Section 2, we recall the main definitions and properties of the theory of species and weighted species [**8**, **43**], and, in particular, we introduce the species of *m-permutations*, the species of *m-derangements*, the species of *m-arrangements* and the *species of m-rencontres*. In Section 3, by a bijective combinatorial approach, we derive several decomposition properties for these species, from which we deduce the corresponding properties for the generalized derangement numbers, the generalized arrangement numbers and the generalized rencontres polynomials. Finally, in Section 4, we obtain some formulas relating the generalized derangement numbers with the r-Bell numbers. In particular, we give an extension of the *Clarke-Sved identity* [**22**]. In the original identity the derangement numbers are related to the Bell numbers, while in our extension the generalized derangement numbers are related to the Stirling numbers of the second kind.

[1]Not to be confused with the r-derangements studied in [**72**].

2 - Combinatorial species

A (*combinatorial*) *species* is a functor $\mathbf{F} : \mathcal{B} \to \mathcal{S}$, where $\mathcal{B}$ is the category of finite sets and bijections and $\mathcal{S}$ is the category of finite sets and functions [**8**,**43**]. Given a finite set S, we denote by $\mathbf{F}[S]$ the set of all structures of species $\mathbf{F}$ (also called $\mathbf{F}$-structures) on the set S. Two species $\mathbf{F}$ and $\mathbf{G}$ are *isomorphic* when there exists a natural isomorphism between them. In this case, we simply write $\mathbf{F} = \mathbf{G}$. Given two species $\mathbf{F}$ and $\mathbf{G}$, we have the following operations, defined, on a finite set S, by

sum	$(\mathbf{F}+\mathbf{G})[S] = \mathbf{F}[S] + \mathbf{G}[S]$
product	$(\mathbf{F}\cdot\mathbf{G})[S] = \sum_{I\subseteq S} \mathbf{F}[I] \times \mathbf{G}[S \setminus I]$
Hadamard product	$(\mathbf{F}\odot\mathbf{G})[S] = \mathbf{F}[S] \times \mathbf{G}[S]$
composition	$(\mathbf{F}\circ\mathbf{G})[S] = \sum_{\pi\in\Pi[S]} \mathbf{F}[\pi] \times \prod_{B\in\pi} \mathbf{G}[B] \qquad (\mathbf{G}[\varnothing] = \varnothing)$

where $\Pi[S]$ denotes the set of partitions of S. The *derivative* of a species $\mathbf{F}$ is the species $\mathbf{F}'$ defined, for every finite set S, by $\mathbf{F}'[S] = \mathbf{F}[S + \{*\}]$, where $*$ denotes an arbitrary element (not in S). More generally, the m-*derivative* of a species $\mathbf{F}$ is the species $\mathbf{F}^{(m)}$ defined, for every finite set S, by $\mathbf{F}^{(m)}[S] = \mathbf{F}[S + M]$, where M is any m-set (disjoint from S).

The *cardinality* of a species $\mathbf{F}$ is the exponential formal series

$$\mathbf{Card}(\mathbf{F}; t) = \sum_{n\geq 0} f_n \frac{t^n}{n!}\,,$$

where $f_n = |\mathbf{F}[S]|$ is the number of all $\mathbf{F}$-structures on any n-set S (this number does not depend on the choice of the n-set S). All the operations recalled above are preserved by cardinality.

More generally, we have the *weighted species* [**43**, p. 54] [**8**, p. 75]. Here, however, we will consider only the weighted species with weights in the algebra $\mathbb{R}[x]$ of the real polynomials in an indeterminate x.

A *weighted set* is a pair (S, w), where S is a finite set and $w : S \to \mathbb{R}[x]$ is a map, called *weight function*, which associates a *weight* $w(s) \in \mathbb{R}[x]$ to each element $s \in S$. The *total weight* of S is the sum of the weights of all elements of S, i.e. $|S|_w = w(S) = \sum_{s\in S} w(s)$. A morphism of weighted sets $f : (S, w) \to (T, v)$ is a function $f : S \to T$ preserving the weights, i.e., $w = v \circ f$.

We can define some operations on weighted sets. The *sum* of two weighted sets (S, w) and (T, v) is the weighted set $(S+T, w+v)$ where $(w+v)(x) = w(x)$

if $x \in S$ and $(w+v)(x) = v(x)$ if $x \in T$. The *product* of (S, w) and (T, v) is the weighted set $(S \times T, w \times v)$, where $(w \times v)(x, y) = w(x)v(y)$. These operations are preserved by the total weight.

A *weighted species* is a functor $\mathbf{F}_x : \mathcal{B} \to \mathcal{S}_x$, where $\mathcal{S}_x$ is the category of finite weighted sets and weight-preserving maps (as defined above). For every finite set S, we have the weighted set $\mathbf{F}_x[S] = (\mathbf{F}[S], w)$, where $\mathbf{F}[S]$ is the set of structures of species $\mathbf{F}$ on the set S and $w : \mathbf{F}[S] \to \mathbb{R}[x]$ is a weight function. An ordinary species is a weighted species where every weight function w is the constant function 1. The *cardinality* of the weighted species $\mathbf{F}_x$ is the exponential formal series

$$\mathbf{Card}(\mathbf{F}_x; t) = \sum_{n\geq 0} f_n^w \frac{t^n}{n!},$$

where f_n^w is the total weight of the set $\mathbf{F}[S]$, for any n-set S, that is, $f_n^w = w(\mathbf{F}[S]) = \sum_{\varphi\in\mathbf{F}[S]} w(\varphi)$. The operations defined for ordinary species can be extended to weighted species and, also in this case, they are preserved by cardinality.

We now recall the definition of the species and weighted species used in the rest of the paper. Fixed $k \in \mathbb{N}$, let $\frac{\mathbf{X}^k}{k!}$ be the *species of k-sets*, defined, for every finite set S, by $\frac{\mathbf{X}^k}{k!}[S] = \{S\}$ whenever $|S| = k$ and $\frac{\mathbf{X}^k}{k!}[S] = \varnothing$ whenever $|S| \neq k$, and let $\frac{\mathbf{X}_x^k}{k!}$ be the *weighted species of k-sets* defined, for every finite set S, by $\frac{\mathbf{X}_x^k}{k!}[S] = (\frac{\mathbf{X}^k}{k!}[S], w)$, where $w(S) = x^{|S|}$ when $|S| = k$. The cardinalities of these species are

$$\mathbf{Card}\Big(\frac{\mathbf{X}^k}{k!}; t\Big) = \frac{t^k}{k!} \qquad \text{and} \qquad \mathbf{Card}\Big(\frac{\mathbf{X}_x^k}{k!}; t\Big) = x^k \frac{t^k}{k!}.$$

Let $\mathbf{Exp}$ be the *exponential species*, or the *uniform species*, defined by $\mathbf{Exp}[S] = \{S\}$ for every finite set S, and having cardinality

$$\mathbf{Card}(\mathbf{Exp}; t) = e^t. \tag{9}$$

Notice that $\mathbf{Exp}' = \mathbf{Exp}$ and that, more generally,

$$(\mathbf{Exp}^n)' = n\mathbf{Exp}^{n-1} \cdot \mathbf{Exp}' = n\mathbf{Exp}^n,$$

for every $n \in \mathbb{N}$, where $n\mathbf{F} = \mathbf{F} + \cdots + \mathbf{F}$ (n times).

Similarly, let $\mathbf{Exp}_x$ be the *weighted exponential species* defined, for every finite set S, by $\mathbf{Exp}_x[S] = (\mathbf{Exp}[S], w) = (\{S\}, w)$, where $w(S) = x^{|S|}$. To give a structure of species $\mathbf{Exp}_x$ on a finite set S simply means to give the weight x to each element of S. Equivalently, we can give a structure of species $\mathbf{Exp}_x$ on a set S by assigning a partition π of S in singletons and the weight x to each block of π. Hence, we have the relation $\mathbf{Exp}_x = \mathbf{Exp} \circ \mathbf{X}_x$. In both interpretations, we obtain the generating series

$$\mathbf{Card}(\mathbf{Exp}_x; t) = \mathrm{e}^{xt} \,. \tag{10}$$

Let **Perm** be the *species of permutations* and let $\mathbf{Perm}^{(m)}$ be the *species of m-permutations*, i.e., the species defined by $\mathbf{Perm}^{(m)}[S] = \mathbf{Perm}[S+M]$, for every finite sets S and M, with $|M| = m$. Hence, by definition, $\mathbf{Perm}^{(m)}$ is the m-derivative of the species **Perm**, and then

$$\mathbf{Card}(\mathbf{Perm}^{(m)}; t) = \frac{d^m}{dt^m}\,\mathbf{Card}(\mathbf{Perm}; t) = \frac{d^m}{dt^m}\,\frac{1}{1-t} = \frac{m!}{(1-t)^{m+1}} \,. \tag{11}$$

Clearly, we also have directly $|\mathbf{Perm}^{(m)}[S]| = (m+n)!$, for every n-set S.

Let **Der** be the *species of derangements* and let $\mathbf{Der}^{(m)}$ be the *species of m-derangements*, i.e., the species defined, for every finite set S, by

$$\mathbf{Der}^{(m)}[S] = \{\sigma \in \mathbf{Perm}^{(m)}[S] \;:\; \mathrm{Fix}(\sigma) \cap S = \varnothing\}$$

where $\mathrm{Fix}(\sigma)$ is the set of fixed points of σ. Notice that an ordinary derangement of $S+M$ is an m-derangement on S, but that the viceversa is not necessarily true. Indeed, an m-derangement on S is a permutation on $S+M$ with no fixed points in S and no restriction on M; so it can have fixed points in M. For instance, if $S = \{1,2,3,4,5\}$ and $M = \{6,7,8,9\}$, then $\sigma = (139)(26)(458)(7)$ is a 4-derangement on S.

Let $\mathbf{Ren}_x^{(m)}$ be the *species of m-rencontres* defined, for every finite set S, by $\mathbf{Ren}_x^{(m)}[S] = (\mathbf{Perm}^{(m)}[S], w)$, where $w : \mathbf{Perm}^{(m)}[S] \to \mathbb{R}[x]$ is the weight function defined by $w(\sigma) = x^{|S \cap \mathrm{Fix}(\sigma)|}$, for every $\sigma \in \mathbf{Perm}^{(m)}[S]$. In other words, a structure of species $\mathbf{Ren}_x^{(m)}$ on a finite set S is a permutation σ of $S+M$ where each fixed point in S has weight x (while the possible fixed points in M have weight 1, as any other point). If S is an n-set, the total weight of $\mathbf{Perm}^{(m)}[S]$ is

$$w(\mathbf{Perm}^{(m)}[S]) = \sum_{k=0}^{n} \binom{n}{k} d_{n-k}^{(m)} x^k = D_n^{(m)}(x) \,.$$

Clearly, when every weight function w is the constant function 1 (i.e. for $x = 1$), then the weighted species $\mathbf{Ren}_x^{(m)}$ reduces to the ordinary species $\mathbf{Perm}^{(m)}$.

Let **Arr** be the species of *arrangements* defined, for every finite set S, by

$$\mathbf{Arr}[S] = \{(I, \sigma) \;:\; I \subseteq S \,,\; \sigma \in \mathbf{Perm}[I]\} \,.$$

The number of all arrangements of an n-set [**61**, p. 16] [**65**, A000522] is

$$a_n = \sum_{k=0}^{n} \binom{n}{k} k!\,.$$

More generally, let $\mathbf{Arr}^{(m)}$ be the species of *m-arrangements* defined, for every finite set S, by

$$\mathbf{Arr}^{(m)}[S] = \{(I,\sigma)\ :\ I \subseteq S\,,\ \sigma \in \mathbf{Perm}^{(m)}[I]\}\,.$$

Let $\mathbf{Lin}$ be the species of *linear orders* (totally ordered sets) and let $\mathbf{Lin}-1$ be the species of *non-empty linear orders*.

Let $\mathbf{Part}$ be the *species of partitions*, i.e., the species defined by $\mathbf{Part}[S] = \Pi[S]$ for any finite set S. Let $\mathbf{F}$ be a species with $\mathbf{F}[\varnothing] = \varnothing$ and $|\mathbf{F}[\{*\}]| = 1$, and let $f(t) = \mathbf{Card}(\mathbf{F};t)$. An $\mathbf{F}$-*enriched partition* of a finite set S is a pair (π,Φ) where $\pi = \{B_1,\dots,B_k\}$ is a partition of S and $\Phi = \{\varphi_1,\dots,\varphi_k\}$ is a family of $\mathbf{F}$-structures, so that each block B_i of π is endowed with a structure φ_i of species $\mathbf{F}$. Let $\mathbf{Part}^{\mathbf{F}}$ be the *species of* $\mathbf{F}$-*enriched partitions*, i.e., the species $\mathbf{Part}^{\mathbf{F}} = \mathbf{Exp} \circ \mathbf{F}$. If $p_n^{\mathbf{F}} = |\mathbf{Part}^{\mathbf{F}}[S]|$, with $|S| = n$, then we have the exponential generating series

$$p^{\mathbf{F}}(t) = \sum_{n\geq 0} p_n^{\mathbf{F}} \frac{t^n}{n!} = \mathbf{Card}(\mathbf{Part}^{\mathbf{F}};t) = \mathbf{Card}(\mathbf{Exp}\circ\mathbf{F};t) = \mathrm{e}^{f(t)}\,.$$

Similarly, let $\widetilde{\mathbf{P}}\mathbf{art}^{\mathbf{F}}$ be the *species of the* $\mathbf{F}$-*enriched partitions without singleton blocks* (i.e., without blocks of size 1). Then $\widetilde{\mathbf{P}}\mathbf{art}^{\mathbf{F}} = \mathbf{Exp}\circ(\mathbf{F}-\mathbf{X})$, where, for simplicity, $\mathbf{F}-\mathbf{X}$ denotes the species which is $\varnothing$ on any singleton $\{*\}$ and which is equal to $\mathbf{F}[S]$ on any other set $S \neq \{*\}$. Furthermore, if $\widetilde{p}_n^{\mathbf{F}} = |\widetilde{\mathbf{P}}\mathbf{art}^{\mathbf{F}}[S]|$, with $|S| = n$, then we have the exponential generating series

$$\widetilde{p}^{\mathbf{F}}(t) = \sum_{n\geq 0} \widetilde{p}_n^{\mathbf{F}} \frac{t^n}{n!} = \mathbf{Card}(\widetilde{\mathbf{P}}\mathbf{art}^{\mathbf{F}};t) = \mathbf{Card}(\mathbf{Exp}\circ(\mathbf{F}-\mathbf{X});t) = \mathrm{e}^{f(t)-t}\,. \tag{12}$$

Let $\mathbf{Part}_x^{\mathbf{F}}$ be the *weighted species of* $\mathbf{F}$-*enriched partitions*, where each singleton block has weight x. Let $p_n^{\mathbf{F}}(x)$ be the polynomial giving the total weight of $\mathbf{Part}^{\mathbf{F}}[S]$, when $|S| = n$. Then, we have the following result.

Lemma 1. *If* $\mathbf{F}$ *is a species with* $\mathbf{F}[\varnothing] = \varnothing$ *and* $|\mathbf{F}[\{*\}]| = 1$, *then we have the relation*

$$\mathbf{Part}_x^{\mathbf{F}} = \mathbf{Exp}_x \cdot \widetilde{\mathbf{P}}\mathbf{art}^{\mathbf{F}} \tag{13}$$

and the exponential generating series

$$p^{\mathbf{F}}(x;t) = \sum_{n\geq 0} p_n^{\mathbf{F}}(x)\frac{t^n}{n!} = \mathrm{e}^{f(t)-t}\,\mathrm{e}^{xt} = \mathrm{e}^{f(t)}\,\mathrm{e}^{(x-1)t}\,. \tag{14}$$

P r o o f. To give a structure of species $\mathbf{Part}_x^{\mathbf{F}}$ on a finite set S means to give a partition $\pi \in \mathbf{Part}[S]$ where each block is endowed with a structure of species $\mathbf{F}$ and where each singleton block has weight x. Since, by hypothesis, there is only one $\mathbf{F}$-structure on a singleton block, this is equivalent to consider a subset I of S, where each element has weight x, and an $\mathbf{F}$-enriched partition without singleton blocks on $S \setminus I$. This remark implies at once relation (13). By this relation, we have

$$\mathbf{Card}(\mathbf{Part}_x^{\mathbf{F}};t) = \mathbf{Card}(\mathbf{Exp}_x \cdot \widetilde{\mathbf{Part}}^{\mathbf{F}};t) = \mathbf{Card}(\mathbf{Exp}_x;t)\cdot \mathbf{Card}(\widetilde{\mathbf{Part}}^{\mathbf{F}};t)\,.$$

Since $p^{\mathbf{F}}(x;t) = \mathbf{Card}(\mathbf{Part}_x^{\mathbf{F}};t)$, by identities (10) and (12), we have series (14). □

By identity (14), the polynomials $p_n^{\mathbf{F}}(x)$ always form an *Appell sequence* [**54**, **62**] and are given by

$$p_n^{\mathbf{F}}(x) = \sum_{k=0}^{n} \binom{n}{k} \widetilde{p}_{n-k}^{\mathbf{F}}\, x^k\,.$$

3 - Basic combinatorial properties

3.1 - *Decomposition properties*

In this section, we derive some basic combinatorial properties of the species of m-permutations, m-derangements, m-rencontres and m-arrangements, and, consequently, of the generalized derangement and arrangement numbers and generalized rencontres polynomials. We start with the following simple result.

T h e o r e m 2. *We have the relations*

$$\mathbf{Perm}^{(m)} = \mathbf{Exp}\cdot \mathbf{Der}^{(m)} \tag{15}$$

$$\mathbf{Ren}_x^{(m)} = \mathbf{Exp}_x \cdot \mathbf{Der}^{(m)} \tag{16}$$

and the cardinalities

$$\mathbf{Card}(\mathbf{Der}^{(m)};t) = \frac{m!\,\mathrm{e}^{-t}}{(1-t)^{m+1}} \tag{17}$$

$$\mathbf{Card}(\mathbf{Ren}_x^{(m)};t) = \frac{m!\, e^{(x-1)t}}{(1-t)^{m+1}}\,. \tag{18}$$

Proof. To give a structure of species $\mathbf{Perm}^{(m)}$ on a finite set S means, by definition, to give a permutation σ of $S+M$. This is equivalent to give a subset $I \subseteq S$ ($I = S \cap \mathrm{Fix}(\sigma)$) and an m-derangement δ on $S \setminus I$ (the restriction of σ to $(S \setminus I) + M$). Then, we have the identity

$$\mathbf{Perm}^{(m)}[S] = \sum_{I \subseteq S} \mathbf{Exp}[I] \times \mathbf{Der}^{(m)}[S \setminus I]$$

which yields relation (15).

Similarly, to give a structure of species $\mathbf{Ren}_x^{(m)}$ on a finite set S means to give an m-permutation σ on S and to assign the weight x to each fixed point of σ in S. This is equivalent to give a subset $I \subseteq S$ ($I = S \cap \mathrm{Fix}(\sigma)$), where each element has weight x, and an m-derangement δ on $S \setminus I$ (the restriction of σ to $(S \setminus I) + M$). This implies relation (16).

By relation (15), we have

$$\mathbf{Card}(\mathbf{Perm}^{(m)};t) = \mathbf{Card}(\mathbf{Exp}\cdot\mathbf{Der}^{(m)};t) = \mathbf{Card}(\mathbf{Exp};t)\cdot\mathbf{Card}(\mathbf{Der}^{(m)};t).$$

Then, by identities (11) and (9), we have

$$\frac{m!}{(1-t)^{m+1}} = e^t\, \mathbf{Card}(\mathbf{Der}^{(m)};t)$$

from which we have at once identity (17). Then, by relation (16) and identity (17), we have

$$\begin{aligned}\mathbf{Card}(\mathbf{Ren}_x^{(m)};t) &= \mathbf{Card}(\mathbf{Exp}_x \cdot \mathbf{Der}^{(m)};t)\\ &= \mathbf{Card}(\mathbf{Exp}_x;t) \cdot \mathbf{Card}(\mathbf{Der}^{(m)};t) = e^{xt}\frac{m!\, e^{-t}}{(1-t)^{m+1}}\end{aligned}$$

which simplifies in series (18). □

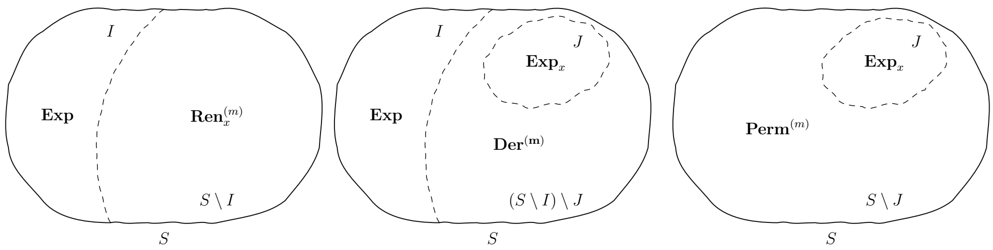
Furthermore, we have the following "*exchange property*".

Theorem 3. *We have the relation*

$$\mathbf{Exp} \cdot \mathbf{Ren}_x^{(m)} = \mathbf{Exp}_x \cdot \mathbf{Perm}^{(m)} \tag{19}$$

and the identity

$$\sum_{k=0}^{n} \binom{n}{k} D_k^{(m)}(x) = \sum_{k=0}^{n} \binom{n}{k} (m+k)!\, x^{n-k}\,. \tag{20}$$



(a) $\mathbf{Exp} \cdot \mathbf{Ren}_x^{(m)}$. (b) $\mathbf{Exp} \cdot (\mathbf{Exp}_x \cdot \mathbf{Der}^{(m)})$. (c) $\mathbf{Exp}_x \cdot \mathbf{Perm}^{(m)}$.

Fig. 1. Decompositions in the proof of Theorem 3.

Proof. To give a structure of species $\mathbf{Exp} \cdot \mathbf{Ren}_x^{(m)}$ on a finite set S means to assign a subset $I \subseteq S$, an m-permutation σ on $S \setminus I$ and a weight x to each fixed point of σ in S (see Figure 1a). This is equivalent to assign a first subset $I \subseteq S$, a second subset $J \subseteq S \setminus I$ ($J = \mathrm{Fix}(\sigma)$), a weight x to each point of J and an m-derangement σ' on $(S \setminus I) \setminus J$ (see Figure 1b). Exchanging the roles of I and J (i.e. considering the points in I as fixed points and the points in J as simple points with weight x), this is equivalent to assign a subset $J \subseteq S$, a weight x to each point of J and an m-permutation σ'' on $S \setminus J$ (see Figure 1c). These bijections imply relation (19), and this relation implies identity (20). □

Theorem 3 immediately implies the following general result.

Theorem 4. *If* $\mathbf{F}$ *is a species with* $\mathbf{F}[\varnothing] = \varnothing$ *and* $|\mathbf{F}[\{*\}]| = 1$, *then we have the relation*

$$\widetilde{\mathbf{Part}}^{\mathbf{F}} \cdot \mathbf{Ren}_x^{(m)} = \mathbf{Part}_x^{\mathbf{F}} \cdot \mathbf{Der}^{(m)} \tag{21}$$

and the identity

$$\sum_{k=0}^{n} \binom{n}{k} \widetilde{p}_{n-k}^{\mathbf{F}} \, D_k^{(m)}(x) = \sum_{k=0}^{n} \binom{n}{k} d_{n-k}^{(m)} \, p_k^{\mathbf{F}}(x)\,. \tag{22}$$

In particular, for $x = 1$, *we have the identity*

$$\sum_{k=0}^{n} \binom{n}{k} (m+k)! \, \widetilde{p}_{n-k}^{\mathbf{F}} = \sum_{k=0}^{n} \binom{n}{k} d_{n-k}^{(m)} \, p_k^{\mathbf{F}}\,. \tag{23}$$

Proof. By relations (16) and (13), we have

$$\begin{aligned}\widetilde{\mathbf{Part}}^{\mathbf{F}} \cdot \mathbf{Ren}_x^{(m)} &= \widetilde{\mathbf{Part}}^{\mathbf{F}} \cdot (\mathbf{Exp}_x \cdot \mathbf{Der}^{(m)}) \\ &= (\mathbf{Exp}_x \cdot \widetilde{\mathbf{Part}}^{\mathbf{F}}) \cdot \mathbf{Der}^{(m)} = \mathbf{Part}_x^{\mathbf{F}} \cdot \mathbf{Der}^{(m)}\,.\end{aligned}$$

This is relation (21), which, in turn, immediately implies identity (22). □

Examples.

1. If $\mathbf{F} = \mathbf{Exp} - 1$, then $\mathbf{Part}^{\mathbf{F}} = \mathbf{Exp} \circ (\mathbf{Exp} - 1) = \mathbf{Part}$ is the species of partitions. So, we have the *Bell numbers* $p_n^{\mathbf{F}} = b_n$ [**65**, A000110], the numbers $\widetilde{p}_n^{\mathbf{F}} = b_n^*$ of partitions without singleton blocks [**65**, A000296] and the polynomials $p_n^{\mathbf{F}}(x) = B_n(x) = \sum_{k=0}^{n} \binom{n}{k} b_{n-k}^* x^k$. Then, identities (22) and (23) become

$$\sum_{k=0}^{n} \binom{n}{k} b_{n-k}^* \, D_k^{(m)}(x) = \sum_{k=0}^{n} \binom{n}{k} d_{n-k}^{(m)} \, B_k(x) \tag{24}$$

$$\sum_{k=0}^{n} \binom{n}{k} (m+k)! \, b_{n-k}^* = \sum_{k=0}^{n} \binom{n}{k} d_{n-k}^{(m)} \, b_k \, . \tag{25}$$

2. If $\mathbf{F} = \mathbf{Cyc}$ is the species of cycles, then $\mathbf{Part}^{\mathbf{F}} = \mathbf{Exp} \circ \mathbf{Cyc} = \mathbf{Perm}$ is the species of permutations, and we have the *factorial numbers* $p_n^{\mathbf{F}} = n!$, the *derangement numbers* $\widetilde{p}_n^{\mathbf{F}} = d_n$ and the *rencontres polynomials* $p_n^{\mathbf{F}}(x) = D_n(x) = \sum_{k=0}^{n} \binom{n}{k} d_{n-k} x^k$. Identities (22) and (23) become

$$\sum_{k=0}^{n} \binom{n}{k} d_{n-k} D_k^{(m)}(x) = \sum_{k=0}^{n} \binom{n}{k} d_{n-k}^{(m)} D_k(x) \tag{26}$$

$$\sum_{k=0}^{n} \binom{n}{k} (m+k)! \, d_{n-k} = \sum_{k=0}^{n} \binom{n}{k} k! \, d_{n-k}^{(m)} \, . \tag{27}$$

3. If $\mathbf{F} = \mathbf{Lin} - 1$ is the species of non-empty linear orders, then $\mathbf{Part}^{\mathbf{F}} = \mathbf{Exp} \circ (\mathbf{Lin} - 1)$ is the species of *Lah partitions*. So, we have the *cumulative Lah numbers* $p_n^{\mathbf{F}} = \ell_n$ [**65**, A000262]) the numbers $\widetilde{p}_n^{\mathbf{F}} = \ell_n^*$ of Lah partitions without singleton blocks [**65**, A052845] and the polynomials $p_n^{\mathbf{F}}(x) = L_n(x) = \sum_{k=0}^{n} \binom{n}{k} \ell_{n-k}^* x^k$. Identities (22) and (23) become

$$\sum_{k=0}^{n} \binom{n}{k} \ell_{n-k}^* D_k^{(m)}(x) = \sum_{k=0}^{n} \binom{n}{k} d_{n-k}^{(m)} L_k(x) \tag{28}$$

$$\sum_{k=0}^{n} \binom{n}{k} (m+k)! \, \ell_{n-k}^* = \sum_{k=0}^{n} \binom{n}{k} d_{n-k}^{(m)} \ell_k \, . \tag{29}$$

4. If $\mathbf{F} = \mathbf{X} + \mathbf{X}^2/2$, then $\mathbf{Part}^{\mathbf{F}} = \mathbf{Exp} \circ (\mathbf{X} + \mathbf{X}^2/2) = \mathbf{Inv}$ is the species of *involutions*. So, we have the *involution numbers* $p_n^{\mathbf{F}} = i_n$ [**65**,

A000085], the numbers of involutions without singletons $\widetilde{p}^{\mathbf{F}}_{2n} = \binom{2n}{n}\frac{n!}{2^n}$ or $\widetilde{p}^{\mathbf{F}}_{2n+1} = 0$ [**65**, A123023] and the *involution polynomials* $p^{\mathbf{F}}_n(x) = I_n(x) = \sum_{k=0}^{\lfloor n/2\rfloor}\binom{n}{k}\binom{n-k}{k}\frac{k!}{2^k}x^{n-2k}$. After some simplifications, identities (22) and (23) become

$$\sum_{k=0}^{\lfloor n/2\rfloor}\binom{n}{k}\binom{n-k}{k}\frac{k!}{2^k}\,D^{(m)}_{n-2k}(x) = \sum_{k=0}^{n}\binom{n}{k}d^{(m)}_{n-k}I_k(x) \tag{30}$$

$$\sum_{k=0}^{\lfloor n/2\rfloor}\binom{n}{k}\binom{n-k}{k}\frac{k!}{2^k}\,(m+n-2k)! = \sum_{k=0}^{n}\binom{n}{k}d^{(m)}_{n-k}i_k\,. \tag{31}$$

For the generalized derangement numbers, we also have the following result.

Theorem 5. *The generalized derangement numbers can be expressed in terms of the ordinary derangement numbers, by means of the formula*

$$d^{(m)}_n = \sum_{k=0}^{m}\binom{m}{k}d_{n+k}\,. \tag{32}$$

Proof. To give an m-derangement δ on an n-set S is equivalent to assign a subset $I \subseteq M$ ($I = \mathrm{Fix}(\delta) \subseteq M$) and an ordinary derangement on $S+(M\setminus I)$. This remark implies at once the claimed identity. □

For the generalized rencontres polynomials the situation is more complicated. Indeed, to express these polynomials in terms of the ordinary rencontres polynomials, we need to consider the species $\mathbf{H}^{(m)}$ defined as follows: for any m-set M and for any finite set S, $\mathbf{H}^{(m)}[S]$ is the set of all permutations on $S+M$ with no cycles entirely contained in S. Clearly, all permutations in $\mathbf{H}^{(m)}[S]$ are particular m-derangements on S. For instance, if $S = \{1,2,3,4,5\}$ and $M = \{6,7,8,9\}$, the permutation $\sigma = (162)(37)(459)(8)$ belongs to $\mathbf{H}^{(4)}[S]$.

The *multiset coefficients* [**66**, pp. 25, 26] are defined by $\left(\!\binom{x}{k}\!\right) = \frac{(x)_k}{k!}$, where $(x)_n = x(x+1)(x+2)\cdots(x+n-1)$ is the *Pochhammer symbol.*

Theorem 6. *We have the relations*

$$\mathbf{Perm}^{(m)} = \mathbf{Perm}\cdot\mathbf{H}^{(m)} \tag{33}$$

$$\mathbf{Ren}^{(m)}_x = \mathbf{Ren}_x\cdot\mathbf{H}^{(m)} \tag{34}$$

and the identity

$$D^{(m)}_n(x) = m!\sum_{k=0}^{n}\binom{n}{k}(m)_k\,D_{n-k}(x)\,. \tag{35}$$

In particular, for $x = 0$, we have the identity

$$d_n^{(m)} = m! \sum_{k=0}^{n} \binom{n}{k} (m)_k \, d_{n-k} \,. \tag{36}$$

Proof. To give a structure of species $\mathbf{Ren}_x^{(m)}$ on a finite set S means to give an m-permutation σ on S and to assign the weight x to each fixed point of σ in S. This is equivalent to give a subset $I \subseteq S$, a permutation σ' on I (formed by the cycles of σ entirely contained in S) where each fixed point has weight x, and an m-permutation σ'' on $S \setminus I$ with no cycles entirely contained in $S \setminus I$. This implies relation (34). In particular, in the ordinary case (when each weight function w is the constant function 1, i.e., when $x = 1$), we have relation (33). Then, from relation (34) (or from relation (33)), we have

$$\mathbf{Card}(\mathbf{Ren}_x^{(m)}; t) = \mathbf{Card}(\mathbf{Ren}_x \cdot \mathbf{H}^{(m)}; t) = \mathbf{Card}(\mathbf{Ren}_x; t) \cdot \mathbf{Card}(\mathbf{H}^{(m)}; t)$$

that is $D^{(m)}(x;t) = D(x;t)\ \mathbf{Card}(\mathbf{H}^{(m)};t)$ or $\mathbf{Card}(\mathbf{H}^{(m)};t) = \frac{D^{(m)}(x;t)}{D(x;t)}$. So, by identity (8), we have

$$\mathbf{Card}(\mathbf{H}^{(m)}; t) = \frac{m!}{(1-t)^m} = m! \sum_{n \geq 0} \left(\!\!\binom{m}{n}\!\!\right) t^n = \sum_{n \geq 0} m!(m)_n \frac{t^n}{n!} \,.$$

Now, by this result and by relation (34), we obtain identity (35). □

Remark 7. From the proof of the previous theorem we have that the number of permutations in $\mathbf{H}^{(m)}[S]$, when $|S| = n$, is

$$H_n^{(m)} = m!n! \left(\!\!\binom{m}{n}\!\!\right) = m!\,(m)_n = m\,(n + m - 1)! \,.$$

For instance, for $m = n = 2$, we have $H_2^{(2)} = 12$. Indeed, we have the permutations (13)(24), (14)(23), (123)(4), (132)(4), (124)(3), (142)(3), (1234), (1243), (1324), (1342), (1423), (1432).

3.2 - *Recursive properties*

The generalized recontres polynomials have the following property.

Theorem 8. *The polynomials $D_n^{(m)}(x)$ are Appell polynomials, that is*

$$\frac{d}{dx} D_n^{(m)}(x) = n D_{n-1}^{(m)}(x) \,. \tag{37}$$

Proof. The derivative

$$\frac{\mathrm{d}}{\mathrm{d}x} D_n^{(m)}(x) = \sum_{k=0}^{n} \binom{n}{k} k d_{n-k}^{(m)} x^{k-1}$$

is the weight of the set of all m-permutations on an n-set S where every fixed point in S, except one, has weight x. To give a permutation of this kind is equivalent to give a point p of S and an m-permutation on $S \setminus \{p\}$ where each fixed point (in $S \setminus \{p\}$) has weight x. This yields at once identity (37). $\square$

For the derivative of the species of derangements, we have the following result.

Theorem 9. *We have the relation*

$$\mathbf{Der}' = (\mathbf{Lin} - 1) \cdot \mathbf{Der} \tag{38}$$

and the identity

$$d_{n+1} = \sum_{k=1}^{n} \binom{n}{k} k!\, d_{n-k}\,. \tag{39}$$

Moreover, we have the recurrence

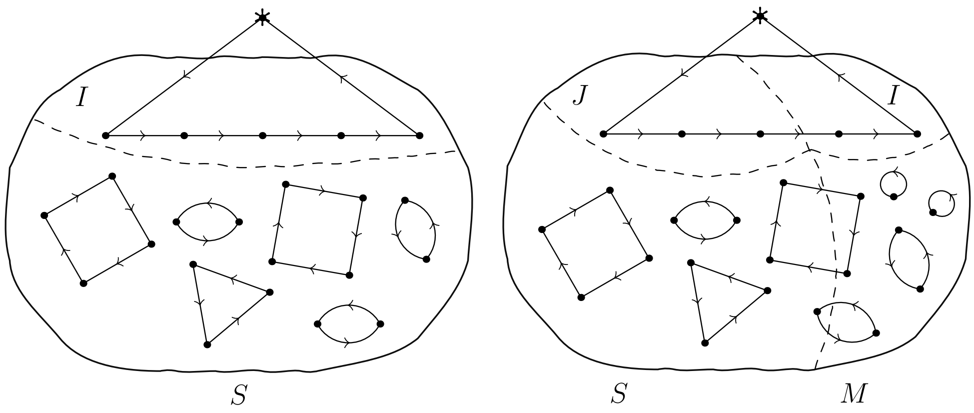
$$d_{n+1} = n d_n + n d_{n-1}\,. \tag{40}$$

Proof. A structure of species $\mathbf{Der}'$ on a finite set S is a derangement δ of $S + \{*\}$ (where $*$ is any point not belonging to S). If we remove $*$, then the cycle containing such a point breaks in a non-empty linear order and δ turns out to be equivalent to a non-empty linear order λ on a subset $I \subseteq S$ and a derangement δ' on $S \setminus I$ (see Figure 2a). This yields relation (38) and consequently identity (39).

This approach can be slightly modified in order to obtain recurrence (40). Indeed, if we remove $*$, without breaking the cycle containing $*$, we obtain a new cycle γ of length at least 1. So, δ becomes a derangement δ' on S (if γ has length at least 2) or a permutation σ on S with exactly one fixed point v (if γ has length 1). By removing v, then σ becomes a derangement δ'' on $S \setminus \{v\}$.

Viceversa, to recover δ from δ', we have to insert $*$ in one cycle of δ' and this can be done in n different ways. Moreover, to recover δ from δ'', we have only to choose one point of S, and this can be done in n different ways. All this implies recurrence (40). $\square$

These results can be easily extended to the species $\mathbf{Der}^{(m)}$ of m-derangements.



(a) The derivative $\mathbf{Der}'$. (b) The derivative $(\mathbf{Der}^{(m)})'$.

Fig. 2. Decompositions in the proof of Theorems 9 and 10.

Theorem 10. *We have the relation*

$$(41)\qquad \big(\mathbf{Der}^{(m)}\big)' = \sum_{i=0}^{m} \binom{m}{i} (\mathbf{Lin}-1)^{(i)} \cdot \mathbf{Der}^{(m-i)}$$

where $(\mathbf{Lin}-1)^{(i)}$ *is the* i*-derivative of the species* $\mathbf{Lin}-1$*, and the identity*

$$(42)\qquad d_{n+1}^{(m)} = \sum_{i=0}^{m}\sum_{j=0}^{n} \binom{m}{i}\binom{n}{j} (i+j)!\, d_{n-j}^{(m-i)} - d_n^{(m)} .$$

Moreover, we have the recurrence

$$(43)\qquad d_{n+1}^{(m)} = (m+n)\, d_n^{(m)} + n\, d_{n-1}^{(m)} .$$

Proof. Given a finite set S, we have $\big(\mathbf{Der}^{(m)}\big)'[S] = \mathbf{Der}^{(m)}[S+\{*\}]$. So, a structure of species $\big(\mathbf{Der}^{(m)}\big)'$ on S is a permutation δ of $(S+\{*\})+M$ without fixed points in $S+\{*\}$. If we remove $*$, then the cycle containing this point breaks in a non-empty linear order. So δ is equivalent to a non-empty linear order λ on a subset $I+J$, where $I \subseteq M$ and $J \subseteq S$, and to a permutation δ' on $(S \setminus J)+(M \setminus I)$ without fixed points in $S \setminus J$ (see Figure 2b). Hence

$$\begin{aligned}\big(\mathbf{Der}^{(m)}\big)'[S] &= \sum_{I\subseteq M}\sum_{J\subseteq S} (\mathbf{Lin}-1)[I+J] \times \mathbf{Der}^{(|M\setminus I|)}[S\setminus J]\\ &= \sum_{I\subseteq M}\sum_{J\subseteq S} (\mathbf{Lin}-1)^{(|I|)}[J] \times \mathbf{Der}^{(m-|I|)}[S\setminus J]\end{aligned}$$

$$= \sum_{I\subseteq M} ((\mathbf{Lin}-1)^{(|I|)} \cdot \mathbf{Der}^{(m-|I|)})[S]$$

$$= \left(\sum_{I\subseteq M} (\mathbf{Lin}-1)^{(|I|)} \cdot \mathbf{Der}^{(m-|I|)} \right) [S]$$

that is

$$\big(\mathbf{Der}^{(m)}\big)' = \sum_{I\subseteq M} (\mathbf{Lin}-1)^{(|I|)} \cdot \mathbf{Der}^{(m-|I|)} .$$

This is relation (41) and such a relation immediately implies identity (42) (just notice that $i+j \neq 0$, being λ a non-empty linear order).

This approach can be slightly modified to obtain recurrence (43). Indeed, if we just remove $*$, then we have a new cycle γ of length at least 1. So, δ reduces to an m-derangement δ' on S (if γ is not a 1-cycle in S) or to a permutation σ on $S+M$ with exactly one fixed point in S (if γ is a 1-cycle in S). Removing this fixed point, say v, σ becomes an m-derangement δ'' on $S \setminus \{v\}$.

Viceversa, to recover δ from δ', we have to insert $*$ in one cycle of δ' and this can be done in $m+n$ different ways. Moreover, to recover δ from δ'', we have to choose one point of S, and this can be done in n different ways. All this implies recurrence (43). □

The results of Theorems 9 and 10 can be extended to the generalized rencontres polynomials. Here, we only prove the next theorem, where recurrence (45) corresponds to recurrence (10) obtained in [**15**] by a formal argument.

Theorem 11. *We have the identity*

$$D^{(m)}_{n+1}(x) = (x-1)\, D^{(m)}_{n}(x) + \sum_{i=0}^{m}\sum_{j=0}^{n} \binom{m}{i}\binom{n}{j} (i+j)!\, D^{(m-i)}_{n-j}(x) \tag{44}$$

and the recurrence

$$D^{(m)}_{n+1}(x) = (x+m+n)\, D^{(m)}_{n}(x) - n(x-1) D^{(m)}_{n-1}(x) . \tag{45}$$

Proof. Given a finite set S, we have $\big(\mathbf{Ren}_x^{(m)}\big)'[S] = \mathbf{Ren}_x^{(m)}[S+\{*\}]$. So, a structure of species $\big(\mathbf{Ren}_x^{(m)}\big)'$ on S is a permutation σ on $(S+\{*\})+M$, where each fixed point in $S+\{*\}$ has weight x. If $*$ is a fixed point, then $*$ has weight x and σ reduces to a permutation σ_1 on $S+M$. Otherwise, if $*$ is not a fixed point, then, by removing $*$, we have that σ is equivalent to a non-empty linear order λ on a subset $I+J$, where $I\subseteq M$ and $J\subseteq S$, and to a permutation σ' on $(S\setminus J)+(M\setminus I)$. This implies formula (44).

Again, we can adapt this argument to deduce recurrence (45). Let σ be a permutation on $(S+\{*\})+M$, where every fixed points in $S+\{*\}$ has weight x. When we remove $*$, we have the following cases. (i) If $*$ is a fixed point, then $*$ has weight x and σ reduces to a permutation σ_1 on $S+M$. (ii) If $*$ is not a fixed point, but swaps with an element v of S, then σ reduces to a permutation σ_2 on $(S\setminus\{v\})+M$. (iii) If $*$ is not a fixed point and does not swap with an element of S, then the cycle containing this element reduces to a cycle γ which is never a 1-cycle in S, and so σ reduces to a permutation σ_3 in $S+M$.

Viceversa, we have the following cases. (i) To recover σ from σ_1, we have to add a new fixed point in S with weight x. (ii) To recover σ from σ_2, we have to choose one element of S, and this can be done in n different ways. (iii) To recover σ from σ_3, we have to insert $*$ in one cycle of σ_3, and this can be done in $m+n-k$ different ways, where k is the number of fixed points of σ_3 in S.

From this analysis, we have

$$\begin{aligned} D_{n+1}^{(m)}(x) &= x\,D_n^{(m)}(x)+nD_{n-1}^{(m)}(x)+\sum_{k=0}^{n}\binom{n}{k}(m+n-k)d_{n-k}^{(m)}x^k \\ &= x\,D_n^{(m)}(x)+nD_{n-1}^{(m)}(x)+(m+n)\,D_n^{(m)}(x)-\sum_{k=0}^{n}\binom{n}{k}kd_{n-k}^{(m)}x^k \\ &= (x+m+n)\,D_n^{(m)}(x)+nD_{n-1}^{(m)}(x)-x\frac{\mathrm{d}}{\mathrm{d}x}D_n^{(m)}(x)\,. \end{aligned}$$

By the Appell identity (37), we have at once recurrence (45). □

3.3 - *Properties of m-arrangements*

We conclude this section by establishing some basic relations between m-arrangements, m-permutations and m-derangements. Let $a_n^{(m)}$ be the *generalized arrangement numbers*, counting the m-arrangements of an n-set.

Theorem 12. *We have the relations*

$$\mathbf{Arr}^{(m)}=\mathbf{Exp}\cdot\mathbf{Perm}^{(m)} \tag{46}$$

$$\mathbf{Arr}^{(m)}=\mathbf{Exp}^2\cdot\mathbf{Der}^{(m)} \tag{47}$$

and the identities

$$a_n^{(m)}=\sum_{k=0}^{n}\binom{n}{k}(m+k)! \quad \textit{or} \quad (m+n)!=\sum_{k=0}^{n}\binom{n}{k}(-1)^{n-k}a_k^{(m)} \tag{48}$$

$$a_n^{(m)} = \sum_{k=0}^{n} \binom{n}{k} 2^{n-k} d_k^{(m)} \quad \textit{or} \quad d_n^{(m)} = \sum_{k=0}^{n} \binom{n}{k} (-2)^{n-k} a_k^{(m)} . \tag{49}$$

Proof. To give a structure of species $\mathbf{Arr}^{(m)}$ on a finite set S means to give a subset I of S, an m-permutation σ on I and the uniform structure on $S \setminus I$. This immediately implies relation (46). Then, this relation and relation (15) implies relation (47). Relations (46) and (47) immediately yield the first identities in (48) and (49), respectively. The other identities are simply the inverse of the previous ones. □

Theorem 13. *The numbers $a_n^{(m)}$ have exponential generating series*

$$a^{(m)}(t) = \sum_{n\geq 0} a_n^{(m)} \frac{t^n}{n!} = \frac{m!\, e^t}{(1-t)^{m+1}} . \tag{50}$$

Proof. By relation (46), we have

$$\begin{aligned}\mathbf{Card}(\mathbf{Arr}^{(m)};t) &= \mathbf{Card}(\mathbf{Exp}\cdot\mathbf{Perm}^{(m)};t)\\ &= \mathbf{Card}(\mathbf{Exp};t)\cdot\mathbf{Card}(\mathbf{Perm}^{(m)};t) .\end{aligned}$$

By identities (9) and (11), we obtain series (50). □

For the derivative of the species of arrangements, we have the following result.

Theorem 14. *We have the relation*

$$\mathbf{Arr}' = 2\mathbf{Arr} + (\mathbf{Lin} - 1)\cdot\mathbf{Arr} \tag{51}$$

and the identity

$$a_{n+1} = a_n + \sum_{k=0}^{n} \binom{n}{k} k!\, a_{n-k} . \tag{52}$$

Proof. By relation (47) with $m = 0$, we have $\mathbf{Arr} = \mathbf{Exp}^2 \cdot \mathbf{Der}$. By differentiating this relation by the Leibniz rule (valid also for species), we get

$$\mathbf{Arr}' = \big(\mathbf{Exp}^2\cdot\mathbf{Der}\big)' = 2\,\mathbf{Exp}^2\cdot\mathbf{Der} + \mathbf{Exp}^2\cdot\mathbf{Der}' = 2\,\mathbf{Arr} + \mathbf{Exp}^2\cdot\mathbf{Der}' .$$

Then, by relation (38), we have

$$\mathbf{Arr}' = 2\mathbf{Arr} + \mathbf{Exp}^2\cdot(\mathbf{Lin}-1)\cdot\mathbf{Der} = 2\mathbf{Arr} + (\mathbf{Lin}-1)\cdot\mathbf{Arr} .$$

This is relation (51), which immediately yields identity (52). □

Theorem 14 can be extended by using relations (47) and (41).

Theorem 15. *We have the relation*

$$(\mathbf{Arr}^{(m)})' = 2\mathbf{Arr}^{(m)} + \sum_{i=0}^{m} \binom{m}{i} (\mathbf{Lin} - 1)^{(i)} \cdot \mathbf{Arr}^{(m-i)} \tag{53}$$

and the identity

$$a_{n+1}^{(m)} = a_n^{(m)} + \sum_{i=0}^{m} \sum_{j=0}^{n} \binom{m}{i} \binom{n}{j} (i+j)!\, a_{n-j}^{(m-i)} \,. \tag{54}$$

Remark 16. The generalized arrangement numbers can be expressed by means of the generalized rencontres polynomials:

$$a_n^{(m)} = D_n^{(m)}(2) \,. \tag{55}$$

So, by replacing x by 2 in recurrences (10) and (11) obtained in [**15**] for the generalized rencontres polynomials, we immediately have the recurrences

$$\begin{aligned} a_{n+2}^{(m)} &= (m+n+3)\, a_{n+1}^{(m)} - (n+1)\, a_n^{(m)} \\ a_{n+1}^{(m+1)} &= (n+1)\, a_n^{(m+1)} + (m+1)\, a_{n+1}^{(m)} \,. \end{aligned}$$

Moreover, again for $x = 2$, identities (26) and (35) become

$$\sum_{k=0}^{n} \binom{n}{k} a_k^{(m)} d_{n-k} = \sum_{k=0}^{n} \binom{n}{k} a_k d_{n-k}^{(m)} \tag{56}$$

$$a_n^{(m)} = m! \sum_{k=0}^{n} \binom{n}{k} (m)_k\, a_{n-k} \,. \tag{57}$$

Similarly, identity (44) specializes in identity (54). Other identities of this kind can be obtained by specializing any identity involving the polynomials $D_n^{(m)}(x)$.

4 - An extension of the Clarke-Sved identity

There are some formulas relating the ordinary derangement numbers d_n and the Bell numbers b_n. For instance, we have the identities

$$\sum_{k=0}^{n} \begin{bmatrix} n+1 \\ k+1 \end{bmatrix} (-1)^k b_k = d_n \tag{58}$$

$$\sum_{k=0}^{n} \begin{Bmatrix} n+1 \\ k+1 \end{Bmatrix} (-1)^k d_k = b_n \tag{59}$$

where the coefficients $\begin{bmatrix} n \\ k \end{bmatrix}$ are the *Stirling numbers of the first kind* and the coefficients $\begin{Bmatrix} n \\ k \end{Bmatrix}$ are the *Stirling numbers of the second kind* [**38**, Section 6.1] (sequences A132393 and A008277 in [**65**]), or the identities

$$\sum_{k=0}^{n} \binom{n}{k} k^r d_{n-k} = n!\, b_r \qquad (r \le n) \tag{60}$$

$$\sum_{k=0}^{n} \binom{n}{k} k^r d_k = n! \sum_{k=0}^{r} \binom{r}{k} (-1)^k n^{r-k} b_k \qquad (r \le n)\,, \tag{61}$$

or the nice congruence [**68**]

$$\sum_{k=1}^{p-1} \frac{b_k}{(-s)^k} \equiv (-1)^{s-1} d_{s-1} \pmod p \tag{62}$$

valid for every positive integer s and any prime p not dividing s.

There are also formulas relating the Bell numbers and the generalized derangement numbers. For instance, the identities ((39) and (40) in [**15**])

$$\sum_{k=0}^{n} \begin{Bmatrix} n \\ k \end{Bmatrix} (-1)^k d_k^{(m)} = m! \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} (m+1)^{n-k} b_k \tag{63}$$

$$\sum_{k=0}^{n} \begin{Bmatrix} n+1 \\ k+1 \end{Bmatrix} (-1)^k d_k^{(m)} = m! \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} m^{n-k} b_k \tag{64}$$

have been obtained by using the formal technique of Sheffer matrices. By the same approach, we can extend formulas (58) and (59). Recall that the *r-Stirling numbers of the first kind* $\begin{bmatrix} n \\ k \end{bmatrix}_r$ and the *r-Stirling numbers of the second kind* $\begin{Bmatrix} n \\ k \end{Bmatrix}_r$ are defined [**11**] by the exponential generating series

$$\sum_{n\ge k} \begin{bmatrix} n \\ k \end{bmatrix}_r \frac{t^n}{n!} = \frac{1}{(1-t)^r} \frac{1}{k!} \left(\ln \frac{1}{1-t} \right)^k \tag{65}$$

$$\sum_{n\ge k} \begin{Bmatrix} n \\ k \end{Bmatrix}_r \frac{t^n}{n!} = \mathrm{e}^{rt} \frac{(\mathrm{e}^t - 1)^k}{k!}\,, \tag{66}$$

and that the *r-Bell numbers* [**49**] have exponential generating series

$$b^{(r)}(t) = \sum_{n\ge 0} b_n^{(r)} \frac{t^n}{n!} = \mathrm{e}^{rt} \mathrm{e}^{\mathrm{e}^t - 1}\,. \tag{67}$$

Theorem 17. *For every $m, r \in \mathbb{N}$, with $r \leq m$, we have the identities*

$$\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{m+1} (-1)^k\, b_k^{(r)} = \frac{d_n^{(m-r)}}{(m-r)!} \tag{68}$$

$$\sum_{k=0}^{n} \begin{Bmatrix} n \\ k \end{Bmatrix}_{m+1} (-1)^k\, d_k^{(m-r)} = (m-r)!\, b_n^{(r)}\,. \tag{69}$$

Proof. By series (65) and (67), we have the generating series

$$\sum_{n\geq 0} \left[\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{m+1} (-1)^k\, b_k^{(r)} \right] \frac{t^n}{n!} = \frac{b^{(r)}(\ln(1-t))}{(1-t)^{m+1}} = \frac{\mathrm{e}^{-t}}{(1-t)^{m-r+1}} = \frac{d^{(m-r)}(t)}{(m-r)!}\,.$$

This proves the first identity (68). Similarly, by series (17) and (66), we have the generating series

$$\sum_{n\geq 0} \left[\sum_{k=0}^{n} \begin{Bmatrix} n \\ k \end{Bmatrix}_{m+1} (-1)^k\, d_k^{(m-r)} \right] \frac{t^n}{n!}$$
$$= \mathrm{e}^{(m+1)t} d^{(m-r)}(-\mathrm{e}^t + 1) = (m-r)!\, \mathrm{e}^{rt} \mathrm{e}^{\mathrm{e}^t - 1} = (m-r)! b^{(r)}(t)\,.$$

This proves the second identity (69). □

Congruence (62) has been generalized to the exponential polynomials [**68**], and these congruences have been further generalized in [**67**] and then in [**50**] to the r-Bell numbers and to the r-Bell polynomials.

Here, we are interested in the Clarke-Sved identity (61), obtained in [**22**] and generalized in [**41**], and in the related identity (60). We give an extension of these formulas to the generalized derangement numbers and to the generalized rencontres polynomials.

Given an r-set R, let $\mathbf{Map}_R$ be the species of maps from R, i.e., the species defined by $\mathbf{Map}_R[S] = \{f : R \to S\}$, for every finite set S. Similarly, let $\mathbf{Sur}_R$ be the species of surjective maps from R. Then, let $\mathbf{F}_R^{(m)}$ be the species defined, for every finite set S, by

$$\mathbf{F}_R^{(m)}[S] = \{\, (f, \sigma) \in \mathbf{Map}_R[S] \times \mathbf{Perm}^{(m)}[S] \;:\; \mathrm{Im} f \subseteq S \cap \mathrm{Fix}(\sigma) \,\}\,.$$

The species $\mathbf{F}_R^{(m)}$ can be decomposed in two different ways, as proved in the following theorem.

Theorem 18. *We have the relations*

$$\mathbf{F}_R^{(m)} = \mathbf{Map}_R \cdot \mathbf{Der}^{(m)} \tag{70}$$

$$\mathbf{F}_R^{(m)} = \mathbf{Sur}_R \cdot \mathbf{Perm}^{(m)} . \tag{71}$$

Equivalently, we have the relation

$$\mathbf{Map}_R \cdot \mathbf{Der}^{(m)} = \mathbf{Sur}_R \cdot \mathbf{Perm}^{(m)} \tag{72}$$

and the identity

$$\sum_{k=0}^{n} \binom{n}{k} k^r d_{n-k}^{(m)} = m!n! \sum_{k=0}^{\min(n,r)} \left\{ {r \atop k} \right\} \binom{m+n-k}{m} . \tag{73}$$

Proof. To give a structure of species $\mathbf{F}_R^{(m)}$ on a finite set S means to give a map $f : R \to S$ and an m-permutation σ on S such that the image of f is contained in the set of fixed points of σ in S. This is equivalent to give a triple (I, g, δ), where I is a subset of S $(I = S \cap \mathrm{Fix}(\sigma))$, g is a map from R to I and δ is an m-derangement on $S \setminus I$. Hence, we have the relation

$$\mathbf{F}_R^{(m)}[S] = \sum_{I \subseteq S} \mathbf{Map}_R[I] \times \mathbf{Der}^{(m)}[S \setminus I]$$

which is equivalent to relation (70).

Equivalently, to give a structure (f, σ) of species $\mathbf{F}_R^{(m)}$ on a finite set S means to give a triple (I, g, σ'), where I is a subset of S such that $I \subseteq S \cap \mathrm{Fix}(\sigma)$ $(I = \mathrm{Im} f)$, g is a surjective map from R to I and σ' is an m-permutation of $S \setminus I$ (the restriction of σ to $(S \setminus I) + M$). This implies relation (71).

From relations (70) and (71) we have relation (72). Finally, by evaluating such a relation on an n-set, we have identity (73). □

For $m = 0$ and $r \le n$, identity (73) reduces to identity (60). Now, to obtain an extension of identity (61), we can proceed as follows. Given an r-set R, let $\mathbf{G}_R^{(m)}$ be the species defined, for every finite set S, by

$$\mathbf{G}_R^{(m)}[S] = \{ (f, \sigma) \in \mathbf{Map}_R[S] \times \mathbf{Perm}^{(m)}[S] \ : \ \mathrm{Im} f \subseteq S \setminus \mathrm{Fix}(\sigma) \} .$$

This species can be decomposed as follows.

Theorem 19. *We have the relation*

$$\mathbf{G}_R^{(m)} = \mathbf{Exp} \cdot (\mathbf{Map}_R \odot \mathbf{Der}^{(m)}) . \tag{74}$$

Proof. To give a structure of species $\mathbf{G}_R^{(m)}$ on a finite set S means to give a map $f : R \to S$ and an m-permutation σ on S such that the image of f is contained in the set of non-fixed points of σ belonging to S. This is equivalent to give a triple (I, g, δ), where I is a subset of S ($I = \mathrm{Fix}(\sigma)$), g is a map from R to $S \setminus I$ and δ is an m-derangement of $S \setminus I$ (the restriction of σ to $(S \setminus I) + M$). So, we have

$$\begin{aligned}\mathbf{G}_R^{(m)}[S] &= \sum_{I \subseteq S} \mathbf{Exp}[I] \times \mathbf{Map}_R[S \setminus I] \times \mathbf{Der}^{(m)}[S \setminus I] \\ &= \sum_{I \subseteq S} \mathbf{Exp}[I] \times (\mathbf{Map}_R \odot \mathbf{Der}^{(m)})[S \setminus I]\end{aligned}$$

from which we have relation (74). □

By Theorems 18 and 19, we have the following result.

Theorem 20. *We have the identity*

$$\sum_{k=0}^{n} \binom{n}{k} k^r d_k^{(m)} = m!n! \sum_{k=0}^{r} \binom{r}{k} (-1)^k n^{r-k} \sum_{i=0}^{\min(n,k)} \left\{ {k \atop i} \right\} \binom{m+n-i}{m}. \tag{75}$$

Proof. The left-hand side of identity (75) is an immediate consequence of relation (74) when it is evaluated on an n-set. To obtain the right-hand side of (75), we determine the size of $\mathbf{G}_R^{(m)}[S]$ by using the principle of inclusion-exclusion. Let S be an n-set and let σ be an m-permutation on S. Then, let $\mathbf{G}_{R,\sigma}^{(m)}[S] = \{f \in \mathbf{Map}_R[S] \ : \ (f, \sigma) \in \mathbf{G}_R^{(m)}[S]\}$. Let $R = \{x_1, x_2, \ldots, x_r\}$ and let $A_i = \{f \in \mathbf{Map}_R[S] \ : \ f(x_i) \in S \cap \mathrm{Fix}(\sigma)\}$. Let $[r] = \{1, 2, \ldots, r\}$. Then

$$|\mathbf{G}_{R,\sigma}^{(m)}[S]| = |A_1' \cap A_2' \cap \cdots \cap A_r'| = \sum_{I \subseteq [r]} (-1)^{|I|} \left| \bigcap_{i \in I} A_i \right| .$$

The elements of the set $\bigcap_{i \in I} A_i$ are the functions $f : R \to S$ mapping each element x_i, with $i \in I$, to a fixed point of σ belonging to S, and mapping each further element of R to an arbitrary element of S. This means that

$$\left| \bigcap_{i \in I} A_i \right| = |S \cap \mathrm{Fix}(\sigma)|^{|I|} \, n^{r-|I|} .$$

So, we have

$$|\mathbf{G}_{R,\sigma}^{(m)}[S]| = \sum_{I \subseteq [r]} (-1)^{|I|} |S \cap \mathrm{Fix}(\sigma)|^{|I|} n^{r-|I|}$$

$$= \sum_{k=0}^{r} \binom{r}{k} (-1)^k |S \cap \mathrm{Fix}(\sigma)|^k n^{r-k}$$

and

$$\begin{aligned} |\mathbf{G}_R^{(m)}[S]| &= \sum_{\sigma \in \mathbf{Perm}^{(m)}[S]} |\mathbf{G}_{R,\sigma}^{(m)}[S]| \\ &= \sum_{\sigma \in \mathbf{Perm}^{(m)}[S]} \sum_{k=0}^{r} \binom{r}{k} (-1)^k |S \cap \mathrm{Fix}(\sigma)|^k n^{r-k} \\ &= \sum_{k=0}^{r} \binom{r}{k} (-1)^k n^{r-k} \sum_{\sigma \in \mathbf{Perm}^{(m)}[S]} |S \cap \mathrm{Fix}(\sigma)|^k . \end{aligned}$$

By identity (73), we have

$$\begin{aligned} \sum_{\sigma \in \mathbf{Perm}^{(m)}[S]} |S \cap \mathrm{Fix}(\sigma)|^k &= \sum_{I \subseteq S} \sum_{\delta \in \mathbf{Der}^{(m)}[S \setminus I]} |I|^k \\ &= \sum_{i=0}^{n} \binom{n}{i} i^k d_{n-i}^{(m)} = m!n! \sum_{i=0}^{\min(n,k)} \left\{ {k \atop i} \right\} \binom{m+n-i}{m} . \end{aligned}$$

Consequently, we have

$$|\mathbf{G}_R^{(m)}[S]| = m!n! \sum_{k=0}^{r} \binom{r}{k} (-1)^k n^{r-k} \sum_{i=0}^{\min(n,k)} \left\{ {k \atop i} \right\} \binom{m+n-i}{m} .$$

This is the right-hand side of (75). □

For $m = 0$ and $r \leq n$, identity (75) reduces to identity (61).

Remark 21. The results obtained in Theorems 18, 19 and 20 can be generalized to the weighted case. Let $\mathbf{F}_{R,x}^{(m)}$ be the weighted species given by the species $\mathbf{F}_R^{(m)}$ where each fixed point in S has weight x. Let $\mathbf{Map}_{R,x}$ be the weighted species of maps from R where every element of the codomain has weight x. Similarly, let $\mathbf{Sur}_{R,x}$ be the weighted species of surjective maps from R where every element of the codomain has weight x. Then, the results obtained in Theorem 18 can be generalized to the relations

$$\mathbf{F}_{R,x}^{(m)} = \mathbf{Map}_{R,x} \cdot \mathbf{Der}^{(m)} , \qquad \mathbf{F}_{R,x}^{(m)} = \mathbf{Sur}_{R,x} \cdot \mathbf{Ren}_x^{(m)}$$

$$\mathbf{Map}_{R,x} \cdot \mathbf{Der}^{(m)} = \mathbf{Sur}_{R,x} \cdot \mathbf{Ren}_x^{(m)}$$

and identity (73) becomes the polynomial identity

$$\sum_{k=0}^{n}\binom{n}{k}k^r d_{n-k}^{(m)}x^k = \sum_{k=0}^{\min(n,r)}\left\{{r \atop k}\right\}\binom{n}{k}k!\,x^k D_{n-k}^{(m)}(x)\,. \tag{76}$$

Now, let $\mathbf{G}_{R,x}^{(m)}$ be the weighted species given by the species $\mathbf{G}_R^{(m)}$ where each fixed point in S has weight x. The relation obtained in Theorem 19 becomes

$$\mathbf{G}_{R,x}^{(m)} = \mathbf{Exp}_x \cdot (\mathbf{Map}_R \odot \mathbf{Der}^{(m)})\,. \tag{77}$$

Moreover, the proof of Theorem 20 can be easily extended to the weighted case. Just, notice that

$$w(\mathbf{G}_R^{(m)}[S]) = \sum_{k=0}^{r}\binom{r}{k}(-1)^k n^{r-k}\sum_{\sigma\in\mathbf{Perm}^{(m)}[S]}|S\cap\mathrm{Fix}(\sigma)|^k x^{|S\cap\mathrm{Fix}(\sigma)|}$$

and

$$\sum_{\sigma\in\mathbf{Perm}^{(m)}[S]}|S\cap\mathrm{Fix}(\sigma)|^k x^{|S\cap\mathrm{Fix}(\sigma)|} = \sum_{i=0}^{n}\binom{n}{i}i^k d_{n-i}^{(m)}x^i = \sum_{i=0}^{\min(n,k)}\left\{{k \atop i}\right\}\binom{n}{i}i!\,x^i D_{n-i}^{(m)}(x)\,.$$

So, in conclusion, identity (75) becomes

$$\sum_{k=0}^{n}\binom{n}{k}k^r d_k^{(m)}x^{n-k} = \sum_{k=0}^{r}\binom{r}{k}(-1)^k n^{r-k}\sum_{i=0}^{\min(n,k)}\left\{{k \atop i}\right\}\binom{n}{i}i!\,x^i D_{n-i}^{(m)}(x)\,. \tag{78}$$

Identity (73) can also be generalized in the following other way. To do that, we will use the differential operator $\Theta = t\frac{\mathrm{d}}{\mathrm{d}t}$, for which we have [**38**, p. 310]

$$\Theta^n = \sum_{k=0}^{n}\left\{{n \atop k}\right\}t^k D^k\,.$$

Furthermore, for an exponential series $a(t) = \sum_{n\geq 0}a_n\frac{t^n}{n!}$, we have

$$\Theta^m a(t) = \sum_{n\geq 0}n^m a_n\frac{t^n}{n!} \tag{79}$$

and, in particular, we have

$$\Theta^n\,\mathrm{e}^{\alpha t} = S_n(\alpha t)\,\mathrm{e}^{\alpha t} \tag{80}$$

Theorem 22. *We have the identity*

$$\text{(81)}\quad \sum_{k=0}^{n}\binom{n}{k}k^r(1-x)^k D_{n-k}^{(m)}(x)=m!n!\sum_{k=0}^{\min(n,r)}\left\{{r \atop k}\right\}\binom{m+n-k}{m}(1-x)^k\,.$$

Proof. By series (8) and formula (80), the exponential generating series of the left-hand side of identity (81) is

$$\begin{aligned}\sum_{n\geq 0}\left[\sum_{k=0}^{n}\binom{n}{k}k^r(1-x)^k D_{n-k}^{(m)}(x)\right]\frac{t^n}{n!} &= \sum_{n\geq 0}n^r(1-x)^n\frac{t^n}{n!}\cdot\sum_{n\geq 0}D_n^{(m)}(x)\frac{t^n}{n!}\\ &=(\Theta^r \mathrm{e}^{(1-x)t})\cdot D^{(m)}(x;t)=S_r((1-x)t)\,\mathrm{e}^{(1-x)t}\cdot\frac{m!\mathrm{e}^{(x-1)t}}{(1-t)^{m+1}}\\ &=S_r((1-x)t)\,\frac{m!}{(1-t)^{m+1}}=m!\sum_{k=0}^{r}\left\{{r \atop k}\right\}(1-x)^k\,\frac{t^k}{(1-t)^{m+1}}\\ &=m!\sum_{k=0}^{r}\left\{{r \atop k}\right\}(1-x)^k\sum_{n\geq k}\binom{n+m-k}{m}t^n\\ &=\sum_{n\geq 0}\left[m!n!\sum_{k=0}^{\min(n,r)}\left\{{r \atop k}\right\}\binom{n+m-k}{m}(1-x)^k\right]\frac{t^n}{n!}\,.\end{aligned}$$

Taking the coefficients of $\frac{t^n}{n!}$ in the first and last series, we get identity (81). □

Identities (76) and (81) can be generalized to the polynomials associated with the enriched partitions. Indeed, we have the following result.

Theorem 23. *Let* $\mathbf{F}$ *be a species with* $\mathbf{F}[\varnothing]=\varnothing$ *and* $|\mathbf{F}[\{*\}]|=1$*. Then, we have the identities*

$$\text{(82)}\qquad \sum_{k=0}^{n}\binom{n}{k}k^r\widetilde{p}_{n-k}^{\mathbf{F}}x^k=\sum_{k=0}^{\min(n,r)}\left\{{r \atop k}\right\}\binom{n}{k}k!x^k p_{n-k}^{\mathbf{F}}(x)$$

$$\text{(83)}\qquad \sum_{k=0}^{n}\binom{n}{k}k^r(1-x)^k p_{n-k}^{\mathbf{F}}(x)=\sum_{k=0}^{\min(n,r)}\left\{{r \atop k}\right\}\binom{n}{k}k!p_{n-k}^{\mathbf{F}}\cdot(1-x)^k\,.$$

Proof. By series (12) and (14) and by formulas (79) and (80), the exponential generating series of the left-hand side of identity (82) is

$$\sum_{n\geq 0}\left[\sum_{k=0}^{n}\binom{n}{k}k^r\widetilde{p}^{\mathbf{F}}_{n-k}x^k\right]\frac{t^n}{n!}=\sum_{n\geq 0}n^r x^n\frac{t^n}{n!}\cdot\sum_{n\geq 0}\widetilde{p}^{\mathbf{F}}_n\frac{t^n}{n!}=(\Theta^r \mathrm{e}^{xt})\cdot\widetilde{p}^{\mathbf{F}}_n(t)$$

$$=S_r(xt)\,\mathrm{e}^{xt}\cdot\mathrm{e}^{f(t)-t}=S_r(xt)\,p^{\mathbf{F}}(x;t)=\sum_{k=0}^{r}\left\{{r \atop k}\right\}x^k\,t^k p^{\mathbf{F}}(x;t)$$

$$=\sum_{k=0}^{r}\left\{{r \atop k}\right\}x^k\sum_{n\geq k}\binom{n}{k}k!p^{\mathbf{F}}_{n-k}(x)\frac{t^n}{n!}=\sum_{n\geq 0}\left[\sum_{k=0}^{\min(n,r)}\left\{{r \atop k}\right\}\binom{n}{k}k!x^k p^{\mathbf{F}}_{n-k}(x)\right]\frac{t^n}{n!}.$$

Taking the coefficients of $\frac{t^n}{n!}$ in the first and last series, we have identity (82).

Similarly, by series (14) and by formulas (79) and (80), the exponential generating series of the left-hand side of identity (83) is

$$\sum_{n\geq 0}\left[\sum_{k=0}^{n}\binom{n}{k}k^r(1-x)^k p^{\mathbf{F}}_{n-k}(x)\right]\frac{t^n}{n!}=\sum_{n\geq 0}n^r(1-x)^n\frac{t^n}{n!}\cdot\sum_{n\geq 0}p^{\mathbf{F}}_n(x)\frac{t^n}{n!}$$

$$=(\Theta^r\mathrm{e}^{(1-x)t})\cdot p^{\mathbf{F}}_n(x;t)=S_r((1-x)t)\,\mathrm{e}^{(1-x)t}\cdot\mathrm{e}^{f(t)}\,\mathrm{e}^{(x-1)t}$$

$$=S_r((1-x)t)\,\mathrm{e}^{f(t)}=S_r((1-x)t)\,p^{\mathbf{F}}(t)=\sum_{k=0}^{r}\left\{{r \atop k}\right\}(1-x)^k\,t^k p^{\mathbf{F}}(t)$$

$$=\sum_{k=0}^{r}\left\{{r \atop k}\right\}(1-x)^k\sum_{n\geq k}\binom{n}{k}k!p^{\mathbf{F}}_{n-k}\frac{t^n}{n!}=\sum_{n\geq 0}\left[\sum_{k=0}^{r}\left\{{r \atop k}\right\}\binom{n}{k}k!p^{\mathbf{F}}_{n-k}(1-x)^k\right]\frac{t^n}{n!}.$$

Taking the coefficients of $\frac{t^n}{n!}$ in the first and last series, we get identity (83). □

Finally, for the generalized arrangement numbers, we have the following result.

Theorem 24. *We have the identities*

$$\sum_{k=0}^{n}\binom{n}{k}2^k k^r d^{(m)}_{n-k}=\sum_{k=0}^{\min(n,r)}\left\{{r \atop k}\right\}\binom{n}{k}k!\,2^k a^{(m)}_{n-k}\tag{84}$$

$$\sum_{k=0}^{n}\binom{n}{k}2^{n-k}k^r d^{(m)}_{k}=\sum_{k=0}^{r}\binom{r}{k}(-1)^k n^{r-k}\sum_{i=0}^{\min(n,k)}\left\{{k \atop i}\right\}\binom{n}{i}i!\,2^i a^{(m)}_{n-i}\tag{85}$$

$$\sum_{k=0}^{n}\binom{n}{k}(-1)^k k^r a^{(m)}_{n-k}=m!n!\sum_{k=0}^{\min(n,r)}\left\{{r \atop k}\right\}\binom{m+n-k}{m}(-1)^k.\tag{86}$$

Proof. By formula (55), just set $x = 2$ in identities (76), (78) and (81). □

For $r = 0$, both identities (84) and (85) reduce to the first identity in (49).

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