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Partially complex ranks for real projective varieties

Abstract. Let $X(\mathbb{C}) \subset \mathbb{P}^r(\mathbb{C})$ be an integral non-degenerate variety defined over $\mathbb{R}$. For any $q \in \mathbb{P}^r(\mathbb{R})$ we study the existence of $S \subset X(\mathbb{C})$ with small cardinality, invariant for the complex conjugation and with q contained in the real linear space spanned by S. We discuss the advantages of these additive decompositions with respect to the $X(\mathbb{R})$-rank, i.e. the rank of q with respect to $X(\mathbb{R})$. We describe the case of hypersurfaces and Veronese varieties.

Keywords. Tensor rank, real tensor rank, real symmetric tensor rank, additive decomposition of polynomials, typical rank.

Mathematics Subject Classification: 14N05, 15A69.

1 - Introduction

In recent years a lot of effort is devoted to the study of secant varieties of projective varieties which are defined over $\mathbb{R}$ ([**4**, **7**, **9**, **10**, **11**, **23**, **24**]). As in [**7**] a *label* is a pair $(a, b) \in \mathbb{N}^2 \setminus \{(0, 0)\}$. The weight of a label (a, b) is the integer $2a + b$. To know a label it is sufficient to know its weight and one of its entries. Let $\sigma : \mathbb{P}^r(\mathbb{C}) \to \mathbb{P}^r(\mathbb{C})$ denote the complex conjugation. Let $X(\mathbb{C}) \subset \mathbb{P}^r(\mathbb{C})$ be an integral and non-degenerate complex projective variety. The pair consisting of $X(\mathbb{C})$ and the embedding $X(\mathbb{C}) \hookrightarrow \mathbb{P}^r(\mathbb{C})$ is defined over $\mathbb{R}$ if and only if $\sigma(X(\mathbb{C})) = X(\mathbb{C})$ (the reader may take the latter as the definition of a real embedded variety). Note that $\mathbb{P}^r(\mathbb{R}) = \{x \in \mathbb{P}^r(\mathbb{C}) \mid \sigma(x) = x\}$ and $X(\mathbb{R}) = X(\mathbb{C}) \cap \mathbb{P}^r(\mathbb{R}) = \{x \in X(\mathbb{C}) \mid \sigma(x) = x\}$.

Definition 1.1. A finite set $S \subset X(\mathbb{C})$, $S \neq \emptyset$, is said to have a *label* (resp. to have (a, b) as its label, resp. to have a label of weight k) if $\sigma(S) = S$ (resp. $\sigma(S) = S$, $b = |S \cap X(\mathbb{R})|$ and $|S| = 2a + b$, resp. $\sigma(S) = S$ and $|S| = k$).

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For any finite subset $S \subset X(\mathbb{C})$ let $\langle S\rangle_{\mathbb{C}}$ denote the minimal complex linear subspace of $\mathbb{P}^r(\mathbb{C})$ containing S. Set $\langle S\rangle_{\mathbb{R}} := \langle S\rangle_{\mathbb{C}} \cap \mathbb{P}^r(\mathbb{R})$. Note the $\sigma(\langle S\rangle_{\mathbb{C}}) = \langle S\rangle_{\mathbb{C}}$ if $\sigma(S) = S$. Thus $\dim_{\mathbb{C}}\langle S\rangle_{\mathbb{C}} = \dim_{\mathbb{R}}\langle S\rangle_{\mathbb{R}}$ if $\sigma(S) = S$. For any integer $k > 0$ the k-secant variety $\sigma_k(X(\mathbb{C}))$ of $X(\mathbb{C})$ is the closure in $\mathbb{P}^r(\mathbb{C})$ of all linear spaces $\langle S\rangle_{\mathbb{C}}$ with $S \subset X(\mathbb{C})$ and $|S| = k$. If $X(\mathbb{C}) \subset \mathbb{P}^r(\mathbb{C})$ is defined over $\mathbb{R}$, then the variety $\sigma_k(X(\mathbb{C}))$ is defined over $\mathbb{R}$ and $\sigma_k(X(\mathbb{C})) \cap \mathbb{P}^r(\mathbb{R})$ is the set $\sigma_k(X(\mathbb{C}))(\mathbb{R})$ of the real points of $\sigma_k(X(\mathbb{C}))$. Quite often $\sigma_k(X(\mathbb{C}))(\mathbb{R})$ is much bigger than the set (sometimes called $\sigma_k(X(\mathbb{R}))$ which is the closure in $\mathbb{P}^r(\mathbb{R})$ of all points with $X(\mathbb{R})$-rank k (see Observation 4 in Remark 2.2, Example 2.3 and Remark 1.3). To see in a concrete example how to use labels we give the following example.

Proposition 1.2. *Let $X(\mathbb{C}) \subset \mathbb{P}^{n+1}(\mathbb{C})$ be an integral hypersurface defined over $\mathbb{R}$. Every $q \in \mathbb{P}^{n+1}(\mathbb{R})$ has either $(1, 0)$ or $(0, 2)$ or $(0, 1)$ as a label.*

Remark 1.3. Take $X(\mathbb{C})$ as in Proposition 1.2. The proposition shows that each $q \in \mathbb{P}^{n+1}(\mathbb{R})$ has a label with weight equal to $r_{X(\mathbb{C})}(q)$. In general not all points of $\mathbb{P}^{r+1}(\mathbb{R}) \setminus X(\mathbb{R})$ have real rank 2, even when $X(\mathbb{R})$ is Zariski dense in $X(\mathbb{C})$ ([**23, 24**]). Compare the very interesting geometry in [**9, 11, 23, 24**] with the very simple description given by Proposition 1.2.

To show that labels help to get cheap additive decompositions of "general" points of $\mathbb{P}^r(\mathbb{R})$ (or of $\sigma_k(X(\mathbb{C}))(\mathbb{R})$), i.e. all except a part with smaller real dimension (and in particular smaller Hausdorff dimension) we prove the following Theorem 1.4. Let $X_{\mathrm{reg}}(\mathbb{C})$ denote the set of smooth points of $X(\mathbb{C})$. Set $X_{\mathrm{reg}}(\mathbb{R}) := X_{\mathrm{reg}}(\mathbb{C}) \cap X(\mathbb{R})$. Since $X(\mathbb{C})$ is an integral variety, $X_{\mathrm{reg}}(\mathbb{C})$ is a connected complex manifold and $\dim X_{\mathrm{reg}}(\mathbb{C}) = \dim X(\mathbb{C})$. If $X_{\mathrm{reg}}(\mathbb{R}) \neq \emptyset$, then $X_{\mathrm{reg}}(\mathbb{R})$ is a real analytic manifold of real dimension $\dim X_{\mathrm{reg}}(\mathbb{C})$, which is Zariski dense in $X_{\mathrm{reg}}(\mathbb{C})$ (Remark 2.1), but closed in $X_{\mathrm{reg}}(\mathbb{C})$ in the euclidean topology. We recall that generic uniqueness holds for a secant variety $\sigma_k(X(\mathbb{C}))$ if for a general $q \in \sigma_k(X(\mathbb{C}))$ there is a unique set $S \subset X(\mathbb{C})$ such that $|S| = k$ and $q \in \langle S\rangle$; here "general" means "for all q in a non-empty Zariski open subset of $\sigma_k(X(\mathbb{C}))$", but to test this condition it is sufficient to prove it for all points q of a non-empty open subset of $\sigma_k(X(\mathbb{C}))$ for the euclidean topology.

Theorem 1.4. *Let $X(\mathbb{C}) \subset \mathbb{P}^r(\mathbb{C})$ be an integral and non-degenerate variety defined over $\mathbb{R}$ and such that $X_{\mathrm{reg}}(\mathbb{R}) \neq \emptyset$. Let g be the minimal integer such that $\sigma_g(X(\mathbb{C})) = \mathbb{P}^r(\mathbb{C})$. Assume that generic uniqueness holds for $\sigma_{g-1}(X(\mathbb{C}))$. Then there exists an open subset $U \subset \mathbb{P}^r(\mathbb{R})$ (for the Zariski topology) such that $\dim \mathbb{P}^r(\mathbb{R}) \setminus U \le r - 1$ (and in particular $\mathbb{P}^r(\mathbb{R}) \setminus U$ has measure 0 and contains no euclidean open subset) and each $q \in U$ has a label of weight $g + 1$.*

The main application of Theorem 1.4 is when $X(\mathbb{C})$ is a Veronese embedding of $\mathbb{P}^n$. Let $\nu_d : \mathbb{P}^n(\mathbb{C}) \to \mathbb{P}^r$, $r := -1+\binom{n+d}{n}$, be the order d Veronese embedding of $\mathbb{P}^n$, i.e. the embedding associated to the vector space $S^d\mathbb{C}^{n+1}$ of all homogeneous degree d complex polynomials in $n+1$ variables. Set $X(\mathbb{C}) := \nu_d(\mathbb{P}^n(\mathbb{C}))$. In this case for any finite $S \subset \mathbb{P}^n(\mathbb{C})$ the linear space $\langle \nu_d(S)\rangle_{\mathbb{C}}$ is the $\mathbb{C}$-linear span of all ℓ_p^d, where $p \in S$ and ℓ_p is the linear form whose equivalence class corresponds to $p \in \mathbb{P}^n(\mathbb{C})$. Both $X(\mathbb{C})$ and the embedding $X(\mathbb{C}) \hookrightarrow \mathbb{P}^n(\mathbb{C})$ are defined over $\mathbb{R}$. If d is odd and $S \subset \mathbb{P}^n(\mathbb{R})$ (with $|S|$ minimal) the interpretation above gives the definition of $X(\mathbb{R})$-rank. If d is even we allow signs, i.e. to define the $X(\mathbb{R})$-rank or real rank of $f \in S^d\mathbb{R}^{n+1}$ we allow as addenda $\pm\ell_p^d$. As an immediate consequence of Theorem 1.4 and taking $r := g-1$ in [**18**, Theorem 1.1] we get the following result.

Theorem 1.5. *Fix integers $n \geq 1$ and $d \geq 3$. Assume $(n,d) \notin \{(2,6), (3,4), (5,3)\}$. Set $r := -1+\binom{n+d}{n}$ and $g := \lceil (r+1)/(n+1)\rceil$. Let $X(\mathbb{C}) \subset \mathbb{P}^r(\mathbb{C})$ be the order d Veronese embedding. Let $\mathcal{B}$ be the set of $x \in \mathbb{P}^n(\mathbb{R})$ without a label of weight $g+1$. Then $\mathcal{B}$ is contained in a real hypersurface of $\mathbb{P}^r(\mathbb{R})$ and in particular it has measure 0 and contains no non-zero euclidean open subset.*

We stress that our bounds do not depend on the real algebraic geometry of $X(\mathbb{R})$ or $X(\mathbb{C})$. In our opinion they give a very strong minimal way (minimal number of real parameters) to represents almost all $\mathbb{P}^r(\mathbb{R})$ using finitely many charts with minimal number of parameters. The number of needed charts is upper bounded in an explicit way. Easy examples (the real rational normal curve) shows that the bound is sharp.

If $\dim X(\mathbb{C}) = 1$ generic uniqueness always holds over $\mathbb{C}$ for any submaximal secant variety ([**16**, Corollary 2.8]) and hence as an easy consequence of Theorem 1.4 we get the following result.

Proposition 1.6. *Let $X(\mathbb{C}) \subset \mathbb{P}^r(\mathbb{C})$ be an integral and non-degenerate curve defined over $\mathbb{R}$ and with $X(\mathbb{R})$ infinite. The set of all $p \in \mathbb{P}^r(\mathbb{R})$ without a label of weight $\lfloor (r+5)/2\rfloor$ has real dimension $< r$.*

We extend Theorem 1.4 to arbitrary secant varieties in the following way.

Theorem 1.7. *Let $X(\mathbb{C}) \subset \mathbb{P}^r(\mathbb{C})$ be an integral and non-degenerate variety defined over $\mathbb{R}$ and such that $X_{\mathrm{reg}}(\mathbb{R}) \neq \emptyset$. Fix an integer $k \geq 2$ such that generic uniqueness holds for $\sigma_{k-1}(X(\mathbb{C}))$. Then there exists a Zariski open subset $U \subset \sigma_k(X(\mathbb{C}))(\mathbb{R})$ such that $\dim_{\mathbb{R}} \sigma_k(X(\mathbb{C}))(\mathbb{R})\setminus U \leq \dim_{\mathbb{R}} \sigma_k(X(\mathbb{C}))(\mathbb{R})-1$ and each $q \in U$ has a label of weight $k+1$.*

We recall that if $\dim X(\mathbb{C}) = n$ and $\sigma_{k+n-2}(X(\mathbb{C}))$ has dimension $(k+n-2)(n+1)-1$, then generic uniqueness holds for $\sigma_{k-1}(X(\mathbb{C}))$ ([**8**, Theorem 5.1]).

Thus Theorem 1.7 may be applied in a huge number of cases ([**2**,**5**,**13**,**17**,**18**,**22**]; see [**8**] for a partial, but longer, list). In particular it applies to several Segre embeddings of multiprojective spaces, i.e. to tensors, and of Segre-Veronese embeddings of a multiprojective space, i.e. to partially symmetric tensors. We mention that if $a > b > 0$ and generic uniqueness holds for $\sigma_a(X(\mathbb{C}))$, then it holds for $\sigma_b(X(\mathbb{C}))$ ([**8**, Proposition 2.3]) and in particular it holds for the Veronese embeddings of a projective space, except for a very short list. The drawback of Theorem 1.7 is that for these examples it covers just one case not covered in [**7**]. Indeed if generic uniqueness holds for $\sigma_k(X(\mathbb{C}))$ it is sufficient to use labels of weight k by [**7**, Theorem 3] and all of them are necessary, since different labels cover disjoint non-empty euclidean open subsets.

Remark 1.8. We explain here why we think that our approach is computationally promising. We just consider $\mathbb{P}^r(\mathbb{R})$, but Theorem 1.7 may be used for the real parts of arbitrary k-secants varieties. Let g be the first positive integer such that $\sigma_g(X(\mathbb{C})) = \mathbb{P}^r(\mathbb{C})$. For each $q \in \mathbb{P}^r(\mathbb{R})$ let $r_{X(\mathbb{R})}(q)$ denote the smallest cardinality of a set $S \subset X(\mathbb{R})$ such that $q \in \langle S\rangle_{\mathbb{R}}$ ([**3**,**4**,**7**,**9**,**10**,**11**,**19**,**24**]). An integer x is a *typical rank* of $X(\mathbb{R})$ if there is a non-empty open subset Δ_x for the euclidean topology such that $r_{X(\mathbb{R})}(q) = x$ for all $q \in \Delta_x$. Let E be the set of all typical ranks of $X(\mathbb{R})$. Taking each Δ_x maximal (i.e. taking the interior for the euclidean topology of the set of all points with $X(\mathbb{R})$-rank x) the set $\mathbb{P}^r(\mathbb{R}) \setminus \cup_{x\in E}\Delta_x$ has measure zero and usually the game is to handle all $q \in \cup_{x\in E}\Delta_x$. For each $x \in E$ to describe each of Δ_x we need a subset of $X(\mathbb{R})$ with cardinality x and hence roughly speaking we need xn real parameters, where $n := \dim_{\mathbb{C}} X(\mathbb{C})$ (unless of course we know that a much smaller subset of $X(\mathbb{R})^x$ will do, but each case needs a detailed study to restrict the subsets of $X(\mathbb{R})$ with cardinality x which must be used to give Δ_x). Call g' the maximal typical rank. Roughly speaking, we need ng' real parameter for at least one euclidean open subset of $\mathbb{P}^r(\mathbb{R})$. The integer g is the minimal typical rank. It is known that $g' \le 2g$ and often $g' \le 2g-1$ or $g' \le 2g-2$ in some cases ([**9**,**12**]). In Theorem 1.4 (and hence in its corollaries like Theorem 1.5 and Proposition 1.6), we need to take labels of weight $g+1$. There are $\lfloor (g+1)/2\rfloor$ labels of weight $g+1$ (resp. g). For any label (a, b) of weight $2a + b$ we only need $(2a + b)n$ real parameters. Indeed we take b distinct points $p_1, \dots, p_b \in X(\mathbb{R})$ and a sufficiently general points $q_1, \dots, q_a \in X(\mathbb{C}) \setminus X(\mathbb{R})$. The choice of $q_1, \dots, q_a$ depends on $2an$ real parameters. Then we add the complex conjugate of $q_1, \dots, q_a$. On the contrary, to use typical ranks we sometimes need almost $2gn$ real parameters (e.g. for the typical ranks of bivariate polynomial of the bivariate polynomials, a problem settled by G. Blekherman in [**10**]).

Using the labels we may also handle real varieties $X(\mathbb{C}) \subset \mathbb{P}^r(\mathbb{R})$ with $X_{\mathrm{reg}}(\mathbb{R}) = \emptyset$ (even with $X(\mathbb{R}) = \emptyset$) for which the notion of typical rank (or even the real rank if $X(\mathbb{R}) = \emptyset$) is not very interesting. We only consider labels $(a, 0)$, i.e. corresponding to a points of $X(\mathbb{C}) \setminus X(\mathbb{R})$ and their complex conjugates. We prove the following result.

Theorem 1.9. *Assume $X_{\mathrm{reg}}(\mathbb{R}) = \emptyset$. Let g be the generic rank of $X(\mathbb{C})$. Fix an even integer k such that $4 \le k \le g$. Assume that generic uniqueness holds for $\sigma_{k-2}(X(\mathbb{C}))$. Then $\sigma_k(X(\mathbb{C}))(\mathbb{R})$ is Zariski dense in $\sigma_k(X(\mathbb{C}))$ and there is an open subset U of $\sigma_k(X(\mathbb{C}))(\mathbb{R})$ such that $\sigma_k(X(\mathbb{C}))(\mathbb{R}) \setminus U$ is a real hypersurface of $\sigma_k(X(\mathbb{C}))(\mathbb{R})$ and for each $q \in U$ there is $S \subset X_{\mathrm{reg}}(\mathbb{C})$ with $\sigma(S) = S$, $|S| = k + 2$ and S of label $(\frac{k+2}{2}, 0)$.*

Remark 1.10. Since we allow labels with weight $> r_{X(\mathbb{C})}(q)$ to cover $q \in \mathbb{P}^r(\mathbb{R})$, the open subsets covering almost all $\mathbb{P}^r(\mathbb{R})$ in Theorems 1.4 and 1.5 and Proposition 1.6 (or almost all $\sigma_k(X(\mathbb{C}))(\mathbb{R})$ in Theorem 1.7) may overlap. In Theorem 1.9 we have a unique label, but even in this case many $q \in \mathbb{P}^r(\mathbb{R})$ may be in the linear span of different σ-invariant sets with label $(\frac{k+2}{2}, 0)$.

Remark 1.11. As in [**7**] the interested reader may work over an arbitrary real closed field $\mathcal{R}$ instead of the field $\mathbb{R}$, just using $\mathcal{C} := \mathcal{R}(i)$ (an algebraic closure of $\mathcal{R}$) instead of $\mathbb{C}$.

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2 - The proofs

Remark 2.1. Let $X(\mathbb{C})$ be an integral projective variety defined over $\mathbb{R}$. Set $n := \dim_{\mathbb{C}} X(\mathbb{C})$. The set $X(\mathbb{R})$ is a closed subset of the compact r-dimensional complex space $\mathbb{P}^r(\mathbb{R})$. Let $X_{\mathrm{reg}}(\mathbb{C})$ be the set of all smooth points of $X(\mathbb{C})$. Since $X(\mathbb{C})$ is an integral variety, $X_{\mathrm{reg}}(\mathbb{C})$ is a non-empty connected smooth n-dimensional complex manifold. Let $M(\mathbb{C})$ be a smooth and connected n-dimensional complex manifold defined over $\mathbb{R}$ and let $M(\mathbb{R})$ denote the set of all its real points. Assume $M(\mathbb{R}) \neq \emptyset$ and fix $p \in M(\mathbb{R})$. Call $\mathcal{A}_{M(\mathbb{C}),p}$ the local ring at p of the complex analytic manifold $M(\mathbb{C})$. Obviously $M(\mathbb{R})$ is closed in $M(\mathbb{C})$. It is well-known that $M(\mathbb{R})$ is a smooth (but not necessarily connected) differential manifold of pure dimension n ([**21**, Ch. II, Corollary 4.11]). Now assume that $M(\mathbb{C})$ is the complex analytic manifold associated to a complex algebraic variety defined over $\mathbb{R}$ and that the real structure of the complex analytic manifold $M(\mathbb{C})$ is the one induced by the assumption that this algebraic variety is defined over $\mathbb{R}$. The set $M(\mathbb{R})$ is the same in the algebraic and in the analytic category. We claim that $M(\mathbb{R})$ is dense for the Zariski topology of the algebraic variety $M(\mathbb{C})$. To prove the claim it is sufficient to prove that if $f \in \mathcal{A}_{M(\mathbb{C}),p}$ and if f vanishes at the germ of $M(\mathbb{R})$ at p, then

$f = 0$. Suppose for instance that $M(\mathbb{C})$ (as an analytic manifold) contains a euclidean neighborhood U of $0 \in \mathbb{C}$ with the usual complex conjugation involution with $\mathbb{R}$ and write $\mathcal{A}_{\mathbb{C}^n,0}$ for the local ring of convergent power series in n complex variables. If $f \in \mathcal{A}_{\mathbb{C}^n,0}$ vanishes at each point of $U \cap \mathbb{R}^n$, then all coefficients of the power series of f vanishes (this is easily reduced to the case $n = 1$ in which it is obvious that if $f \neq 0$, then f has at most countably many zeros). By [**21**, Proposition 4.2] the involution $\sigma : M(\mathbb{C}) \to M(\mathbb{C})$ is uniquely determined by the set of its fixed points (when non-empty).

Remark 2.2. Let $X(\mathbb{C}) \subset \mathbb{P}^r(\mathbb{C})$ be an integral and non-degenerate variety defined over $\mathbb{R}$ (no assumption on $X(\mathbb{R})$ and/or $X_{\mathrm{reg}}(\mathbb{R})$). Fix an even integer $k \geq 2$. Just to simplify the language we assume $k \leq r+1$, so that k general points of $X(\mathbb{C})$ are linearly independent. For any positive integer t set $X(\mathbb{C})^{(t)} := \{(a_1, \dots, a_t) \in X(\mathbb{C})^t \mid a_i \neq a_j \text{ for all } i \neq j\}$. Since $X(\mathbb{C})^{(t)}$ is a non-empty Zariski open subset of $X(\mathbb{C})^t$, it is an irreducible quasi-projective variety. It is invariant for the complex conjugation σ and the fixed point set is just the set $X(\mathbb{R})^t \cap X(\mathbb{C})^{(t)}$. The symmetric group S_t of permutations of $\{1, \dots, t\}$ acts on $X(\mathbb{C})^{(t)}$ and this action commutes with the complex conjugation. Since $X(\mathbb{C})^{(t)}$ is a quasi-projective variety, there is a quasi-projective variety $X(\mathbb{C})_{(t)}$ parametrizing the orbits for this action of S_t ([**25**, p. 111]) and $X(\mathbb{C})_{(t)}$ is defined over $\mathbb{R}$. $X(\mathbb{C})_{(t)}$ parametrizes the subsets $S \subset X(\mathbb{C})$ with cardinality t. An element $S \in X(\mathbb{C})_{(t)}$ is an element of $X(\mathbb{C})_{(t)}(\mathbb{R})$ if and only if $\sigma(S) = S$. Thus if $q \in X(\mathbb{C}) \setminus X(\mathbb{R})$, the set $\{q, \sigma(q)\} \in X(\mathbb{C})_{(2)}(\mathbb{R})$, while $\{q, \sigma(q)\} \nsubseteq X(\mathbb{R})$. Thus for any even $t \geq 2$ there are many $S \in X(\mathbb{C})_{(t)}(\mathbb{R})$ which are not contained in $X(\mathbb{R})$. If $X(\mathbb{R}) \neq \emptyset$ this is also true for all odd integers $t \geq 3$.

Claim 1. For all even integers $t \geq 2$ the set $X(\mathbb{C})_{(t)}(\mathbb{R})$ is Zariski dense in $X(\mathbb{C})_{(t)}$.

Proof of Claim 1. Since $X(\mathbb{C})_{(t)}$ is an irreducible complex variety, to prove Claim 1 it is sufficient to prove that the Zariski closure of $X(\mathbb{C})_{(t)}(\mathbb{R})$ in $X(\mathbb{C})_{(t)}$ contains a non-empty euclidean open subset. First assume $t = 2$, take some $q \in X_{\mathrm{reg}}(\mathbb{C}) \setminus X_{\mathrm{reg}}(\mathbb{R})$ and a euclidean neighborhood U of $q \in X_{\mathrm{reg}}(\mathbb{C}) \setminus X_{\mathrm{reg}}(\mathbb{R})$. All points $\{q, \sigma(q)\}$, $q \in U$, are contained in $X(\mathbb{C})_{(2)}(\mathbb{R})$ and their union give a euclidean neighborhood of $\{q, \sigma(q)\}$ in $X(\mathbb{C})_{(2)}$. Now assume $t \geq 4$ and set $b := t/2$. We take $\{q_1, \dots, q_b\} \in X(\mathbb{C})_{(b)}$ with the additional restriction that $q_i \neq \sigma(q_j)$ for any i, j and use all these points $\{q_1, \dots, q_b, \sigma(q_1), \dots, \sigma(q_b) \in X(\mathbb{C})_{(t)}(\mathbb{R})$.

Observation 1. If $X_{\mathrm{reg}}(\mathbb{R}) \neq \emptyset$ for all integers $t \geq 1$ the set $X(\mathbb{C})_{(t)}(\mathbb{R})$ is Zariski dense in $X(\mathbb{C})_{(t)}$.

Claim 2. The set $\sigma_k(X(\mathbb{C}))(\mathbb{R})$ is Zariski dense in $\sigma_k(X(\mathbb{C}))$.

P r o o f o f C l a i m 2. For all integers $t \geq 1$ set

$$X(\mathbb{C})_{\langle t\rangle} := \{S \in X(\mathbb{C})_{(t)} \mid S \textit{ is linearly independent}\}.$$

$X(\mathbb{C})_{\langle t\rangle)}$ is a non-empty Zariski open subset of $X(\mathbb{C})_{(t)}$ defined over $\mathbb{R}$ and $X(\mathbb{C})_{\langle t\rangle}(\mathbb{R}) = X(\mathbb{C})_{\langle t\rangle} \cap X(\mathbb{C})_{(t)}(\mathbb{R})$. Thus $X(\mathbb{C})_{\langle t\rangle}(\mathbb{R})$ is Zariski dense in $X(\mathbb{C})_{\langle t\rangle}$. For each $S \in X(\mathbb{C})_{\langle t\rangle}(\mathbb{R})$ the linear space $\langle S\rangle_{\mathbb{C}}$ is defined over $\mathbb{R}$ and hence $\langle S\rangle_{\mathbb{C}} \cap \mathbb{P}^r(\mathbb{R})$ is a real projective space of dimension $t-1$. Since k is even, Claim 1 shows that the union of these real projective spaces is Zariski dense in $\sigma_k(X(\mathbb{C}))$.

O b s e r v a t i o n 2. Take $X(\mathbb{C})$ with $X(\mathbb{R}) = \emptyset$. Claim 2 gives that $\sigma_2(X(\mathbb{C}))(\mathbb{R})$ is large. For instance we may take as $X(\mathbb{C})$ a smooth plane conic $X(\mathbb{C}) \subset \mathbb{P}^2(\mathbb{C})$ defined over $\mathbb{R}$ but with no real point (the smooth conic $x_0^2+x_1^2+x_2^2=0$). We get $\sigma_2(X(\mathbb{C})) = \mathbb{P}^2(\mathbb{R})$. Proposition 1.2 shows that each $q \in \mathbb{P}^2(\mathbb{R})$ has $(1,0)$ as a label.

O b s e r v a t i o n 3. If $X_{\mathrm{reg}}(\mathbb{R}) \neq \emptyset$, then $\sigma_x(X(\mathbb{C}))(\mathbb{R})$ is Zariski dense in $\sigma_x(X(\mathbb{C}))$ also for odd x. In many cases with $x \geq 2$ it is larger than the union of all linear spaces $\langle S\rangle_{\mathbb{R}}$ with $S \subset X(\mathbb{R})$ and $\sharp(S) \leq x$.

O b s e r v a t i o n 4. Fix an integer $x \geq 2$ and assume that each subset of $X(\mathbb{C})$ with cardinality $2x$ is linearly independent. This is sufficient to get that each $q \in \mathbb{P}^r(\mathbb{C})$ with $r_{X(\mathbb{C})}(q) = x$ is in the linear span of a unique set $S \subset X(\mathbb{C})$ with cardinality x. Thus $q \in \mathbb{P}^r(\mathbb{R})$ is and only if $\sigma(S) = S$. Since $x \geq 2$, many such points q are not in the linear span of the union of x points of $X(\mathbb{C})$.

E x a m p l e 2.3. Let $X(\mathbb{C}) \subset \mathbb{P}^3(\mathbb{C})$ be the rational normal curve. Let $\tau(X(\mathbb{C}))$ be its tangent develobale. By Sylvester's theorem ([**20**]) a point $q \in \mathbb{P}^3(\mathbb{C})$ has $r_{X(\mathbb{C})}(q) = 2$ if and only if $q \in \mathbb{P}^3(\mathbb{C}) \setminus \tau(X(\mathbb{C}))$. Fix $q \in \mathbb{P}^3(\mathbb{R}) \setminus \tau(X(\mathbb{C})) \cap \mathbb{P}^3(\mathbb{R})$ and fix $S \subset X(\mathbb{C})$ such that $q \in \langle S\rangle_{\mathbb{C}}$ and $|S| = 2$. Since any 4 points of $X(\mathbb{C})$ are linearly independent. Thus S is unique. Since $\sigma(q) = q$ we get $\sigma(S) = S$. Thus S has a label. Call U (resp. U') the set of all $q \in \mathbb{P}^3(\mathbb{R}) \setminus \tau(X(\mathbb{C})) \cap \mathbb{P}^3(\mathbb{R})$ such that S has label $(2,0)$ (resp. $(0,1)$). We have $U \cap U' = \emptyset$, $U \cup U' = \mathbb{P}^3(\mathbb{R}) \setminus \tau(X(\mathbb{C})) \cap \mathbb{P}^3(\mathbb{R})$ and U and U' are semialgebraic sets of real dimension 3.

P r o o f o f P r o p o s i t i o n 1.2. If $q \in X(\mathbb{R})$, then $(1,0)$ is a label for q. Fix $q \in \mathbb{P}^{n+1}(\mathbb{R}) \setminus X(\mathbb{R})$. Let $T(\mathbb{C})$ be the set of all lines $L \subset \mathbb{P}^{n+1}(\mathbb{C})$ containing o. The set $T(\mathbb{C})$ is an n-dimensional complex space. Since $q \in \mathbb{P}^{n+1}(\mathbb{R})$, $T(\mathbb{C})$ is the complexification of a real n-dimensional projective space $T(\mathbb{R})$. For any $x \in \mathbb{P}^{n+1}(\mathbb{R}) \setminus \{q\}$ the lines spanned by $\{q, x\}$ is an element of $T(\mathbb{R})$. Since $q \notin X(\mathbb{C})$, no element of $T(\mathbb{C})$ is contained in $X(\mathbb{C})$. The set of all $L \in T(\mathbb{C})$ tangent to $X(\mathbb{C})$ is a proper closed algebraic subset $\Delta(\mathbb{C}) \subsetneq T(\mathbb{C})$ defined over $\mathbb{R}$. Since $X_{\mathrm{reg}}(\mathbb{R}) \neq \emptyset$ and each element of $T(\mathbb{C})$ meets $X(\mathbb{C})$, there is $o \in X_{\mathrm{reg}}(\mathbb{R})$ such that the line L spanned by $\{q, o\}$ is transversal to $X(\mathbb{C})$. Set $d := \deg(X(\mathbb{C}))$ and $S := L \cap X(\mathbb{C})$. Since $L \in T(\mathbb{R})$, we have $\sigma(S) = S$. Since $L \notin \Delta$, we have $|S| = d$. Since $\sigma(S) = S$, there is a set $S' \subseteq S$ such that $\sigma(S') = S'$ and $|S'| = 2$. The line L is spanned by S'. The set S' has either label $(1, 0)$ or label $(0, 2)$. $\square$

Proof of Theorem 1.4. Set $n := \dim X(\mathbb{C})$. Since generic uniqueness holds for the secant variety $\sigma_{g-1}(X(\mathbb{C}))$, $\sigma_{g-1}(X(\mathbb{C}))$ is not defective, i.e. it has dimension $(n+1)(g-1)-1$. Let $\mathcal{V} \subset \sigma_{g-1}(X(\mathbb{C}))(\mathbb{R})$ be the set of all points with a label of weight $g-1$. Since generic uniqueness holds for $\sigma_{g-1}(X(\mathbb{C}))$, the set $\sigma_{g-1}(X(\mathbb{C})) \setminus \mathcal{V}$ has real dimension $< (n+1)(g-1)$ ([**7**, Theorem 3]), i.e., since it is a semialgebraic set, it does not contain a non-empty open subset of $\mathbb{P}^r(\mathbb{R})$ for the euclidean topology. A Zariski dense subset of $\mathbb{P}^r(\mathbb{C})$ is obtained taking the union of all $\langle\{x, y\}\rangle_{\mathbb{C}}$ with $x \in \mathcal{V}$ and $y \in X(\mathbb{C}) \setminus X(\mathbb{R})$ (Claims 1 and 2 of Remark 2.2). Thus we get a Zariski dense subset of $\mathbb{P}^r(\mathbb{C})$ taking the union of all complex linear spaces $\langle\{x, y, \sigma(y)\}\rangle_{\mathbb{C}}$. The complex linear space $\langle\{x, y, \sigma(y)\}\rangle_{\mathbb{C}}$ is the linear span of σ-invariant set, i.e. a finite set with a label: if x has (a, b) as one of its label, then we may take $(a+1, b)$ as the label. $\square$

Remark 2.4. The proof of Theorem 1.4 shows that we may omit the label $(0, g+1)$.

Proof of Theorem 1.5. By [**18**, Theorem 1.1] generic uniqueness holds for $\sigma_{g-1}(X(\mathbb{C}))$, with the exceptions listed in the statement of Theorem 1.5. $\square$

Proof of Proposition 1.6. Since $X(\mathbb{C})$ is a curve, its secant varieties are non-degenerate ([**1**, Corollary 1.5], [**15**, Remark 3.1 (i)]). Thus if r is odd we have $\sigma_{(r+1)/2}(X(\mathbb{C})) = \mathbb{P}^r(\mathbb{C})$, while if r is even we have $\sigma_{(r+2)/2}(X(\mathbb{C})) = \mathbb{P}^r(\mathbb{C})$ and $\sigma_{r/2}(X(\mathbb{C}))$ is a hypersurface of $\mathbb{P}^r(\mathbb{C})$. Thus in the set-up of Theorem 1.4 we have $g = \lfloor (r+2)/2 \rfloor$. By [**16**, Corollary 2.8] generic uniqueness holds for $\sigma_{\lfloor r/2 \rfloor}(X(\mathbb{C}))$. Apply Theorem 1.4. $\square$

Proof of Theorem 1.7. Use that $\sigma_k(X(\mathbb{C}))$ is the join of $\sigma_{k-1}(X(\mathbb{C}))$ and $X(\mathbb{C})$. $\square$

Proof of Theorem 1.9. Let W be a non-empty Zariski open subset of $\sigma_{k-2}(X(\mathbb{C}))$ such that for each $q \in W$ there is a unique $S_q \subset X(\mathbb{C})$ with $|S_q| = k-2$ and $q \in \langle S_q \rangle$. The uniqueness of S_q gives $\sigma(S_q) = S_q$ if $q \in W \cap \mathbb{P}^r(\mathbb{R})$. Restricting if necessary W we may assume that all S_q are contained in X_{reg}. Thus $S_q \cap X(\mathbb{R}) = \emptyset$ for all q. Thus if q has a label, then the label is of type $(\frac{k}{2}-1, 0)$. Then we take the join with 2 copies of $X(\mathbb{C})$,i.e, to some $q \in W$ we associate all sets $S_q \cup \{y, z, \sigma(y), \sigma(z)\} \subset X(\mathbb{C})$ with $y, z \in$

$X(\mathbb{C}) \setminus X(\mathbb{R})$, $|\{y, z, \sigma(y), \sigma(z)\}| = 4$ and $S_q \cap \{y, z, \sigma(y), \sigma(z)\} = \emptyset$. Note that $S_q \cup \{y, z, \sigma(y), \sigma(z)\}$ is a σ-invariant set with label $(\frac{k}{2} + 1, 0)$. $\square$

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