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Non vanishing of Dirichlet series of completely multiplicative functions

Abstract. Let

$$L(s) = \sum_{n=1}^{+\infty} \frac{a(n)}{n^s}$$

be a Dirichlet series where $a(n)$ is a bounded completely multiplicative function. We prove that if $L(s)$ extends to a holomorphic function on the open half space $\operatorname{Re} s > 1 - \delta$, $\delta > 0$ and $L(1) = 0$ then such a half space is a zero free region of the Riemann zeta function $\zeta(s)$. Similar results are proven for completely multiplicative functions defined on the space of the ideals of the ring of the algebraic integers of a number field of finite degree.

Keywords. Riemann zeta function, Dirichlet series, Absolutely/completely monotone functions.

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1 - Introduction

The prime number theorem and the Dirichlet theorem on primes in arithmetic progressions (and their generalizations to number fields) rely respectively on the fact that

$$\zeta(1+it) \neq 0 \tag{1}$$

for each real $t \neq 0$ and that

$$L(\chi, 1) \neq 0 \tag{2}$$

for each non principal character $\chi : \mathbb{N}^+ \to \mathbb{C}$.

Further results on the density of the primes on arithmetic progressions follow from the inequality

$$L(\chi, 1+it) \neq 0 \tag{3}$$

for each real t.

In the above statements $\zeta(s)$ is the famous Riemann zeta function, which is meromorphic on the whole complex plane and coincides with the Dirichlet series

$$\sum_{n=1}^{+\infty} \frac{1}{n^s}$$

when $\operatorname{Re} s > 1$, while $L(\chi, s)$ is the *Dirichlet $L-$function* associated to the character χ, which (when χ is not principal) is a holomorphic entire function which when $\operatorname{Re} s > 1$ coincides with the series

$$\sum_{n=1}^{+\infty} \frac{\chi(n)}{n^s}.$$

Let us recall that the Riemann zeta function is meromorphic on the whole complex plane, has a unique simple pole at $s = 1$ with residue 1.

The zeroes of $\zeta(s)$ are the so called *trivial zeroes*

$$\zeta(-2m) = 0, \ m = 1, 2, \dots$$

and the remaining ones ρ satisfy

$$0 < \operatorname{Re} \rho < 1.$$

The famous *Riemann hypothesis* is that all of them are on the *critical line*

$$\operatorname{Re} s = \frac{1}{2}.$$

Nevertheless it is known that infinitely many of them are on the critical line.

Here, and in the rest of the paper, $\mathbb{N}^+$ is the set of the positive integers and if

$$a : \mathbb{N}^+ \to \mathbb{C}$$

is an arbitrary bounded arithmetic function then

$$L(a,s) = \sum_{n=1}^{+\infty} \frac{a(n)}{n^s}$$

is the corresponding associated Dirichlet series.

We recall that a *Dirichlet character* is an arithmetic function, that is a function

$$\chi : \mathbb{N}^+ \to \mathbb{C}$$

which is *completely multiplicative*, that is $\chi(1) = 1$ and

$$\chi(mn) = \chi(m)\chi(n)$$

for each pair of $m, n \in \mathbb{N}^+$ and which also is periodic of period $q > 1$, that is $\chi(n+q) = \chi(n)$ for each $n \in \mathbb{N}^+$ and $\chi(k) = 0$ if k and q are not relatively prime.

Clearly any Dirichlet character χ is a bounded function and it is easy to show that values of a bounded completely multiplicative functions are complex numbers z which satisfy $|z| \leq 1$.

There are several methods in literature to achieve (3).

One of them is an easy consequence of the following remarkable result of Ingham [**6**].

Theorem 1.1. *Let*

$$L(a,s) = \sum_{n=1}^{+\infty} \frac{a(n)}{n^s}$$

be a Dirichlet series where $a : \mathbb{N}^+ \to \mathbb{C}$ *is an arbitrary bounded completely multiplicative arithmetic function.*

Assume that $L(a,s)$ *extends to a holomorphic function on the open half space*

$$\operatorname{Re} s > \frac{1}{2} - \delta$$

with $\delta > 0$. *Then*

$$L(a, 1+it) \neq 0$$

for each $t \in \mathbb{R}$.

When $a = \chi$, a non principal Dirichlet character, one obtains (3).

The Ingham proof of Theorem 1.1 is quite involved, but a very simple proof is given by Bateman in [**2**].

The first result of this paper is the following.

Theorem 1.2. *Let $\sigma_0 < 1$ be given.*

If there exists a completely multiplicative bounded arithmetic function

$$a : \mathbb{N}^+ \to \mathbb{C}$$

such that the Dirichlet series

$$L(a, s) = \sum_{n=1}^{+\infty} \frac{a(n)}{n^s}$$

extends holomorphically on the open half space

$$\operatorname{Re} s > \sigma_0$$

and

$$L(a, 1) = 0$$

then

$$\zeta(s) \neq 0$$

for each s satisfying $\operatorname{Re} s > \sigma_0$.

The theorem above, together with the fact that the Riemann zeta function $\zeta(s)$ has (infinitely many) zeroes on the line $\operatorname{Re} s = 1/2$, easily implies the Ingham result; see at the end of Section 4 for details.

When $a = \chi$, where χ is a non principal Dirichlet character it is easy to see that the corresponding Dirichlet series $L(\chi, s)$ extends holomorphically on the half space $\operatorname{Re} s > 0$ (actually $L(\chi, s)$ extends holomorphically as an entire function). If

$$L(\chi, 1) = 0$$

for some non principal Dirichlet character χ then Theorem 1.2 implies that the Riemann zeta function $\zeta(z)$ wouldn't have any zero on the half space $\operatorname{Re} s > 0$, which is clearly absurd.

The author is not able to give any example of a Dirichlet series $L(a, s)$ as in Theorem 1.2 such that satisfies $L(a, 1) = 0$ and is holomorphic on a half space

$$\operatorname{Re} s > \sigma_0$$

with

$$\frac{1}{2} \leq \sigma_0 < 1.$$

Indeed the existence of such a series for $\sigma_0 = 1/2$, combined with our Theorem 1.2, would imply the Riemann hypothesis, and as far I know the existence of an half space $\operatorname{Re} s > \sigma_0$ with $1/2 < \sigma_0 < 1$ which is a zero free region for the Riemann zeta function also is an open conjecture.

Nevertheless we observe that the converse of Theorem 1.2 also holds.

Indeed if $\lambda(n)$ denotes the *Liouville function* (see next section for details) then the meromorphic function

$$L(\lambda, s) = \sum_{n=1}^{+\infty} \frac{\lambda(n)}{n^s} = \frac{\zeta(2s)}{\zeta(s)}$$

satisfies $L(\lambda, 1) = 0$ and obviously $L(\lambda, s)$ is holomorphic on the half space $\operatorname{Re} s > \sigma, 1/2 \leq \sigma < 1$ if, and only if, such a half space is a zero free region for $\zeta(s)$.

The proof of Theorem 1.2 is obtained as an elementary consequence of a general non vanishing principle for holomorphic functions which are analytic continuations of exponentials of completely monotone functions which we think of independent interest.

Let us recall that a C^∞ function

$$u : I \to \mathbb{R}$$

where I is a interval of $\mathbb{R}$ is *completely monotone* if for $k = 0, 1, \ldots$

$$(-1)^k u^{(k)}(x) \geq 0$$

for each $x \in I$.

Then our result is the following.

Theorem 1.3. *Let $\sigma_1, \sigma_2 \in \mathbb{R}$ with $\sigma_1 < \sigma_2$ and let $f(s)$ be a holomorphic function defined on the open half space*

$$\operatorname{Re} s > \sigma_2.$$

Assume that the restriction of $f(s)$ to the half line $]\sigma_2, +\infty[$ is a real completely monotone function and

$$F(s) := \exp f(s)$$

extends holomorphically on the open half space

$$\operatorname{Re} s > \sigma_1.$$

Then the function $f(s)$ also extends holomorphically on the open half space

$$\operatorname{Re} s > \sigma_1$$

and hence

$$F(s) = \exp f(s) \neq 0$$

when $\operatorname{Re} s > \sigma_1$.

It is quite surprising that our approach makes it unnecessary to use anywhere the standard Euler product expansion of such Dirichlet series.

Actually the Euler product expansion is necessary to prove the non vanishing in the half space of absolute convergence of the Dirichlet series associated to *multiplicative* but not completely multiplicative functions, that is to functions

$$a : \mathbb{N}^+ \to \mathbb{C}$$

such that

$$a(mn) = a(m)a(n)$$

when m and n are relatively prime. But in this paper we do not consider multiplicative arithmetic function. For *completely multiplicative* arithmetic functions we obtain directly their representation as exponential of Dirichlet series without using its product expansion; see (the proofs of) Lemma 4.1, and Proposition 6.2 for details.

Let us now describe the contents of the paper.

In Section 2 we recall basic facts on arithmetic functions and the associated Dirichlet series that we need in the rest of the paper.

The proof of Theorem 1.3 and Theorem 1.1 are given respectively in Section 3 and Section 4.

In Section 5 we give a refined version of Theorem 1.3 which is used in Section 6 to prove Theorem 6.1, the main result of this paper, which extends to a class of generalized Dirichlet series the above Theorem 1.1, including, among the others, the Dirichlet series associated to completely multiplicative functions defined on the ideals of the ring of the integers of a number fields.

Non vanishing theorems for general $L-$type functions on the boundary of the half space of absolute convergence are then obtained in Section 7.

We end this introduction with a "toy application" of Theorem 1.3 giving three proofs of $\zeta(1 + it) \neq 0$. Each of them contains "themes" which will be expanded in the rest of the paper.

All of them start observing that when $\operatorname{Re} s > 1$ we have

$$\zeta(s) = \exp\left(\sum_{n=1}^{+\infty} \frac{\Lambda(n)}{n^s \log n}\right)$$

where $\Lambda(n)$ is the *von Mangoldt function* (see, e.g., next section).

Assume that

$$\zeta(1+it)=0$$

for some $t>0$. Following [**17**, pag. 199] consider the function

$$F(s)=\zeta(s)^2\zeta(s+it)\zeta(s-it).$$

Then $F(s)$ has removable singularities at $s=1$, $s=1+it$ and $s=1-it$ and hence is a holomorphic entire function.

When $\operatorname{Re} s>1$ we have then

$$F(s)=\exp f(s)$$

where

$$f(s)=\sum_{n=1}^{+\infty}\frac{2\big(1+\operatorname{Re}(n^{-it})\big)\Lambda(n)}{n^s\log n}.$$

Since $\operatorname{Re} n^{-it}=\cos\big(t\log(n)\big)\geq -1$ the Dirichlet series $f(s)$ has non negative coefficients and hence the function

$$]1,+\infty[\ \ni\sigma\mapsto f(\sigma)\in\mathbb{R}$$

is completely monotone.

Theorem 1.3 implies that $f(s)$ extends to an entire holomorphic function and $F(s)=\exp f(s)$ is a non vanishing entire holomorphic function; a classical Landau's theorem (see Theorem 2.1) implies that the series $f(s)$ then converges for all $s\in\mathbb{C}$.

Now we have three ways to conclude the proof.

The first one begins by observing that at $s=1+it$ the factor $\zeta(s)^2$ of the function $F(s)$ has a zero of the second order while the factor $\zeta(s-it)$ has a simple pole.

Since $F(1+it)\neq 0$ then the remaining factor $\zeta(s+it)$ must necessarily have a simple pole at $s=1+it$, that is the Riemann zeta function $\zeta(s)$ would have also another pole at $s=1+2it$, which is absurd.

The second one follows from the fact that if $F(s)$ never vanishes then $\zeta(s)$ also never vanishes when $s\neq 1$. In particular it follows that $\zeta(s)$ does not vanish neither when $s=-2m$, $m=1,2,\ldots$ nor when $\operatorname{Re} s=1/2$ and this is not the case ...

For the third let us denote by $P = \{2, 3, 5, \dots\}$ the set of the positive prime numbers and set $a(n) = n^{-it}$. If $\sigma \in \mathbb{R}$ then

$$\begin{aligned} f(\sigma) &= \sum_{n=1}^{+\infty} \frac{2\big(1 + \operatorname{Re} a(n)\big)\Lambda(n)}{n^{\sigma}\log(n)} = \sum_{p \in P}\sum_{m=1}^{+\infty} \frac{2\big(1 + \operatorname{Re} a(p)^m\big)}{mp^{m\sigma}} \\ &\geq \sum_{p \in P}\sum_{m=1}^{2} \frac{2\big(1 + \operatorname{Re} a(p)^m\big)}{mp^{m\sigma}} \\ &\geq \sum_{p \in P} \frac{\big(2 + \operatorname{Re} a(p) + \operatorname{Re} a(p)^2\big)}{p^{2\sigma}}. \end{aligned}$$

Observing that

$$\boxed{|w| \leq 1 \implies \operatorname{Re} w + \operatorname{Re} w^2 \geq -\frac{9}{8}}$$

with equality at

$$w = \frac{1}{4} \pm i\frac{\sqrt{15}}{4}$$

we then obtain

$$f(\sigma) \geq \frac{7}{8}\sum_{p \in P} \frac{1}{p^{2\sigma}}.$$

Since the series of the reciprocal of the prime numbers diverges then the series $f(s)$ also diverges at $s = 1/2$.

For a fourth (easy) proof see Corollary 7.1.

2 - Prerequisites

In this paper we need only the basic results on (multiplicative) arithmetic function and their associated Dirichlet series. Basic references include the first chapters of [**1**], [**14**], [**15**], [**10**], [**18**] and [**24**].

Here we review basic material that we need in this paper.

The already mentioned *von Mangoldt function* is the arithmetic function

$$\Lambda(n) = \begin{cases} \log p & \text{if } n = p^k \text{ for some prime } p \text{ and integer } k \geq 1, \\ 0 & \text{otherwise.} \end{cases}$$

It satisfies the identity

$$\sum_{d \mid n} \Lambda(d) = \log n. \tag{4}$$

The *Liouville function* is the completely multiplicative function

$$\lambda(n) = (-1)^{\Omega(n)}$$

where $\Omega(n)$ is the number of prime factors of n, counted with multiplicity.

The Liouville function and the Riemann zeta function are related by the identity

$$\frac{\zeta(2s)}{\zeta(s)} = \sum_{n=1}^{+\infty} \frac{\lambda(n)}{n^s}.$$

We will need the following elementary result on Dirichlet series.

Lemma 2.1. *Let*

$$F(s) = \sum_{n=1}^{+\infty} \frac{a(n)}{n^s}$$

where $a : \mathbb{N}^+ \to \mathbb{C}$ *is a bounded arithmetic function. Then,*

$$\lim_{\mathbb{R} \ni \sigma \to \infty} F(\sigma) = a(1).$$

The following statement is a classical result of Landau (see e.g. [**10**, Theorem 1.7, pag. 16] or [**9**, Lemma 1, pag. 314]).

Theorem 2.1. *Let* $\sigma_0, \sigma_1 \in \mathbb{R}$ *with* $\sigma_0 < \sigma_1$ *and let*

$$f(s) = \sum_{n=1}^{+\infty} \frac{a_n}{n^s}$$

be a Dirichlet series with non negative coefficients $a_n \geq 0$.

Assume that $f(s)$ *converges when* $\operatorname{Re} s > \sigma_1$ *and extends to a holomorphic function on* $\operatorname{Re} s > \sigma_0$.

Then the series $f(s)$ *also converges when* $\operatorname{Re} s > \sigma_0$.

3 - A non vanishing principle

In this section we prove Theorem 1.3.

We begin recalling a theorem due to Pringsheim which appeared in [**19**] also known as the Pringsheim-Vivanti theorem. See also [**5**, Theorem 5.7.1], and [**20**, Theorem 8.2.2].

Theorem 3.1. *Let*

$$f(z) = \sum_{n=0}^{+\infty} a_n z^n$$

be a convergent power series with radius of convergence R, with $0 < R < +\infty$.

If $a_n \geq 0$ for each n then it is not possible to extend $f(z)$ holomorphically in a neighbourhood of $z = R$.

The following proposition is the basic technical tool of the paper.

Proposition 3.1. *Let*

$$E : \mathbb{C} \to \mathbb{C}$$

be a holomorphic entire function such that $E(\mathbb{R}) \subset \mathbb{R}$, $E'(t) > 0$ for each $t > 0$ and

$$\lim_{t \to +\infty} E(t) = +\infty.$$

Let

$$f(z) = \sum_{n=1}^{+\infty} a_n z^n$$

and

$$F(z) = \sum_{n=1}^{+\infty} A_n z^n$$

be two convergent powers series with real coefficients such that

$$F(z) = E\big(f(z)\big)$$

in a neighbourhood of $z = 0$.

If the coefficients a_n of $f(z)$ are non negative then the series $f(z)$ and $F(z)$ have the same radius of convergence.

Proof. If the function f is constant then F also is constant and in this case the assertion is obvious.

We hence assume that f is not constant.

Let r and R denote the radius of convergence respectively of the series $f(z)$ and $F(z)$.

Then $F(z) = E\big(f(z)\big)$ is holomorphic on the disc $|z| < r$. Standard theorems of one complex variable imply that $R \geq r$.

We now assume that $R > r$ and derive a contradiction.

Since f is not constant $a_{n_0} > 0$ for some $n_0 > 0$ and hence

$$\begin{aligned} f(x) &= \sum_{n=0}^{+\infty} n a_n x^n \geq a_{n_0} x^{n_0} > 0, \\ f'(x) &= \sum_{n=1}^{+\infty} n a_n x^{n-1} \geq n_0 a_{n_0} x^{n_0 - 1} > 0 \end{aligned}$$

when $0 < x < r$.

Then if $0 < x < r$ we have

$$F'(x) = E'\big(f(x)\big)f'(x) > 0$$

and hence, by the Lagrange theorem, the function $F(x)$ is strictly increasing on the closed interval $[0, r]$. Since $F(r) > F(0)$ then

$$F(x) > F(0),\ \ 0 < x < r'$$

for some r' with

$$r < r' < R.$$

The hypotheses on E implies that the inverse function

$$E^{-1} :]E(0), +\infty[\to]0, +\infty]$$

is well defined and real analytic.

Being

$$F(x) > F(0) = E\big(f(0)\big) \geq E(0)$$

when $0 < x < r'$, it follows that the function

$$]0, r'[\ \ni x \mapsto u(x) := E^{-1}\big(F(x)\big) \in \mathbb{R}$$

is a well defined real analytic function which coincides with $f(x)$ when $0 < x < r$.

The power expansion of $u(x)$ at $x = r$ then defines a holomorphic extension of the function $f(z)$ in a neighbourhood of $z = r$ and this contradicts Theorem 3.1. □

We are now ready to prove Theorem 1.3.

So, let $\sigma_1,\ \sigma_2 \in \mathbb{R}$ and $f(s)$, $F(s)$ be given as in Theorem 1.3. Let us fix $a > \sigma_2$. Then the functions

$$f_a(z) = f(a - z)$$

and

$$F_a(z) = F(a - z)$$

are holomorphic respectively on the disks $|z| < a - \sigma_2$ and $|z| < a - \sigma_1$.

The relation $F_a(z) = \exp\big(f_a(z)\big)$ holds on the smaller disk $|z| < a - \sigma_2$ and for each integer $n \geq 0$

$$f_a^{(n)}(0) = (-1)^n f^{(n)}(a) \geq 0.$$

Theorem 3.1 (with $E = \exp$) implies then that the radius of convergence of the power series expansions of $f_a(z)$ and $F_a(z)$ at $z = 0$ are the same.

Since $F_a(z)$ is holomorphic on the disk $|z| < a - \sigma_1$ such a common radius of convergence is at least $a - \sigma_1$.

Hence the function $f_a(z)$, defined in the smaller disk $|z| < a - \sigma_2$, extends holomorphically on the bigger disk $|z| < a - \sigma_1$.

The formula

$$f(s) = f_a(a - s)$$

then defines a holomorphic extension of the function $f(s)$ on the disk $|z| < a - \sigma_1$.

The observation that the union of all the disks

$$|s - a| < a - \sigma_1,\ a > \sigma_2$$

is the open half space

$$\operatorname{Re} s > \sigma_1$$

completes the proof.

4 - Proof of Theorem 1.2

Here is the proof of Theorem 1.2.

So, let $\sigma_0 < 1$ and let $a : \mathbb{N}^+ \to \mathbb{C}$ be a bounded completely multiplicative function. Assume that the Dirichlet series

$$L(a, s) = \sum_{n=1}^{+\infty} \frac{a(n)}{n^s}$$

extends holomorphically in the half space $\operatorname{Re} s > \sigma_0$ and

$$L(a, 1) = 0.$$

We denote the function $L(a, s)$ simply with $L(s)$.

Since $a(n)$ is completely multiplicative and bounded then necessarily

$$|a(n)| \leq 1$$

for each $n \in \mathbb{N}^+$ and hence the series $L(s)$ converges absolutely when $\operatorname{Re} s > 1$.

The following lemma is well known, but we include a very elementary proof.

Lemma 4.1. *The series $L(s)$ satisfies*

$$L(s) = \exp\left(\sum_{n=2}^{+\infty} \frac{a(n)\Lambda(n)}{n^s \log n}\right)$$

when $\operatorname{Re} s > 1$.

Proof. Let

$$f(s) = \sum_{n=2}^{+\infty} \frac{a(n)\Lambda(n)}{n^s \log n}.$$

Then

$$f'(s) = -\sum_{n=2}^{+\infty} \frac{a(n)\Lambda(n)}{n^s}.$$

Since the arithmetic functions $a(n)$ and $\lambda(n)$ are completely multiplicative then the identity (4) easily implies that

$$f'(s)L(s) = L'(s).$$

Then the derivative of the function

$$s \mapsto \exp\bigl(-f(s)\bigr)L(s)$$

vanishes and hence is constant, that is

$$L(s) = c\exp\bigl(f(s)\bigr)$$

for some constant c. Now put $s = \sigma \in \mathbb{R}$; and take the limit as $\sigma \to +\infty$; Lemma 2.1 then implies

$$1 = a(1) = c\exp(0) = c.$$

□

When $a(n) = 1$ then $L(s) = \zeta(s)$ and hence we obtain

$$\zeta(s) = \exp\left(\sum_{n=1}^{+\infty} \frac{\Lambda(n)}{n^s \log n}\right).$$

Consider the function

$$F(s) = \zeta(s)^2 L(s)\overline{L(\overline{s})}.$$

Since $L(1) = 0$, then the function $F(s)$ is holomorphic on the open half space $\operatorname{Re} s > \sigma_0$ and when $\operatorname{Re} s > 1$ also satisfies

$$F(s) = \exp f(s)$$

where

$$f(s) = \sum_{n=1}^{+\infty} \frac{2\bigl(1 + \operatorname{Re} a(n)\bigr)\Lambda(n)}{n^s \log n}.$$

Clearly $f(s)$ is a Dirichlet series (absolutely) convergent when $\operatorname{Re} s > 1$ with non negative coefficients and hence the restriction of $f(s)$ to the real half line $]1, +\infty[$ is a completely monotone function.

Theorem 1.3 implies that $F(s)$ never vanishes on the half space $\operatorname{Re} s > \sigma_0$. This forces $\zeta(s) \neq 0$ when $\operatorname{Re} s > \sigma_0$ and the proof of Theorem 1.2 is completed.

We observe that the Ingham Theorem 1.1 easily follows from the Theorem 1.2.

Indeed let

$$F(s) = \sum_{n=1}^{+\infty} \frac{a(n)}{n^s}$$

be as in Theorem 1.1 and assume that $F(1 + it) = 0$ for some $t \in \mathbb{R}$.

Then the function $L(s) = F(s + it)$ satisfies the hypotheses of Theorem 1.2. Indeed $L(1) = 0$ and when $\operatorname{Re} s > 1$ we have

$$L(s) = F(s + it) = \sum_{n=1}^{+\infty} \frac{a(n)n^{-it}}{n^s}\,.$$

The map $n \mapsto a(n)n^{-it}$ is a bounded completely multiplicative arithmetic function. Theorem 1.2 then implies that the open half space $\operatorname{Re} s > 1/2 - \delta$ is a zero free region for the Riemann zeta function and this is not the case.

5 - A refined non vanishing principle

In order to extend Theorem 1.3 we need a refined version of Theorem 3.1.

Theorem 5.1. *Let $R, \delta > 0$ be positive real numbers. Let*

$$u :] - \delta, R[\to \mathbb{R}$$

be a real analytic function. If

$$a_n := \frac{u^{(n)}(0)}{n!} \geq 0, \ n = 0, 1, \ldots,$$

then the radius of convergence of the series

$$\sum_{n=0}^{+\infty} a_n z^n$$

is greater than R.

P r o o f. Let us denote by $r > 0$ the radius of convergence of the series

$$f(z) = \sum_{n=0}^{+\infty} a_n z^n.$$

Then $f(z)$ is holomorphic on the disc $|z| < r$ and $f(x) = u(x)$ when $0 \leq x < r$.

Assume that $r < R$. As $u(x)$ is real analytic it follows that the power series expansion of $u(x)$ at $x = r$ defines a holomorphic extension of the function $f(z)$ in a neighbourhood of $z = r$. But this is not allowed by Theorem 3.1.

It necessarily follows that $r \geq R$ and we are done. □

Let us recall that a C^∞ function

$$u : I \to \mathbb{R}$$

where I is a interval of $\mathbb{R}$ is *absolutely monotone* if for $k = 0, 1, \ldots$

$$u^{(k)}(x) \geq 0$$

for each $x \in I$.

The absolutely monotone functions were introduced by S. Bernstein in [**3**]. For a comprehensive treatment of the theory of such functions (and their cousins the completely monotone ones) see, e.g., [**25**, Chapter IV]; see also [**21**, Chapter 1].

The following is an extension of Theorem 1.3.

T h e o r e m 5.2. *Let $\sigma_1, \sigma_2 \in \mathbb{R}$ with $\sigma_1 < \sigma_2$ and let $f(s)$ be a holomorphic function defined on the open half space*

$$\operatorname{Re} s > \sigma_2.$$

Assume that the restriction of $f(s)$ to the half line $]\sigma_2, +\infty[$ is a real completely monotone function and

$$F(s) = \exp f(s)$$

extends holomorphically in an open neighbourhood of the real interval $]\sigma_1, \sigma_2]$.

Then both the functions $f(s)$ and $F(s)$ extend holomorphically on the open half space

$$\operatorname{Re} s > \sigma_1,$$

the relation

$$F(s) = \exp f(s)$$

still holds on such a half space and hence

$$F(s) \neq 0, \ \operatorname{Re} s > \sigma_1.$$

P r o o f. As in the proof of Theorem 5.2 we choose $a > \sigma_2$ and consider the functions

$$f_a(z) = f(a - z)$$

and

$$F_a(z) = F(a - z).$$

They are both holomorphic functions on the disk $|z| < a - \sigma_2$ and the function $F_a(z)$ is also holomorphic in a neighbourhood of the segment $[0, a - \sigma_1[$.

Since f is completely monotone on the interval $]\sigma_2, 2a - \sigma_2[$ then f_a is absolutely monotone on the interval $]\sigma_2 - a, a - \sigma_2[$.

Since the composition of absolutely monotone functions is absolutely monotone it follows that the function $F_a(z) = \exp\big(f_a(z)\big)$ is also absolutely monotone on the interval $]\sigma_2 - a, a - \sigma_2[$.

But the function $F_a(z)$ is real analytic on the interval $]\sigma_2 - a, a - \sigma_1[$.

Then Theorem 5.1 implies that the radius of convergence of the powers expansion of $F_a(t)$ at $t = 0$ is greater that $a - \sigma_1$.

Proposition 3.1 implies that both the powers expansions of $f_a(z)$ and $F_a(z)$ at $z = 0$ have the same radius of convergence at least $a - \sigma_1$ and hence provide holomorphic extensions respectively of $f(s)$ and $F(s)$ on the disc $|s - a| < a - \sigma_1$.

As in the proof of Theorem 5.2 we conclude the proof by observing that the union of all the disks

$$|s - a| < a - \sigma_1, \ a > \sigma_2$$

is the open half space $\mathrm{Re}\, s > \sigma_1$. □

6 - The main theorem

In this section we consider a fixed real arithmetic function

$$\nu : \mathbb{N}^+ \to \mathbb{R}$$

satisfying the following properties:

1. ν is completely multiplicative, that is

$$\nu(mn) = \nu(n)\nu(m)$$

for each $m, n \in \mathbb{N}^+$;

2. $\nu(n) > 1$ for each $n > 1$;

3. for a certain $\sigma_1 > 0$ the (generalized) Dirichlet series

$$Z(s) = \sum_{n=1}^{+\infty} \frac{1}{\nu(n)^s}$$

converges for $\operatorname{Re} s > \sigma_1$ and

$$\lim_{\sigma \to \sigma_1^+} Z(\sigma) = +\infty.$$

Observe that condition (3) forces

(5) $$\lim_{n \to +\infty} \nu(n) = +\infty$$

and hence

Proposition 6.1. *There exists $\nu_0 > 1$ such that*

$$\nu(n) \geq \nu_0$$

for each $n > 1$.

Observe that we do not require the sequence $n \mapsto \nu(n)$ to be not decreasing.

The basic example is given by the Dedekind zeta function of number field (see, e.g., [**9**], [**15**]). Let K be a number field, that is a finite extension of the rational field $\mathbb{Q}$ and let $\mathfrak{o}$ be the ring of the algebraic integers in K. Consider a one to one bijection

(6) $$p \mapsto \mathfrak{p}_p$$

between the positive prime numbers $P = \{2, 3, 5, \ldots\}$ and the non zero prime (i.e. maximal) ideals of $\mathfrak{o}$.

Since the non zero ideals of $\mathfrak{o}$ factor uniquely as product of non zero prime ideals then we can extend (6) to a bijection

$$n \mapsto I_n$$

between the positive integers $n \in \mathbb{N}^+$ in such a way that if

$$n = p_1^{\mu_1} \cdots p_k^{\mu_k}$$

then

$$I_n = \mathfrak{p}_{p_1}^{\mu_1} \cdots \mathfrak{p}_{p_k}^{\mu_k}.$$

Then the arithmetic function

$$\nu(n) = \mathfrak{N}(I_n)$$

where $\mathfrak{N}(I_n)$ is the *norm* of the ideal $\mathfrak{p}_n$ satisfies the properties (1), (2) and (3) above.

Then

$$Z(s) = \sum_{n=1}^{+\infty} \frac{1}{\nu(n)^s} = \zeta_K(s),$$

where

$$\zeta_K(s) = \sum_I \frac{1}{\mathfrak{N}(I)^s}$$

is the Dedekind zeta function of the number field K.

Other examples arise from abstract arithmetic semigroups (see [**8**]), Beurling's generalized prime numbers (see [**10**, Section 8.4, pag. 266], [**4**]), and the zeta function associated to an arithmetical scheme (see [**23**]).

The purpose of this section is to extend Theorem 1.2 to (generalized) Dirichlet series of the form

$$\sum_{n=1}^{+\infty} \frac{a(n)}{\nu(n)^s}.$$

Clearly if the arithmetic function $a(n)$ is bounded then such a series defines a holomorphic function on the open half space $\operatorname{Re} s > \sigma_1$.

We need to prove some elementary properties of such a series.

Let us begin extending Lemma 2.1 and Lemma 4.1 to such a series.

Lemma 6.1. *Let*

$$F(s) = \sum_{n=1}^{+\infty} \frac{a(n)}{\nu(n)^s}$$

where $a : \mathbb{N}^+ \to \mathbb{C}$ *is a bounded arithmetic function. Then*

$$\lim_{\sigma\to\infty} F(\sigma) = a(1).$$

Proof. Let $\nu_0 > 1$ such that $\nu(n) \geq \nu_0$ when $n > 1$.

Fix $\sigma_2 > \sigma_1$. Then for each $\sigma > \sigma_2$ we have

$$|F(\sigma) - a(1)| \leq \sum_{n=2}^{+\infty} \frac{|a(n)|}{\nu(n)^\sigma} \leq \frac{1}{\nu_0^{\sigma-\sigma_2}} \sum_{n=2}^{+\infty} \frac{|a(n)|}{\nu(n)^{\sigma_2}}.$$

Since

$$\lim_{\sigma\to+\infty} \frac{1}{\nu_0^{\sigma-\sigma_2}} = 0$$

the assertion follows. □

We define the $\nu-$von Mangoldt function

$$\Lambda_\nu(n) = \begin{cases} \log \nu(p) & \text{if } n = p^k \text{ for some prime } p \text{ and integer } k \geq 1, \\ 0 & \text{otherwise.} \end{cases}$$

As in the classical case it is easy to prove that for each $n > 0$

$$\sum_{d|n} \Lambda_\nu(d) = \log \nu(n). \tag{7}$$

Proposition 6.2. *Let*

$$F(s) = \sum_{n=1}^{+\infty} \frac{a(n)}{\nu(n)^s}$$

where $a : \mathbb{N}^+ \to \mathbb{C}$ *is a bounded completely multiplicative arithmetic function. Then when* $\operatorname{Re} s > \sigma_1$

$$F(s) = \exp\left(\sum_{n=2}^{+\infty} \frac{a(n)\Lambda_\nu(n)}{\nu(n)^s \log \nu(n)} \right).$$

Proof. Let

$$f(s) = \sum_{n=2}^{+\infty} \frac{a(n)\Lambda_\nu(n)}{\nu(n)^s \log \nu(n)}.$$

Then

$$f'(s) = -\sum_{n=2}^{+\infty} \frac{a(n)\Lambda_\nu(n)}{\nu(n)^s}.$$

Since the arithmetic functions $a(n)$ and $\lambda(n)$ are completely multiplicative then (7) easily implies that

$$f'(s)F(s) = F'(s).$$

Then the derivative of the function

$$\exp\bigl(-f(s)\bigr)F(s)$$

vanishes, that is

$$F(s) = c\exp\bigl(f(s)\bigr)$$

for some constant c. Then put $s = \sigma \in \mathbb{R}$ and take the limit as $\sigma \to +\infty$; lemma 6.1 then implies

$$1 = a(1) = c \exp(0) = c.$$

□

Let us denote by P the set of positive prime numbers.

Proposition 6.3.

$$\lim_{\sigma \to \sigma_1^+} \sum_{p \in P} \frac{1}{\nu(p)^\sigma} = +\infty\,.$$

Proof. If $\sigma > \sigma_1$ proposition 6.2 implies

$$\begin{aligned} Z(\sigma) &= \exp\left(\sum_{n=2}^{+\infty} \frac{\Lambda_\nu(n)}{\nu(n)^\sigma \log \nu(n)}\right) = \exp\left(\sum_{p \in P} \sum_{m=1}^{+\infty} \frac{1}{m\nu(p)^{m\sigma}}\right) \\ &= \exp\left(\sum_{p \in P} \frac{1}{\nu(p)^\sigma}\right) \exp\bigl(h(\sigma)\bigr) \end{aligned}$$

where

$$h(\sigma) = \sum_{p \in P} \sum_{m=2}^{+\infty} \frac{1}{m\nu(p)^{m\sigma}}\,.$$

As in the classical case, using the fact that $\nu(n) \geq \nu_0 > 1$ when $n > 1$ it is easy to show that the series $h(\sigma)$ converges for $\sigma > \sigma_1/2$ and hence

$$\lim_{\sigma \to \sigma_1^+} \sum_{p \in P} \frac{1}{\nu(p)^\sigma} = \lim_{\sigma \to \sigma_1^+} \bigl(\log Z(\sigma) - h(\sigma)\bigr) = +\infty.$$

□

We now are ready to state and prove the main theorem of the paper.

Theorem 6.1. *Let $\sigma_0 < \sigma_1$ and let $D \subset \mathbb{C}$ be an open connected subset containing the real half line $]\sigma_0, +\infty[$ and contained in the open half space* $\operatorname{Re} s > \sigma_0$. *Assume also that D is symmetric with respect to the real axis, that is $s \in D \implies \overline{s} \in D$ and contains the open half space* $\operatorname{Re} s > \sigma_1$.

Let

$$L : D \to \mathbb{C}$$

be a meromorphic function on D which is holomorphic on $D \cap \mathbb{R} \setminus \{\sigma_1\}$ and

$$L(s) = \sum_{n=1}^{+\infty} \frac{a(n)}{\nu(n)^s}$$

when $\operatorname{Re} s > 1$, *where $a(n)$ is a bounded completely multiplicative arithmetic function.*

Assume also that

$$Z(s) = \sum_{n=1}^{+\infty} \frac{1}{\nu(n)^s}$$

extends to a meromorphic function on D with a unique simple pole at $s = \sigma_1$.

Let $s_0 \in D \setminus \{\sigma_1\}$ be given. Then:

1. *if σ_1 is a zero or a pole of $L(s)$ and $Z(s_0) \neq 0$ then s_0 is a zero (resp. a pole) of $L(s)$ if, and only if, $\overline{s_0}$ is a pole (resp. a zero) of $L(s)$;*

2. *if $L(\sigma_1) = 0$ and $L(s)$ is holomorphic at $s = s_0$ and $s = \overline{s_0}$ then $Z(s_0) \neq 0$; in particular, if $L(s)$ is holomorphic on D then D is a zero free region for $Z(s)$;*

3. *if $L(\sigma_1) = 0$ then the boundary point σ_0 of D satisfies the constraint*

$$\sigma_0 \geq \frac{\sigma_1}{2}.$$

Proof. Since $a(n)$ is completely multiplicative and bounded then necessarily

$$|a(n)| \leq 1$$

for each $n \in \mathbb{N}^+$ and by Proposition 6.2 we also have

$$L(s) = \exp\left(\sum_{n=1}^{+\infty} \frac{a(n)\Lambda_\nu(n)}{\nu(n)^s \log \nu(n)}\right)$$

and analogously

$$Z(s) = \exp\left(\sum_{n=1}^{+\infty} \frac{\Lambda_\nu(n)}{\nu(n)^s \log \nu(n)}\right).$$

Consider the functions

$$F(s) = Z(s)^2 L(s) \overline{L(\overline{s})}$$

and

$$G(s)=\frac{Z(s)^2}{L(s)\overline{L(\overline{s})}}\,.$$

When $\operatorname{Re} s>1$

$$F(s)=\exp f(s),\;\; G(s)=\exp g(s)$$

where

$$f(s)=\sum_{n=1}^{+\infty}\frac{2\big(1+\operatorname{Re}a(n)\big)\Lambda_\nu(n)}{\nu(n)^s\log\nu(n)}$$

and

$$g(s)=\sum_{n=1}^{+\infty}\frac{2\big(1-\operatorname{Re}a(n)\big)\Lambda_\nu(n)}{\nu(n)^s\log\nu(n)}\,.$$

Clearly $f(s)$ and $g(s)$ are Dirichlet series (absolutely) convergent when $\operatorname{Re} s>\sigma_1$ with non negative coefficients and hence their restriction to the real half line $]\sigma_1,+\infty[$ are completely monotone functions.

Assume now that $L(\sigma_1)=0$. Theorem 5.2 then implies that $F(s)$ never vanishes on D and hence

$$L(s)\overline{L(\overline{s})}=\frac{F(s)}{Z(s)^2}\,.$$

Similarly, if $L(s)$ has a pole at $s=\sigma_1$ we obtain

$$L(s)\overline{L(\overline{s})}=\frac{Z(s)^2}{G(s)}$$

with $G(s)$ never vanishing on D.

In both cases (1) easily follows.

If $L(\sigma_1)=0$ then $F(s)=Z(s)^2L(s)\overline{L(\overline{s})}$ never vanishes on D and (2) also follows.

It remains to prove (3), that is the inequality $\sigma_0\geq\sigma_1/2$ assuming $L(\sigma_1)=0$.

For this purpose observe that Theorem 5.2 also implies that the defined above function

$$f(s)=\sum_{n=1}^{+\infty}\frac{2\big(1+\operatorname{Re}a(n)\big)\Lambda_\nu(n)}{\nu(n)^s\log\nu(n)}$$

extends holomorphically on the open half space $\operatorname{Re} s>\sigma_0$.

The Landau's Theorem 2.1 also holds for these generalized series (see, e.g., [**10**, Lemma 15.1, pag. 463]) and hence the series $f(s)$ also converges when $\operatorname{Re} s>\sigma_0$.

We will obtain the desired inequality $\sigma_0 \geq \sigma_1/2$ showing that the series $f(s)$ diverges at $s = \sigma_1/2$.

An argument similar to the one given at the end of the introduction of the paper yields

$$f(\sigma) \geq \frac{7}{8} \sum_{p \in P} \frac{1}{\nu(p)^{2\sigma}}\,.$$

Proposition 6.3 then implies $f(\sigma) \to +\infty$ as $\sigma \to \sigma_1/2$ from the left, as desired. □

7 - Corollaries

In this section we will prove several immediate corollaries of Theorem 6.1 which in a simple and unified manner gives various non vanishing results for zeta and L like functions on the boundary of the half plane of absolute convergence.

Let

$$\nu : \mathbb{N}^+ \to \mathbb{R}$$

be as in the previous section with the associated "zeta function"

$$Z(s) = \sum_{n=1}^{+\infty} \frac{1}{\nu(n)^s}$$

having $\sigma_1 > 0$ as abscissa of absolute convergence.

Let us begin with the "prime number theorem". Such a theorem is already proved in the literature (see [**12**] and [**11**, Theorem 1.2, pag. 10]) but our proof is very easy.

Corollary 7.1. *If $Z(s)$ extends in an open neighbourhood of the closed half space*

$$\operatorname{Re} s \geq \sigma_1$$

to a meromorphic function with a unique simple pole at $s = \sigma_1$ then

$$Z(\sigma_1 + it) \neq 0$$

for each $t \in \mathbb{R} \setminus \{0\}$.

Proof. Let $t \in \mathbb{R} \setminus \{0\}$ and suppose that $Z(\sigma_1 + it) = 0$. Since $\overline{Z(\overline{s})} = Z(s)$ then also $Z(\sigma_1 - it) = 0$.

Set $L(s) = Z(s + it)$. When $\operatorname{Re} s > \sigma_1$

$$L(s) = \sum_{n=1}^{+\infty} \frac{\nu(n)^{-it}}{\nu(n)^s},$$

and the function $n \mapsto \nu(n)^{-it}$ is bounded and completely multiplicative. We also have $L(\sigma_1) = 0$ and $L(\sigma_1 - 2it) = 0$. Then assertion (1) of Theorem 6.1 implies that $L(s)$ has a pole at $s = \sigma_1 + 2it$, that is $Z(s)$ has (another) pole at $s = \sigma_1 + 3it$ and this contradicts the hypotheses made on $Z(s)$. □

Let also

$$a : \mathbb{N}^+ \to \mathbb{R}$$

be a bounded completely multiplicative function with the associated "$L-$function"

$$L(s) = \sum_{n=1}^{+\infty} \frac{a(n)}{\nu(n)^s}.$$

Most non vanishing theorem for $L-$ function associated to various Dirichlet/Hecke characters follow from the following statement.

Corollary 7.2. *Let $Z(s)$ and $L(s)$ be given. Assume that $L(s)$ is meromorphic on an open neighbourhood of the closed half space*

$$\operatorname{Re} s \geq \frac{\sigma_1}{2}$$

and also $Z(s)$ is holomorphic there with the exception of a simple pole at $s = \sigma_1$.

If $L(\sigma_1 + it) = 0$ for some $t \in \mathbb{R}$ then the fuction $L(s)$ admits at least a pole at $s = \sigma + it$ for some σ satisfying

$$\frac{\sigma_1}{2} \leq \sigma < \sigma_1.$$

In particular, if $L(s)$ is holomorphic on such a neighbourhood then

$$L(\sigma_1 + it) \neq 0$$

for each $t \in \mathbb{R}$.

Proof. Let $t \in \mathbb{R}$ and assume that $L(\sigma_1 + it) = 0$. Then the function

$$L_t(s) := L(s + it)$$

satisfies $L_t(\sigma_1) = 0$.

If $L_t(s)$ has no poles at $s = \sigma \geq \sigma_1/2$ then $L_t(s)$ is holomorphic up to the left of $s = \sigma_1/2$, contradicting (3) of Theorem 6.1. $\square$

Observe that if the Riemann hypothesis holds the constraint $\operatorname{Re} s \geq \sigma_1/2$ in the corollary above is optimal. Indeed, when $\sigma_1 = 1$ and $\nu(n) = n$, that is, $Z(s) = \zeta(s)$, the Riemann zeta function, then the function

$$L(s) = \frac{\zeta(2s)}{\zeta(s)} = \sum_{n=1}^{+\infty} \frac{\lambda(n)}{n^s},$$

where $\lambda(n)$ is the Liouville function, is holomorphic on the open half space $\operatorname{Re} s > 1/2$ and $L(1) = 0$.

We end with a curiosity.

Lemma 7.1. *Let A and B two subset of $\mathbb{R}$. Assume that $0 \in A$ and for each pair of distincts reals x, y whenever*

$$\frac{x+y}{2} \in A$$

then

$$x \in A \Longleftrightarrow y \in B.$$

If $b \in B$ then $3b \in A \cap B$ and if $a \in A$ then $-3a \in A \cap B$.

Proof. Since $0 \in A$ then for each $x \neq 0$

$$x \in A \Longleftrightarrow -x \in B.$$

Let $b \in B$ be given. If $b = 0$ then the assertion is trivially verified. Assume hence that $b \neq 0$.

Then we have $-b \in A$ and

$$\begin{aligned}
\frac{(-2b)+0}{2} &= -b \in A,\ 0 \in A &&\Longrightarrow && -2b \in B,\ 2b \in A,\\
\frac{(-3b)+b}{2} &= -b \in A,\ b \in B &&\Longrightarrow && -3b \in A,\ \boxed{3b \in B},\\
\frac{b+3b}{2} &= 2b \in A,\ b \in B &&\Longrightarrow && \boxed{3b \in A}.
\end{aligned}$$

Thus we see that $3b \in A \cap B$, as required.

Let now $a \in A$. If $a = 0$ then the assertion is trivially verified and if $a \neq 0$ then $-a \in B$ and we have just seen that then $-3a \in A \cap B$, as desired. $\square$

Proposition 7.1. *Let $Z(s)$ and $L(s)$ be meromorphic in an open neighbourhood of the closed half space*

$$\operatorname{Re} s \geq \sigma_1.$$

Assume that $Z(s)$ has a unique simple pole at $s = \sigma_1$. If σ_1 is a pole or a zero of $L(s)$ then $L(s)$ has no poles on the line $\operatorname{Re} s = \sigma_1$ and $L(\sigma_1 + it) \neq 0$ for each real $t \neq 0$.

Proof. Assume first that $L(\sigma_1) = 0$. Let $A, B \subset \mathbb{R}$ the set of $t \in \mathbb{R}$ such that $\sigma_1 + it$ is respectively a zero or a pole of $L(s)$.

We now show that A and B satisfy the hypotheses of Lemma 7.1.

Of course $0 \in A$ being $L(\sigma_1) = 0$. Let $u, v \in \mathbb{R}$ with $u \neq v$ and assume that

$$t := \frac{u + v}{2} \in A,$$

that is $L(\sigma_1 + it) = 0$.

Consider then the function

$$L_t(s) := L(s + it)$$

and set $s_0 = \sigma_1 + i(u - t)$.

Since $L_t(\sigma) = 0$ then assertion (3) of Theorem 6.1 implies that s_0 is a zero of $L_t(s)$ if, and only if, $\overline{s_0}$ is a pole of $L_t(s)$.

Observing that and $\overline{s_0} = \sigma_1 + i(v - t)$ we obtain that $u \in A$ if, and only if, $v \in B$, as desired.

Since obviously $A \cap B = \emptyset$ Lemma 7.1 then forces $B = \emptyset$ and $A = \{0\}$ and this completes the proof of the proposition in the case $L(\sigma_1) = 0$.

Assume now that σ_1 is a pole of $L(s)$. Then it suffices to repeat the aforementioned argument interchanging A with B and observing that the function $L_t(s)$ has a pole at $s = \sigma_1$ instead of a zero.

The proof of the proposition is so completed. □

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