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Existence results for fractional differential inclusion with nonlocal boundary conditions

Abstract. This paper deals with the existence of solutions for nonlinear singular higher fractional differential inclusions supplemented with multi-point boundary conditions. Firstly, we investigate it for L^1-Carathéodory convex, compact valued, multifunctions. Secondly, we investigate it for the case of not necessarily convex valued multifunctions via some conditions by applying Schaefer's fixed point theorem combined with the selection theorem due to Bressan and Colombo. Finally, we investigate it for nonconvex valued multifunctions via a fixed point theorem for multivalued maps due to Covitz and Nadler. Two illustrative examples are presented at the end of the paper to illustrate the validity of our results.

Keywords. Fractional differential inclusion, Existence, Nonlocal boundary, Fixed point theorem.

Mathematics Subject Classification: 47H10, 26A33, 34A08.

1 - Introduction

Due to the fact that the tools of fractional calculus have numerous applications in various disciplines of science and engineering such as physics, mechanics, chemistry, biology, etc, the subject of fractional differential equations has gained a considerable attention by a great deal of researchers, such as in [**25**,**26**]. There have been many papers and books dealing with theoretical development of fractional calculus, the solutions or positive solutions of boundary value problems for nonlinear differential equations and fractional differential inclusions,

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for examples and details see [**34**] and papers [**4**, **15**]. We quote also that realistic problems arising from economics, optimal control, stochastic analysis can be modelled as differential inclusion.

The study of fractional differential inclusions was initiated by EL-Sayad and Ibrahim [**18**]. Also, recently, several qualitative results for fractional differential inclusion were obtained in [**22**, **30**, **31**, **32**, **33**] and the references therein.

On the other hand boundary value problems with local and nonlocal boundary conditions constitute a very interesting and important class of problems. They include two, three and multi-point boundary value problems. The existence and multiplicity of positive solutions for such problems have received a great deal of attention. To identify a few, we refer the reader to [**9**, **10**, **12**, **17**].

In 2015 Alsulami et al. [**6**] studied the existence of solutions of the following nonlinear third-order ordinary differential inclusion with multi-strip boundary conditions

$$\begin{cases} u^{(3)}(t) \in F(t, u(t)), \ t \in (0,1), \\ u(0) = 0, \ u'(0) = 0, \\ u(1) = \sum\limits_{i=1}^{n-2} \alpha_i \int_{\zeta_i}^{\eta_i} u(s)\, ds, \\ 0 < \zeta_i < \eta_i < 1, \ i = 1, 2, \ldots, n-2, \ n \geq 3. \end{cases}$$

In 2017, Resapour et al. [**35**] investigated a Caputo fractional inclusion with integral boundary condition for the following problem

$$\begin{cases} {}^{c}D^{\alpha} u(t) \in F\left(t, u(t), {}^{c}D^{\beta} u(t), u'(t)\right), \\ u(0) + u'(0) + {}^{c}D^{\beta} u(0) = \int_0^{\eta} u(s)\, ds, \\ u(1) + u'(1) + {}^{c}D^{\beta} u(1) = \int_0^{\nu} u(s)\, ds, \end{cases}$$

where $1 < \alpha \leq 2$, $\eta, \nu, \beta \in (0,1)$, $F : [0,1] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to 2^{\mathbb{R}}$ is a compact valued multifunction and ${}^{c}D^{\alpha}$ denotes the Caputo fractional derivative of order α.

In 2018, Bouteraa and Benaicha [**11**] studied the existence of solutions for the Caputo fractional differential inclusion

$${}^{c}D^{\alpha} u(t) \in F\left(t, u(t), u'(t)\right), \quad t \in J = [0,1]$$

subject to three-point boundary conditions

$$\begin{cases} \beta u(0) + \gamma u(1) = u(\eta), \\ u(0) = \int_0^{\eta} u(s)\, ds, \\ \beta {}^{c}D^{p} u(0) + \gamma {}^{c}D^{p} u(1) = {}^{c}D^{p} u(\eta), \end{cases}$$

where $2 < \alpha \leq 3,\ 1 < p \leq 2,\ 0 < \eta < 1,\ \beta, \gamma \in \mathbb{R}^+$, $F : [0,1] \times \mathbb{R} \times \mathbb{R} \to 2^{\mathbb{R}}$ is a compact valued multifunction and ${}^cD^{\alpha}$ denotes the Caputo fractional derivative of order α.

In this paper, we are interested in the existence of solutions for the nonlinear fractional differential inclusion

$$-D_{0^+}^{\alpha} u(t) \in F(t, u(t)), \quad t \in (0,1), \tag{1.1}$$

subject to the boundary conditions

$$u^{(i)}(0) = 0,\ i \in \{0, 1, \ldots, n-2\},\ D_{0^+}^{\beta} u(1) = \sum_{j=1}^{p} a_j D_{0^+}^{\beta} u(\eta_j), \tag{1.2}$$

where $D_{0^+}^{\alpha}$, $D_{0^+}^{\beta}$ are the standard Riemann-Liouville fractional derivative of order

$$\alpha \in (n-1, n],\ \beta \in [1, n-2]\ for\, n \in \mathbb{N}^*\ and\ n \geq 3,$$

$F \in C((0,1) \times \mathbb{R}, \mathbb{R})$ is allowed to be singular at $t = 0$ and/or $t = 1$ and $a_j \in \mathbb{R}^+,\ j = 1, 2, \ldots, p,\ 0 < \eta_1 < \eta_2 < \ldots < \eta_p < 1,\ for\, p \in \mathbb{N}^+$.

The current paper is organized as follows. In Section 2, we introduce some definitions and preliminary results that will be used in the remainder of the paper. In Section 3, we present existence results for the problem $(1.1) - (1.2)$ when the right-hand side is a convex as well as a non-convex compact multifunction. In the first result we consider the case when the right hand side has convex values, and prove an existence result via the nonlinear alternative for Kakutani maps [**21**]. In the second result, we shall combine the Schaefer's fixed point theorem with a selection theorem due to Bressan and Colombo for lower semi-continuous multivalued maps with nonempty closed and decomposable values, while in the third result, we shall use the fixed point theorem for contraction multivalued maps due to Covitz and Nadler. Our results are based on the fixed point theorem contraction multivalued maps due to Govitz and Nadler [**13**] and we give two examples to illustrate our results.

2 - Preliminaries

In this section, first, we introduce some necessary definitions and lemmas of fractional calculus to facilitate the analysis of the problem $(1.1) - (1.2)$. These details can be found in the recent literature; see [**25, 26, 29**] and the references therein.

Let $AC^i([0,1], \mathbb{R})$ denote the space of $i - times$ differentiable functions $u : [0,1] \to \mathbb{R}$ whose $i - th$ derivative $u^{(i)}$ is absolutely continuous.

Definition 2.1. Let $\alpha > 0$, $n-1 < \alpha < n$, $n = [\alpha]+1$ and $u \in AC^n([0,\infty),\mathbb{R})$.

The Caputo derivative of fractional order α for the function $u : [0,+\infty) \to \mathbb{R}$ is defined by

$$^cD^{\alpha}u(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} u^{(n)}(s)\, ds.$$

The Riemann-Liouville fractional derivative order α for the function $u : [0,+\infty) \to \mathbb{R}$ is defined by

$$D_{0^+}^{\alpha}u(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_0^t (t-s)^{n-\alpha-1} u(s)\, ds, \quad t > 0,$$

provided that the right hand side is pointwise defined in $(0,\infty)$ and the function $\Gamma : (0,\infty) \to \mathbb{R}$, defined by

$$\Gamma(u) = \int_0^{\infty} t^{u-1} e^{-t} dt,$$

is called Euler's gamma function.

Definition 2.2. The Riemann-Liouville fractional integral of order $\alpha > 0$ of a function $u : (0,\infty) \to \mathbb{R}$ is given by

$$I^{\alpha}u(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} u(s)\, ds, \quad t > 0,$$

provided that the right hand side is pointwise defined in $(0,\infty)$.

We recall in the following lemma some properties involving Riemann-Liouville fractional integral and Riemann-Liouville fractional derivative or Caputo fractional derivative which are need in Lemma 3.1.

Lemma 2.3 ([**3**, Prop. 4.3], [**4**]). *Let $\alpha, \beta \geq 0$ and $u \in L^1(0,1)$. Then the next formulas hold.*

(i) $\left(D^{\beta} I^{\alpha} u\right)(t) = I^{\alpha-\beta} u(t)$,

(ii) $(D^{\alpha} I^{\alpha} u)(t) = u(t)$, *where D^{α} and D^{β} represents Riemann-Liouville's or Caputo's fractional derivative of order α and β respectively.*

Lemma 2.4 ([**21**]). *Let $\alpha > 0$ and $y \in L^1(0,1)$. Then, the general solution of the fractional differential equation $D^{\alpha}_{0+}u(t) + y(t) = 0,\ 0 < t < 1$ is given by*

$$u(t) = -\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} y(s)\,ds + c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \cdots + c_n t^{\alpha-n}, \quad 0 < t < 1,$$

where $c_0, c_1, ..., c_{n-1}$ are real constants and $n = [\alpha] + 1$.

Next, we will present notations, definitions and preliminary facts from multivalued analysis which are used throughout this paper. For more details on the multivalued maps, see the book of Aubin and Celina [**2**], Demling [**16**], Gorniewicz [**20**] and Hu and Papageorgiou [**23**], see also [**2**, **13**, **14**, **27**, **28**].

Here $(C[0,1], \mathbb{R})$ denotes the Banach space of all continuous functions from $[0,1]$ into $\mathbb{R}$ with the norm $\|u\| = \sup\{|u(t)| : for\ all\ t \in [0,1]\}$, $L^1([0,1], \mathbb{R})$, the Banach space of measurable functions $u : [0,1] \to \mathbb{R}$ which are Lebesgue integrable, normed by $\|u\|_{L^1} = \int_0^1 |u(t)|\,dt$.

Let (X, d) be a metric space induced from the normed space $(X, \|\cdot\|)$. We denote

$$P_0(X) = \{A \in P(X) : A \neq \phi\},$$
$$P_b(X) = \{A \in P_0(X) : A\ is\ bounded\},$$
$$P_{cl}(X) = \{A \in P_0(X) : A\ is\ closed\},$$
$$P_{cp}(X) = \{A \in P_0(X) : A\ is\ compact\},$$
$$P_{cp,cv}(X) = \{A \in P_0(X) : A\ is\ compact\ and\ convex\},$$
$$P_{b,cl}(X) = \{A \in P_0(X) : A\ is\ closed\ and\ bounded\},$$

where $P(X)$ is the family of all subsets of X.

2.1 - *The upper semi-continuous case and the Carathéodory case*

Definition 2.5. A multivalued map $G : X \to P(X)$.

(1) $G(u)$ is convex (closed) valued if $G(u)$ is convex (closed) for all $u \in X$,

(2) is bounded on bounded sets if $G(B) = \underset{u \in B}{\cup} G(u)$ is bounded in X for all $B \in P_b(X)$ i.e. $\sup_{u \in B}\{\sup\{|v|, v \in G(u)\}\} < \infty$,

(3) is called upper semi-continuous (u.s.c) on X if for each $u_0 \in X$, the set $G(u_0)$ is a nonempty closed subset of X and if for each open set N of X containing $G(u_0)$ there exists an open neighborhood N_0 of u_0 such that $G(N_0) \subseteq N$,

(4) is said to be completely continuous if $G(B)$ is relatively compact for every $B \in P_b(X)$,

(5) has a fixed point if there is $u \in X$ such that $u \in G(u)$. The fixed point set of the multivalued operator G will be denote by Fix G.

Remark 2.6 ([**16**, Prop. 1.2]). It is well known that, if the multivalued map G is completely continuous with nonempty compact values, then G is u.s.c if and only if G has closed graph i.e., $u_n \to u$, $v_n \to v$, $v_n \in G(u_n)$ imply $v \in G(u)$.

Definition 2.7. A multivalued map $G : [0,1] \to P_{cl}(\mathbb{R})$ is said to be measurable if for every $y \in \mathbb{R}$ the function

$$t \longmapsto d(y, G(t)) = \inf \{\|y - z\| \,:\, z \in G(t)\},$$

is measurable.

Definition 2.8. A multivalued maps $F : [0,1] \times \mathbb{R} \to 2^{\mathbb{R}}$ is said to be Carathéodory if,

(i) $t \longmapsto F(t,u)$ is measurable for all $u \in \mathbb{R}$,

(ii) $u \longmapsto F(t,u)$ is upper semi-continuous for almost all $t \in (0,1)$. Furthermore a Carathéodory function is called L^1-Carathéodory if,

(iii) for each $\rho > 0$, there exists $\phi_\rho \in L^1([0,1], \mathbb{R}^+)$ such that

$$\|F(t,u)\| = \sup \{|v|\,,\, v \in F(t,u)\} \leq \phi_\rho(t)\,,\ a.e.\ in\ [0,1]\,,$$

for all $|u| < \rho$.

Definition 2.9. Let Y be a nonempty closed subset of a Banach space E and $G : Y \to P_{cl}(E)$ be a multivalued operator with nonempty closed values.

(i) G is said to be lower semi-continuous (l.s.c) if the set $\{x \in X \,:\, G(x) \cap U \neq \phi\}$ is open for any open set U in E.

(ii) G has a fixed point if there is $x \in Y$ such that $x \in G(x)$.

For each $u \in (C[0,1], \mathbb{R})$, define the set of selection of F by

$$S_{F,u} = \{v \in AC([0,1], \mathbb{R}) : v \in F(t, u(t)),\ for\ almost\ all\ t \in [0,1]\}.$$

For $P(X) = 2^X$, consider the Pompeiu-Hausdorff metric (see [**8**]) $H_d : 2^X \times 2^X \to [0, \infty)$ given by

$$H_d(A, B) = \max\left\{\sup_{a\in A} d(a, B), \sup_{b\in B} d(b, A)\right\},$$

where $d(a, B) = \inf_{b\in B} d(a, b)$ and $d(b, A) = \inf_{a\in A} d(a, b)$. Then $(P_{b,cl}(X), H_d)$ is a metric space and $(P_{cl}(X), H_d)$ is a generalized metric space see [**8**].

Lemma 2.10 ([**27**]). *Let X be a Banach space. $F : [0,1] \times X \to P_{cp,cv}(X)$ an L^1-Carathéodory multifunction and Θ a linear continuous mapping from $L^1([0,1], X)$ to $C([0,1], X)$. Then the operator $(\Theta \circ S_F)(u) = \Theta(S_{F,u})$ is a closed graph operator in $C([0,1], X) \times C([0,1], X)$.*

Lemma 2.11 (Nonlinear alternative for Kakutani maps [**21**]). *Let E be a Banach space. C a closed convex subset of E, U an open subset of C and $0 \in U$. Suppose that $F : \overline{U} \to P_{cp,cv}(C)$ is an upper semi-continuous compact map, where $P_{cp,cv}(C)$ denotes the family of nonempty, compact convex subset of C. Then either F has a fixed point in $\overline{U}$ or there exist $u \in \partial U$ and $\lambda \in (0,1)$ such that $u \in \lambda F(U)$.*

Lemma 2.12 ([**2**, Lemma 1.1.9]). *Let $\{K_n\}_{n\in\mathbb{N}} \subset K \subset X$ be a sequence of subsets where K is a compact subset of a separable Banach space X. Then*

$$\overline{co}\left(\limsup_{n\to\infty} K_n\right) = \bigcap_{n\geq 1} \overline{co}\left\{\bigcup_{n\geq 1} K_n\right\},$$

where $\overline{co}A$ refers to the closure of the convex hull of A.

Lemma 2.13 ([**2**, Lemma 1.4.13]). *Let X and Y two metric spaces. If $G : X \to P_{cp}(Y)$ is upper semi-continuous, then for each $x_0 \in X$,*

$$\limsup_{x\to x_0} G(x) = G(x_0).$$

Definition 2.14. Let X be a Banach space. A sequence $\{x_n\}_{n\in\mathbb{N}} \subset L^1([a,b], X)$ is said to be semi-compact if

(i) it is integrably bounded i.e., there exists $q \in L^1([a,b], \mathbb{R}^+)$ such that

$$|x_n(t)|_E \leq q(t),\ for\ a.e.\ t \in [a,b]\ and\ every\ n \in \mathbb{N},$$

(ii) the image sequence $\{x_n\}_{n\in\mathbb{N}}$ is relatively compact in E for $a.e.$ $t \in [a, b]$.

The following important result follows from Dunford-Pettis theorem (see, [**24**, Prop. 4.2.1]).

Lemma 2.15. *Every semi-compact sequence $L^1([a,b], X)$ is weakly compact in the space $L^1([a,b], X)$.*

When the nonlinearity takes convex values, Mazur's Lemma may be useful:

Lemma 2.16 ([**28**, Thm. 21.4]). *Let E be a normed space and $\{x_n\}_{n\in\mathbb{N}} \subset E$ a sequence weakly converging to a limit $x \in E$. Then there exists a sequence of convex combinations $y_m = \sum\limits_{k=1}^{m} \alpha_{mk} x_k$ with $\alpha_{mk} > 0$, for $k = 1, 2, \ldots, m$ and $\sum\limits_{k=1}^{m} \alpha_{mk} x_k = 1$ which coverges strongly to x.*

3 - Existence results

Let $X = \{u : u \in C([0,1], \mathbb{R})\}$ endowed with the norm defined by $\|u\| = \sup\limits_{t\in[0,1]} |u(t)|$ such that $\|u\| < \infty$. Then $(X, \|.\|)$ is a Banach space.

Lemma 3.1. *Let $\sum\limits_{j=1}^{p} a_j \eta_j^{\alpha-\beta-1} \in [0,1)$, $\alpha \in (n-1, n]$, $\beta \in [1, n-2]$, $n \geq 3$ and $y \in (C[0,1])$. Then the solution of the fractional boundary value problem*

$$\left\{\begin{array}{l} D_{0+}^{\alpha} u(t) + y(t) = 0, \\ u^{(i)}(0) = 0, \ i \in \{0, 1, \ldots, n-2\}, \\ D_{0+}^{\beta} u(1) = \sum\limits_{j=1}^{p} a_j D_{0+}^{\beta} u(\eta_j), \end{array}\right. \tag{3.1}$$

is given by

$$u(t) = \int_0^1 G(t,s)\, y(s)\, ds, \tag{3.2}$$

where

$$G(t,s) = g(t,s) + \frac{t^{\alpha-1}}{d} \sum_{j=1}^{p} a_j h(\eta_j, s), \tag{3.3}$$

$$(3.4)\qquad g(t,s)=\frac{1}{\Gamma(\alpha)}\begin{cases} t^{\alpha-1}(1-s)^{\alpha-\beta-1}-(t-s)^{\alpha-1}, & 0\le s\le t\le 1,\\ t^{\alpha-1}(1-s)^{\alpha-\beta-1}, & 0\le t\le s\le 1,\end{cases}$$

(3.5)

$$h(t,s)=\frac{1}{\Gamma(\alpha)}\begin{cases} t^{\alpha-\beta-1}(1-s)^{\alpha-\beta-1}-(t-s)^{\alpha-\beta-1}, & 0\le s\le t\le 1,\\ t^{\alpha-\beta-1}(1-s)^{\alpha-\beta-1}, & 0\le t\le s\le 1,\end{cases}$$

and $d=1-\sum_{j=1}^{p}a_j\eta_j^{\alpha-\beta-1}$.

Proof. By using Lemma 2.2, the solution for the above equation is

$$u(t)=-\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}y(s)\,ds+c_1t^{\alpha-1}+c_2t^{\alpha-2}+\cdots+c_nt^{\alpha-n},$$

where $c_1,c_2...,c_n$ are arbitrary real constants. By $u(0)=0$, we have $c_n=0$. Then

$$(3.6)\quad u(t)=-\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}y(s)\,ds+c_1t^{\alpha-1}+c_2t^{\alpha-2}+\cdots+c_{n-1}t^{\alpha-n+1}.$$

Differentiating (3.6), we have

$$u'(t)=\frac{1-\alpha}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-2}y(s)\,ds+(\alpha-1)c_1t^{\alpha-2}$$
$$+(\alpha-2)c_2t^{\alpha-3}+\cdots+(\alpha-n+1)c_{n-1}t^{\alpha-n}.$$

By $u'(0)=0$, we have $c_{n-1}=0$.

Similarly, we get $c_2=c_3...=c_{n-2}=0$. Hence

$$(3.7)\qquad u(t)=-\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}y(s)\,ds+c_1t^{\alpha-1}.$$

By (3.7) and Lemma 2.1, we get

$$D_{0+}^{\beta}u(t)=\frac{1}{\Gamma(\alpha-\beta)}\left[c_1\Gamma(\alpha)t^{\alpha-\beta-1}-\int_0^t(t-s)^{\alpha-\beta-1}y(s)\,ds\right],$$

this and by $D_{0+}^{\beta}u(1)=\sum_{j=1}^{p}a_jD_{0+}^{\beta}u(\eta_j)$, we have

$$c_1=\frac{1}{d\Gamma(\alpha)}\left[\int_0^1(1-s)^{\alpha-\beta-1}y(s)\,ds-\sum_{j=1}^{p}a_j\int_0^{\eta_j}(\eta_j-s)^{\alpha-\beta-1}y(s)\,ds\right].$$

Substituting c_1 into (3.7), we see that the unique solution of the problem (3.1) is

$$\begin{aligned}
u(t)&=\frac{t^{\alpha-1}}{d\Gamma(\alpha)}\left[\int_0^1(1-s)^{\alpha-\beta-1}y(s)\,ds-\sum_{j=1}^{p}a_j\int_0^{\eta_j}(\eta_j-s)^{\alpha-\beta-1}y(s)\,ds\right]\\
&\quad-\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}y(s)\,ds\\
&=\frac{1}{\Gamma(\alpha)}\left[\int_0^t\left[t^{\alpha-1}(1-s)^{\alpha-\beta-1}-(t-s)^{\alpha-1}\right]y(s)\,ds\right.\\
&\quad+\int_t^1t^{\alpha-1}(1-s)^{\alpha-\beta-1}y(s)\,ds+\frac{1-d}{d}\int_0^1t^{\alpha-1}(1-s)^{\alpha-\beta-1}y(s)\,ds\\
&\quad\left.-\frac{t^{\alpha-1}}{d}\sum_{j=1}^{p}a_j\int_0^{\eta_j}(\eta_j-s)^{\alpha-\beta-1}y(s)\,ds\right]\\
&=\int_0^1g(t,s)y(s)\,ds+\frac{t^{\alpha-1}}{d}\sum_{j=1}^{p}a_j\left[\int_{\eta_j}^1\eta_j^{\alpha-\beta-1}(1-s)^{\alpha-\beta-1}y(s)\,ds\right.\\
&\quad\left.+\int_0^{\eta_j}\left[\eta_j^{\alpha-\beta-1}(1-s)^{\alpha-\beta-1}-(\eta_j-s)^{\alpha-\beta-1}\right]y(s)\,ds\right]\\
&=\int_0^1g(t,s)y(s)\,ds+\frac{t^{\alpha-1}}{d}\sum_{j=1}^{p}a_j\int_0^1h(\eta_j,s)y(s)\,ds\\
&=\int_0^1G(t,s)y(s)\,ds.
\end{aligned}$$

The proof is completed. □

Lemma 3.2. *Let* $\sum_{j=1}^{p} a_j \eta_j^{\alpha-\beta-1} \in [0,1)$, $\alpha \in (n-1,n]$, $\beta \in [1,n-2]$, $n \geq 3$. *Then, the Green functions* $G(t,s)$ *defined by* (3.3) *have the following properties:*

(i) *The function* $G(t,s) \geq 0$ *are continuous on* $[0,1] \times [0,1]$ *and* $G(t,s) > 0$ *for all* $t,\ s \in (0,1)$,

(ii) $\max_{t\in[0,1]} G(t,s) = G(1,s)$, *for all* $s \in [0,1]$, *where*

$$G(1,s) = g(1,s) + \frac{1}{d}\sum_{j=1}^{p} a_j h(\eta_j, s) \leq \frac{(1-s)^{\alpha-\beta-1}}{d\Gamma(\alpha)}. \tag{3.8}$$

Proof. The proof is evident, we omit it. □

In order to investigate the problem (1.1) – (1.2) we shall provide an application of Lemma 2.11.

Theorem 3.3. *Suppose that the Carathéodory multifunction* $F : J \times \mathbb{R} \to P_{cp,cv}(\mathbb{R})$ *has nonempty, compact, convex values and satisfies*

(H_1) *There exist a continuous nondecreasing function* $\psi : [0,\infty) \to (0,\infty)$ *and a continuous function* $p \in L^1(J, \mathbb{R}^+)$ *such that*

$$\|F(t,u(t))\| = \sup\{|v| : v \in F(t,u(t))\} \leq p(t)\,\psi(\|u\|),$$

for all a.e. $t \in J$ *and* $u \in \mathbb{R}$.

(H_2) *There exist* ψ *and* p *as in* (H_1) *and* $R > 0$ *such that*

$$R\left[\psi(\|u\|)\left(\int_0^1 G(1,s)\,p(s)\,ds\right)\right]^{-1} > 1. \tag{3.9}$$

Then the inclusion problem (1.1) – (1.2) *has at least one solution.*

Proof. Define the operator

$$N(u) = \left\{ h \in X,\ \exists y \in S_{F,u} \setminus h(t) = \int_0^1 G(t,s)\,y(s)\,ds,\ t \in J \right\}, \tag{3.10}$$

where $G(t,s)$ defined by (3.3).

We show that the operator N has a fixed point by applying Lemma 2.11. First, we show that N maps bounded sets of X into bounded sets. Suppose that $r>0$ and $B_r=\{u\in X:\|u\|\leq r\}$.

Let $u\in B_r$ and $h\in N(u)$. Choose $y\in S_{F,u}$ such that

$$h(t)=\int_0^1 G(t,s)\,y(s)\,ds,$$

for almost all $t\in J$. Thus and from Lemma 3.2, we have

$$\begin{aligned}|h(t)|&\leq\int_0^1 G(t,s)\,|y(s)|\,ds\\&\leq\|p\|_\infty\,\psi(\|u\|)\int_0^1 G(1,s)\,ds,\end{aligned}$$

where $\|p\|_\infty=\sup\limits_{t\in J}|p(t)|$.

Hence,

$$\begin{aligned}\|h\|&=\max_{t\in J}|h(t)|\\&\leq\|p\|_\infty\,\psi(\|u\|)\int_0^1 G(1,s)\,ds,\end{aligned}$$

for all $h\in N(u)$, i.e., $N(B_r)$ bounded.

Now, we show that N maps bounded sets into equicontinuous subsets of X. Let $u\in B_r$ and $t_1,\,t_2\in J$ with $t_1<t_2$. Then we have

$$\begin{aligned}|h(t_2)-h(t_1)|\leq&\int_0^1|g(t_2,s)-g(t_1,s)|\,|y(s)|\,ds\\&+\frac{|t_2^{\alpha-1}-t_1^{\alpha-1}|}{d}\sum_{j=1}^p a_j\int_0^1 h(\eta_j,s)\,|y(s)|\,ds\\\leq&\int_0^1|g(t_2,s)-g(t_1,s)|\,|p(r)\,\psi(\|u\|)|\,ds\\&+\frac{|t_2^{\alpha-1}-t_1^{\alpha-1}|}{d}\sum_{j=1}^p a_j\int_0^1 h(\eta_j,s)\,\|p\|_\infty\,\psi(\|u\|)\,ds.\end{aligned}$$

On the other hand,

$$
\begin{aligned}
(3.11)\quad & \int_0^1 |g(t_2,s)-g(t_1,s)|\,|y(s)|\,ds \\
&\le \left(\int_0^{t_1}+\int_{t_1}^{t_2}+\int_{t_2}^{1}\right)|g(t_2,s)-g(t_1,s)|\,\|p\|_\infty\,\psi(\|u\|)\,ds \\
&\le \left(\int_0^{t_1}+\int_{t_1}^{t_2}+\int_{t_2}^{1}\right)|g(t_2,s)-g(t_1,s)|\,\|p\|_\infty\,\psi(\|u\|)\,ds \\
&\le \|p\|_\infty\,\psi(\|u\|)\int_0^{t_1}\Big[\left(t_2^{\alpha-1}-t_1^{\alpha-\beta-1}\right)(1-s)^{\alpha-\beta-1} \\
&\qquad\qquad + (t_1-s)^{\alpha-1}-(t_2-s)^{\alpha-1}\Big]\,ds \\
&\quad + \|p\|_\infty\,\psi(\|u\|)\int_{t_1}^{t_2}\left[\left(t_2^{\alpha-1}-t_1^{\alpha-1}\right)(1-s)^{\alpha-\beta-1}+(t_1-s)^{\alpha-1}\right]ds \\
&\quad + \|p\|_\infty\,\psi(\|u\|)\int_{t_2}^{1}\left[\left(t_2^{\alpha-1}-t_1^{\alpha.-1}\right)(1-s)^{\alpha-\beta-1}\right]ds.
\end{aligned}
$$

It is seen that $|h(t_2)-h(t_1)|\to 0$, uniformly as $t_2\to t_1$, we have

$$\|h(t_2)-h(t_1)\|\to 0,$$

as $t_2\to t_1$.

Thus N is equicontinuous and so N is relatively compact on B_r. Consequently the Ascoli-Arzela theorem implies that N is compact on B_r.

Now, we show that N is upper semi-continuous. To do this, it is sufficient to show that N has a closed graph. Let $u_n\to u_0$, $h_n\in N(u_n)$ for all n and $h_n\to h_0$. We prove that $h_0\in N(u_0)$. For each n choose $y_n\in S_{F,u_n}$ such that, for all $t\in J$,

$$h_n(t)=\int_0^1 G(t,s)\,y_n(s)\,ds.$$

Thus it is suffices to show that there exists $y_0\in S_{F,u_0}$ such that for each $t\in J$,

$$h_0(t)=\int_0^1 G(t,s)\,y_0(s)\,ds.$$

The condition (H_1) implies that $y_n(t) \in F(t, u_n(t)))$, hence $|y_n(t)| \leq p(t)\psi(\|u\|)$. Then $\{y_n\}_{n\in\mathbb{N}}$ is integrable bounded in $L^1(J,\mathbb{R})$. Since F has compact values, we deduce that $\{y_n\}_{n\in\mathbb{N}}$ is semi-compact. By using Lemma 2.15, there exists a subsequence, still denoted $\{y_n\}_{n\in\mathbb{N}}$, which converge weakly to some limit $y_0 \in L^1(J,\mathbb{R})$. Moreover, the mapping $\Theta : L^1(J,\mathbb{R}) \to X$ defined

$$\Theta(f)(t) = \int_0^1 G(t,s) f(s)\, ds,$$

is a continuous linear operator. Then it remains continuous if these spaces are endowed with their weak topologies [**28**]. Moreover, for *a.e.* $t \in J$, $u_n(t)$ converges to $u_0(t)$. Then, we have

$$h_0(t) = \int_0^1 G(t,s)\, y_0(s)\, ds.$$

It remains to prove that $y_0 \in F(t, u_0(t))\,, a.e. t \in J$. Mazur's Lemma 2.16 yields the existence of $\alpha_i^n \geq 0$, $i = n, \ldots, k(n)$ such that $\sum\limits_{i=1}^{k(n)} \alpha_i^n = 1$ and the sequence of convex combinations $g_n(\cdot) = \sum\limits_{i=1}^{k(n)} \alpha_i^n y_i(\cdot)$ converges strongly to $y_0 \in L^1(J,\mathbb{R})$. Using Lemma 2.12, we obtain that

$$\begin{aligned} y_0(t) &\in \bigcap_{n\geq 1} \overline{\{g_n(t)\}},\ a.e.\ t \in J \\ &\subset \bigcap_{n\geq 1} \overline{co}\{y_k(t)\,,\ k \geq n\} \\ &\subset \bigcap_{n\geq 1} \overline{co}\left\{\bigcup_{n\geq 1} F(t, u_k(t))\right\} \qquad (3.12) \\ &= \bigcap_{n\geq 1} \overline{co}\left\{\limsup_{k\to\infty} F(t, u_k(t))\right\}. \end{aligned}$$

The fact that the multivalued function $F(\cdot, u)$ is semi-continuous and has compact values, together with Lemma 2.13, implies that

$$\limsup_{n\to\infty} F(t, u_n(t)) = F(t, u(t))\,, \quad a.e.\ t \in J.$$

This with (3.12) yields that $y_0 \in \overline{co}F(t, u_0(t))$.

Finally, $F(\cdot,\cdot)$ has closed, convex values. Hence $y_0(t) \in F(t, u_0(t))$, $a.e. t \in J$. On the other hand, observe that

$$\|h_n(t) - h_0(t)\| = \left\| \int_0^1 G(t,s)(y_n(s) - y_0(s))\, ds \right\| \to 0,$$

as $n \to 0$.

Thus, it follows by Lemma 2.10 that $\Theta \circ S_F$ is a closed graph operator. Further, we have $h_n \in \Theta(S_{F,u_n})$. Since $u_n \to u_0$, we have that

$$h_0(t) = \int_0^1 G(t,s)\, y_0(s)\, ds,$$

for some $y_0 \in S_{F,u_0}$. Hence $h_0 \in N_0(u)$, proving that N has a closed graph. Finally, with Remark 2.6 and the compactness of N, we conclude that N is upper semi-continuous.

In this part of the proof, we show that $N(u)$ is convex for all $u \in X$. Let h_1, $h_2 \in N(u)$ and $\omega \in [0,1]$. Choose y_1, $y_2 \in S_{F,u}$. Then

$$\omega h_1(t) + (1-\omega) h_2(t) = \int_0^1 G(t,s)\left[\omega y_1(s) + (1-\omega) y_2(s)\right] ds$$

for all $t \in J$.

Since F has convex values, so $S_{F,u}$ is convex. Thus $\omega h_1 + (1-\omega) h_2 \in N(u)$. Consequently, N is convex-valued.

We give now an a priori bound of solution. If there exist $\lambda \in (0,1)$ such that $u \in \lambda N(u)$ then there exists $y \in S_{F,u}$ such that

$$u(t) = \lambda \int_0^1 G(t,s)\, y(s)\, ds.$$

In view of (H_1), we obtain

$$|u(t)| \leq \psi(\|u\|) \left[\int_0^1 G(1,s)\, p(s)\, ds \right].$$

Consequently

$$\|u(t)\| \left[\psi(\|u\|) \int_0^1 G(1,s)\, p(s)\, ds \right]^{-1} \leq 1,$$

for almost all $t \in J$. Thus, in view of (H_2), there exists $R > 0$ such that $\|u\| < R$. Put $U = \{u \in X : \|u\| < R\}$. Note there are no $u \in \partial U$ and $\lambda \in (0,1)$ such that $u \in \lambda N(u)$ and the operator $N : \overline{U} \to P_{cp,cv}\left(\overline{U}\right)$ is upper semi-continuous because it is completely continuous. Now, by using Lemma 2.11, N has fixed point in $\overline{U}$ which is solution of the inclusion problem $(1.1)-(1.2)$. This completes the proof. □

3.1 - *The lower semi-continuous case and the Lipschitz case*

We provide another result about the existence of solutions for the problem $(1.1)-(1.2)$ by changing the assumptions of convex values for multifunction. Our strategy to deal with this problem is based on the Schaefer theorem together with the selection theorem of Bressan and Colombo [**7**] for lower semi-continuous maps with decomposable values.

Definition 3.4. Let A be a subset of $[0,1] \times \mathbb{R}$. A is $L \otimes B$ measurable if A belongs to the $\sigma-$algebra generated by all sets of the $J \times D$, where J is Lebesgue measurable in $[0,1]$ and D is Borel measurable in $\mathbb{R}$.

Definition 3.5. A subset A of $L^1([0,1],\mathbb{R})$ is decomposable if all $u, v \in A$ and measurable $J \subset [0,1] = j$, the function $u\chi_J + v\chi_{j\setminus J} \in A$, where χ_J stands for the caracteristic function of J.

Definition 3.6. Let Y be a separable metric space and $N : Y \to P\left(L^1([0,1],\mathbb{R})\right)$ be a multivalued operator. We say N has property (BC) if N is lower semi-continuous (l.s.c) and has nonempty closed and decomposable values.

Let $F : [0,1] \times \mathbb{R} \to P(\mathbb{R})$ be a multivalued map with nonempty compact values. Define a multivalued operator

$$\Phi : C([0,1],\mathbb{R}) \to P\left(L^1([0,1],\mathbb{R})\right),$$

by letting

$$\Phi(u) = \left\{w \in L^1([0,1],\mathbb{R}) : w(t) \in F(t,u(t)) \ for\ a.e.\ t \in [0,1]\right\}.$$

Definition 3.7. The operator Φ is called the Niemytzki operator associated with F. We say F is of the lower semi-continuous type (l.s.c type) if its associated Niemytzki operator Φ has (BC) property.

Next we state a selection theorem due to Bressan and Colombo.

Lemma 3.8 ([7]) (Bressan and Colombo). *Let Y be a separable metric space and $N : Y \to P\left(L^1([0,1],\mathbb{R})\right)$ be a multivalued operator which has the property* (BC). *Then N has a continuous selection, that is, there exists a continuous function (single-valued) $g : Y \to L^1([0,1],\mathbb{R})$ such that $g(u) \in N(u)$ for every $u \in Y$.*

Definition 3.9. A multivalued operator $N : X \to P_{cl}(X)$ is called

(i) $\rho-$Lipschitz if and only if there exists $\rho > 0$ such that $H_d(N(u), N(v)) \leq \rho d(u,v)$ for each $u, v \in X$,

(ii) a contraction if and only if it is $\rho-$Lipschitz with $\rho < 1$.

Lemma 3.10 ([13]) (Covitz-Nadler). *Let (X,d) be a complete metric space. If $N : X \to P_{cl}(X)$ is a contraction, then $FixN \neq \phi$, where $FixN$ is the fixed point of the operator N.*

Definition 3.11. A measurable multivalued function $F : [0,1] \to P(X)$ is said to be integrably bounded if there exists a function $g \in L^1([0,1], X)$ such that, for all $v \in F(t)$, $\|v\| \leq g(t)$ for a.e. $t \in [0,1]$.

With the help of Schaefer's theorem combined with the selection theorem of Bressan and Colombo for lower semicontinuous maps with decomposable values, first we shall present an existence result for the problem $(1.1)-(1.2)$. Before this, let us introduce the following hypotheses which are assumed hereafter

(H_3) $F : [0,1] \times \mathbb{R} \to P_{cp}(\mathbb{R})$ be a multivalued map verifying

(i) $(t,u) \longmapsto F(t,u)$ is $L \otimes B$ measurable.

(ii) $u \longmapsto F(t,u)$ is lower semi-continuous for $a.e.$ $t \in [0,1]$.

(H_4) F is integrably bounded, that is, there exists a function $m \in L^1([0,1], \mathbb{R}^+)$ such that $\|F(t,u)\| = \sup\{|v| : v \in F(t,u)\} \leq m(t)$ for almost all $t \in [0,1]$.

Lemma 3.12 ([19]). *Let $F : [0,1] \times \mathbb{R} \to P_{cp}(\mathbb{R})$ be a multivalued map. Assume (H_3) and (H_4) hold. Then F is of the l.s.c. type.*

Definition 3.13. A function $u \in AC^2([0,1],\mathbb{R})$ is called a solution to the boundary value problem $(1.1)-(1.2)$ if u satisfies the differential inclusion (1.1) $a.e.$ on $[0,1]$ and the condition (1.2).

In the first result, we are concerned with the existence of solutions for the problem $(1.1)-(1.2)$ when the right hand side does not have necessarily convex values. Our strategy to deal with this problem is based on Schaefer's fixed point theorem with the selection theorem of Bressan and Colombo [7] for lower semi-continuous operators.

Theorem 3.14 ([**16**]) (Schaefer). *Let A be a completely continuous mapping of a Banach space X into it self, such that the set $\{u \in X : u = \lambda Au,\ 0 < \lambda < 1\}$ is bounded, then A has a fixed point.*

Now, we are in position to present the first main result of this subsection.

Theorem 3.15. *Suppose that (H_3) and (H_4) hold. Then the problem $(1.1) - (1.2)$ has at least one solution.*

Proof. We note that if (H_3) and (H_4) are satisfied, then, by Lemma 3.12 we have that F is of the lower semi-continuous type. Thus by Lemma 3.8, there exists a continuous function $y : C([0,1],\mathbb{R}) \to L^1([0,1],\mathbb{R})$ such that $y(u) \in \Phi(u)$ for all $u \in C([0,1],\mathbb{R})$. Consider the problem

$$D^{\alpha}_{0^+} u(t) + y(u(t)) = 0, \quad t \in (0,1), \tag{3.13}$$

subject to the boundary conditions

$$u^{(i)}(0) = 0,\ i \in \{0,1,\dots,n-2\},\ D^{\beta}_{0^+} u(1) = \sum_{j=1}^{p} a_j D^{\beta}_{0^+} u(\eta_j). \tag{3.14}$$

If $u \in C([0,1],\mathbb{R})$ is a solution to the problem (3.13)−(3.14), then u is a solution to the problem $(1.1) - (1.2)$. Transform the problem $(3.13) - (3.14)$ into a fixed point problem. Consider the operator $T : C([0,1],\mathbb{R}) \to C([0,1],\mathbb{R})$, defined by

$$Tu(t) = \int_0^1 G(t,s)\, y(s)\, ds,\ t \in J = [0,1].$$

We show that T is a compact operator.

Step 1. T is continuous.

Let (u_n) be a sequence such that $u_n \to u$ in $C([0,1],\mathbb{R})$. Then

$$\begin{aligned} |T(u_n)(t) - T(u)(t)| &= \int_0^1 G(t,s)\, |y_n(s) - y(s)|\, ds \\ &\le \int_0^1 G(1,s)\, |y_n(s) - y(s)|\, ds. \end{aligned}$$

Since y is continuous, then

$$\|(Tu_n)(t) - (Tu)(t)\| \to 0,\ as\ n \to \infty.$$

Step 2. T is bounded into bounded sets of $C([0,1],\mathbb{R})$ (see above Theorem 3.3).

Step 3. T sends bounded sets of $C([0,1],\mathbb{R})$ into equicontinuous sets (see above Theorem 3.3).

As consequence of Step 1 to 3, toghether with the Arzela-Ascoli theorem, we can conclude that T is completely continuous.

In order to apply Schaefer's theorem, it remains to shows next step.

Step 4. The set

$$\Omega = \{u \in C([0,1],\mathbb{R}) : \lambda u = T(u) \text{ } for\ some\ \lambda > 1\},$$

is bounded. Let $u \in \Omega$. Then $\lambda u = T(u)$ for some $\lambda > 1$ and

$$u(t) = \frac{1}{\lambda}\int_0^1 G(t,s)\, y(s)\, ds,\ t \in J,$$

this implies by (H_4) that for each $t \in [0,1]$, we have

$$|u(t)| \leq \int_0^1 G(t,s)\, m(s)\, ds,\ t \in J,$$

thus

$$|u(t)| \leq \int_0^1 G(1,s)\, m(s)\, ds = K.$$

This shows that Ω is bounded.

As consequence of Schaefer's theorem, we deduce that T has a fixed point which is a solution to the problem $(3.13)-(3.14)$, and hence a solution to the problem $(1.1)-(1.2)$. □

Finally, we state and prove the second main result of this subsection. We prove the existence of solutions for the inclusion problem $(1.1)-(1.2)$ with a nonconvex valued right hand side by applying a fixed point theorem for multivalued maps due to Covitz and Nadler. For investigation of the problem $(1.1)-(1.2)$ we shall provide an application of the Lemma 3.10 and the following lemma.

Lemma 3.16 ([**14**]). *A multifunction $F : X \to C(X)$ is called a contraction whenever there exists $\gamma \in (0,1)$ such that $H_d(N(u), N(v)) \leq \gamma d(u,v)$ for all $u, v \in X$.*

Now, we present second main result of this subsection.

Theorem 3.17. *Assume that the following hypothyses hold.*

(H_5) $F : J \times \mathbb{R} \to P_{cp}(\mathbb{R})$ *is an integrable bounded multifunction such that the map* $t \longmapsto F(t,u)$ *is measurable,*

(H_6) $H_d(F(t,u_1), F(t,u_2)) \leq m(t)|u_1 - u_2|$ *for almost all* $t \in J$ *and* $u_1, u_2 \in \mathbb{R}$ *with* $m \in L^1(J, \mathbb{R})$ *and* $d(0, F(t,0)) \leq m(t)$ *for almost all* $t \in J$. *Then the problem* $(1.1)-(1.2)$ *has a solution provided that*

$$l = \int_0^1 G(1,s)\, m(s)\, ds < 1.$$

Proof. We transform problem $(1.1)-(1.2)$ into a fixed point problem. Consider the operator $N : C[0,1] \to P(C[0,1], \mathbb{R})$ defined by (3.10). It is clear that fixed points of N are solution of $(1.1)-(1.2)$.

We shall prove that N fulfills the assumptions of Covitz-Nadler contraction principle.

Note that, the multivalued map $t \longmapsto F(t, u(t))$ is measurable and closed for all $u \in AC^1([0,\infty))$ (e.g., [**14**, Theorem III.6]). Hence, it has a measurable selection and so the set $S_{F,u}$ is nonempty, so, $N(u)$ is nonempty for any $u \in C([0,\infty))$.

First, we show that $N(u)$ is a closed subset of X for all $u \in AC^1([0,\infty), \mathbb{R})$. Let $u \in X$ and $\{u_n\}_{n\geq 1}$ be a sequence in $N(u)$ with $u_n \to u$, *as* $n \to \infty$ in $u \in C([0,\infty))$. For each n, choose $y_n \in S_{F,u}$ such that

$$u_n(t) = \int_0^1 G(t,s)\, y_n(s)\, ds.$$

Since F has compact values, we may pass onto a subsequence (if necessary) to obtain that y_n converges to $y \in L^1([0,1], \mathbb{R})$ in $L^1([0,1], \mathbb{R})$. In particular, $y \in S_{F,u}$ and for any $t \in [0,1]$, we have

$$u_n(t) \to u(t) = \int_0^1 G(t,s)\, y(s)\, ds,$$

i.e., $u \in N(u)$ and $N(u)$ is closed.

Next, we show that N is a contractive multifunction with constant $l < 1$. Let $u,\ v \in C([0,1], \mathbb{R})$ and $h_1 \in N(u)$. Then there exist $y_1 \in S_{F,u}$ such that

$$h_1(t) = \int_0^1 G(t,s)\, y_1(s)\, ds,\ t \in J.$$

By (H_6), we have

$$H_d(F(t,u(t)), F(t,v(t))) \le m(t)(|u(t) - v(t)|),$$

for almost all $t \in J$.

So, there exists $w \in S_{F,v}$ such that

$$|y_1(t) - w| \le m(t)(|u(t) - v(t)|),$$

for almost all $t \in J$.

Define the multifunction $U : J \to P(\mathbb{R})$ by

$$U(t) = \{w \in \mathbb{R} : |y_1(t) - w| \le m(t)(|u(t) - v(t)|)\ \textit{for almost all}\ t \in J\}.$$

It is easy to chek that the multifunction $V(\cdot) = U(\cdot) \cap F(\cdot, v(\cdot))$ is measurable (e.g., [**14**, Theorem III.4]).

Thus, there exists a function $y_2(t)$ which is measurable selection for V. So, $y_2 \in S_{F,v}$ and for each $t \in J$, we have

$$|y_1(t) - y_2(t)| \le m(t)(|u(t) - v(t)|).$$

Now, consider $h_2 \in N(u)$ which is defined by

$$h_2(t) = \int_0^1 G(t,s)\, y_2(s)\, ds,\ t \in J,$$

and one can obtain

$$\begin{aligned} |h_1(t) - h_2(t)| &\le \int_0^1 G(t,s)\, |y_1(s) - y_2(s)|\, ds \\ &\le \int_0^1 G(1,s)\, m(s)\, |u(s) - v(s)|\, ds. \end{aligned}$$

Hence

$$\|h_1(t) - h_2(t)\| \le \|p\|_\infty \|u - v\| \left[\int_0^1 G(1,s)\, m(s)\, ds \right].$$

Analogously, interchanging the roles of u and v, we obtain

$$H_d(N(u), N(v)) \le \|u - v\| \left[\int_0^1 G(1,s)\, m(s)\, ds \right].$$

Since N is a contraction, it follows by Lemma 3.10 (by using the result of Covitz and Nadler) that N has a fixed point which is a solution to problem (1.1) – (1.2). □

We construct two examples to illustrate the applicability of the results presented.

Example 3.18. *Consider the problem*

$$-D^\alpha u(t) \in F(t, u(t)), \quad t \in [0,1], \tag{3.15}$$

subject to the three-point boundary conditions

$$u^{(i)}(0) = 0,\ i \in \{0,1\},\ D_{0+}^\beta u(1) = \sum_{j=1}^{2} a_j D_{0+}^\beta u(\eta_j), \tag{3.16}$$

where $\alpha = \frac{5}{2}$, $\beta = 1$ $a_1 = \frac{1}{2}$, $a_2 = \frac{3}{2}$,$\eta_1 = \frac{1}{16}$, $\eta_2 = \frac{5}{16}$ *and* $F(t, u(t)) : [0,1] \times \mathbb{R} \to 2^{\mathbb{R}}$ *multivalued map given by*

$$u \longmapsto F(t,u) = \left(0, \frac{t|u|}{2(1+|u|)}\right),\ u \in \mathbb{R},$$

verifying (H_5).

Obviously,

$$\sup\{|f| : f \in F(t,u)\} \le \frac{t+1}{2},$$

we have

$$H_d(F(t,u), F(t,v)) \le \left(\frac{t+1}{2}\right)|u - v|,\ u, v \in \mathbb{R},\ t \in [0,1],$$

which shows that (H_6) *holds.*

So, if $m(t) = \frac{t+1}{2}$ *for all* $t \in [0,1]$, *then*

$$H_d(F(t,u), F(t,v)) \leq m(t)|u-v|.$$

It can be easily found that $d = 1 - \frac{1}{2}\left(\frac{1}{16}\right)^{\frac{5}{2}} - \frac{3}{2}\left(\frac{5}{16}\right)^{\frac{5}{2}} = 0,9176244637.$

Finally,

$$l = \int_0^1 G(1,s)\, m(s)\, ds = 0,4636273746 < 1.$$

Hence, all assumptions and conditions of Theorem 3.17 *are satisfied. So, Theorem* 3.17 *implies that the inclusion problem* (3.15) − (3.16) *has at least one solution.*

Example 3.19. *Consider the problem* (3.15) − (3.16), *where* $\alpha = \frac{5}{2}$, $\beta = 1$, $a_1 = \frac{1}{2}$, $a_2 = \frac{3}{2}$, $\eta_1 = \frac{1}{16}$, $\eta_2 = \frac{5}{16}$ *and* $F(t,u) : [0,1] \times \mathbb{R} \to 2^{\mathbb{R}}$ *is a multivalued map given by*

$$u \longmapsto F(t,u) = \left(\left(\frac{3+t^2}{4}\right)\left(\frac{|u|}{1+|u|} + \sin(u)\right), \frac{|u|^3}{2\left(1+|u|^3\right)} + 5t^3 + 4\right), \quad u \in \mathbb{R}.$$

Obviously, for $f \in F$, *we have*

$$\|f\| \leq \max\left(\left(\frac{3+t^2}{4}\right)\left(\frac{|u|}{1+|u|} + \sin(u)\right), \frac{|u|^3}{2\left(1+|u|^3\right)} + 5t^3 + 4\right) \leq \frac{19}{2}, \quad u \in \mathbb{R}.$$

Thus

$$\|F(t,u)\| = \sup\{|f| : f \in F(t,u)\} \leq \frac{19}{2} = p(t)\,\psi(\|u\|), \quad u \in \mathbb{R},$$

with $p(t) = 1$ *and* $\psi(\|u\|) = \frac{19}{2}$. *Further using the condition*

$$R\left[\psi(\|u\|)\left(\int_0^1 G(1,s)\, p(s)\, ds\right)\right]^{-1} > 1,$$

we find that

$$R > \left(\frac{19}{2}\right)(0,4636273746) \cong 4,4044600587,$$

this shows that all the assumptions of Theorem 3.3 *are satisfied. So, the inclusion problem* (3.15) − (3.16) *has at least one solution.*

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