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A family of Cauchy-Riemann type operators

Abstract. A natural question is whether and in which sense the definition of a holomorphic function depends on the choice of the two vectors $\{1, i\}$ that form a basis of $\mathbb{C}$ over $\mathbb{R}$. In fact these two vectors determine both the form of the Cauchy-Riemann operator, and the splitting of a holomorphic function in its harmonic real and imaginary components. In this paper we consider the basis $\{1, e^{i\theta}\}$ of $\mathbb{C}$ over $\mathbb{R}$, and define as *θ-holomorphic* the functions that belong to the kernel of a Cauchy-Riemann type operator determined by this basis. We study properties of these functions, and discuss the relation between them and classical holomorphic functions. This analysis will lead us to discover the special role that $\theta = \pi/2$ plays, that renders the theory of holomorphic functions special among this family of theories.

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1 - Introduction

One of the most successful function theories is the theory of holomorphic functions of a complex variable. Born in the nineteenth century, it was fully developed only during the last century, [**1**].

This paper was inspired by a basic question posed by E. Vesentini while he was at the Scuola Normale Superiore, Pisa, together with the first and third author. The approach and the results we present here are elementary,

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but we think they are worth exploring as they allow us a different look at holomorphicity for functions of a single complex variable.

Put simply, one considers the field $\mathbb{C}$ of complex numbers as the field generated, over the reals, by two independent units, 1, and i, where $i^2 = -1$; its elements are traditionally indicated by $z = x + iy$. One then considers the notion of derivability and differentiability for functions from $\mathbb{C}$ to $\mathbb{C}$, and the interesting (and crucially important) discovery is the fact that a function f admits complex derivative on an open set Ω if and only if, in that same set, it satisfies the so-called Cauchy-Riemann equation:

$$\frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right) = 0 \tag{1}$$

which defines the class of *holomorphic* functions. Since every function $f : \mathbb{C} \to \mathbb{C}$ can be written, using the same basis $(1, i)$, as $f(x + iy) = u(x, y) + iv(x, y)$, where u and v are differentiable functions from $\mathbb{R}^2$ to $\mathbb{R}$, the Cauchy-Riemann equation is actually equivalent (as one can see by recalling that 1 and i are independent over the reals) to the Cauchy-Riemann system

$$\begin{cases} u_x = v_y \\ u_y = -v_x \end{cases}$$

where as customary the subscripts indicate partial derivatives. The special form of this system is of great interest, because it immediately implies that both u and v are harmonic functions, i.e.

$$\Delta u := u_{xx} + u_{yy} = \Delta v = v_{xx} + v_{yy} = 0$$

and we say indeed that the real part u of a holomorphic function and its imaginary part v are *harmonic conjugate* to each other. This is not the place to give a full description of the theory of holomorphic functions, but some of the most fecund consequences of these definitions lie in the fact that the simple calculations above indicate a way to actually *factor* the Laplacian operator by writing

$$\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = \left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right).$$

The approach to the new definition of slice regularity for quaternion valued functions of a quaternionic variable ([**2**, **3**]) has brought a renewed attention on the possibility of varying the imaginary unit which appears in the Cauchy-Riemann operator. Indeed the family of Cauchy-Riemann operators used in the definition of slice regular functions is of the form

$$\frac{1}{2}\left(\frac{\partial}{\partial x} + I\frac{\partial}{\partial y}\right)$$

where I is any element of the 2-sphere of imaginary units in the skew field of quaternions.

In the complex setting, one can then ask what would have happened if, instead of the canonical basis $(1, i)$, one would have taken any other unit vector in $\mathbb{C}$, independent of 1. In other words, one can ask whether and in which sense the definition of a holomorphic function depends on the choice of the two vectors $\{1, i\}$ that form a basis of $\mathbb{C} \cong \mathbb{R}^2$ over $\mathbb{R}$. In fact these two vectors determine both the form of the Cauchy-Riemann operator, and the splitting of a holomorphic function in its harmonic real and imaginary components. More precisely, we consider the basis $\{1, e^{i\theta}\}$ of $\mathbb{R}^2 \cong \mathbb{C}$, and define as *$\theta$-holomorphic* the functions that belong to the kernel of the operator

$$\left(\frac{\partial}{\partial x} + e^{i\theta}\frac{\partial}{\partial y}\right).$$

In this paper we study properties of these functions, and discuss the relation between them and classical holomorphic functions. This analysis will lead us to discover the special role that $\theta = \pi/2$ plays, that renders the theory of holomorphic functions special among this family of theories.

2 - θ-holomorphic functions

For any $\theta \in (0, 2\pi)$, $\theta \neq \pi$, we consider the basis $\{1, e^{i\theta}\}$ of $\mathbb{C} \cong \mathbb{R}^2$. In the next definition we will give a notion of holomorphicity with respect to the chosen basis.

Definition 2.1. *Let Ω be a domain in $\mathbb{C} \cong \mathbb{R}^2$ and let $\theta \in (0, 2\pi)$, $\theta \neq \pi$. A function $f : \Omega \to \mathbb{C}$ having continuous partial derivatives is called* θ-holomorphic *if*

$$\left(\frac{\partial}{\partial x} + e^{i\theta}\frac{\partial}{\partial y}\right) f(x + e^{i\theta}y) = 0$$

on Ω.

As customary any complex valued function f defined on a domain $\Omega \subseteq \mathbb{C} \cong \mathbb{R}^2$ can be split in the new basis as

$$f(x + e^{i\theta}y) = u(x, y) + e^{i\theta}v(x, y)$$

where $u, v : \Omega \to \mathbb{R}$ are real valued functions on Ω.

Proposition 2.2. *Let* $\Omega \subseteq \mathbb{C} \cong \mathbb{R}^2$ *be a domain. A function* $f : \Omega \to \mathbb{C}$, $f = u + e^{i\theta} v$, *is* θ*-holomorphic if and only if the following system of partial differential equations holds for* u, v

$$\begin{cases} u_x = v_y \\ u_y + v_x + 2v_y \cos\theta = 0 \end{cases}$$

on Ω.

Proof. The equality

$$\begin{aligned}\left(\frac{\partial}{\partial x} + e^{i\theta}\frac{\partial}{\partial y}\right)(u + e^{i\theta} v) &= u_x + e^{i\theta} u_y + e^{i\theta} v_x + e^{2i\theta} v_y \\ &= u_x + e^{i\theta} u_y + e^{i\theta} v_x + (-1 + 2\cos\theta\, e^{i\theta}) v_y \\ &= u_x - v_y + e^{i\theta}(u_y + v_x + 2v_y \cos\theta)\end{aligned}$$

directly implies the assertion. □

By means of the previous equivalence it is possible to identify a θ-holomorphic "variable". If for example we choose x to be the first component, then we can solve the system

$$\begin{cases} v_y = 1 \\ v_x = -2\cos\theta \end{cases}$$

obtaining $v = y + c(x)$ with $c'(x) = -2\cos\theta$ which leads to $v = y - 2x\cos\theta$ (up to an additive constant).

Definition 2.3. *The* θ*-holomorphic function* $Z(x, y) = x + e^{i\theta}(y - 2x\cos\theta)$ *is called the* θ-holomorphic variable.

It is worthwhile noticing that the choice of the θ-holomorphic variable is arbitrary (as in the holomorphic case). Instead of x as the first component of Z we could have chosen any linear combination $ax + by$ of x and y. In this general case the second component $w(x, y)$ of Z can be computed by solving (see Proposition 2.2)

$$\begin{cases} w_y = a \\ w_x = -b - 2a\cos\theta\,. \end{cases}$$

Hence $w = ay + c(x)$ with $c'(x) = -b - 2a\cos\theta$, which gives

$$w(x, y) = ay - bx - 2ax\cos\theta$$

(up to an additive constant).

The family of θ-holomorphic functions is closed with respect to pointwise multiplication.

Proposition 2.4. *If $f, g : \Omega \to \mathbb{C}$ are θ-holomorphic functions on a domain Ω, then the pointwise product fg is θ-holomorphic; therefore θ-holomorphic functions have the structure of an algebra.*

Proof. Let $f = u + e^{i\theta}v$ and $g = a + e^{i\theta}b$. Then

$$fg = (u + e^{i\theta}v)(a + e^{i\theta}b) = ua - vb + e^{i\theta}(ub + va + 2vb\cos\theta) = U + e^{i\theta}V.$$

Computing partial derivatives of U and V we get

$$\begin{aligned} U_x &= u_x a + u a_x - v_x b - v b_x, \\ U_y &= u_y a + u a_y - v_y b - v b_y, \\ V_x &= u_x b + u b_x + v_x a + v a_x + 2(v_x b + v b_x)\cos\theta, \\ V_y &= u_y b + u b_y + v_y a + v a_y + 2(v_y b + v b_y)\cos\theta, \end{aligned}$$

so that

$$U_x - V_y = (u_x - v_y)a + u(a_x - b_y) - (u_y + v_x + 2v_y\cos\theta)b - v(b_x + a_y + 2b_y\cos\theta) = 0$$

and

$$\begin{aligned} &U_y + V_x + 2V_y\cos\theta \\ &\quad = u_y a + u a_y - v_y b - v b_y + u_x b + u b_x + v_x a + v a_x + 2(v_x b + v b_x)\cos\theta \\ &\qquad + 2\cos\theta(u_y b + u b_y + v_y a + v a_y + 2(v_y b + v b_y)\cos\theta) \\ &\quad = (u_y + v_x + 2v_y\cos\theta)a + u(a_y + b_x + 2b_y\cos\theta) + (-v_y + u_x)b \\ &\qquad + (u_y + v_x + 2v_y\cos\theta)2b\cos\theta + v(a_x - b_y) + 2v(a_y + b_x + 2b_y\cos\theta)\cos\theta \\ &\quad = 0. \end{aligned}$$

□

As a consequence, complex polynomials and convergent power series in the variable $Z(x, y) = x + e^{i\theta}(y - 2\cos\theta\, x)$ define θ-holomorphic functions.

Lemma 2.5 (Abel). *Let $\{a_n\}_{n\in\mathbb{N}}$ be a sequence of complex numbers such that $\liminf_{n\to\infty} \dfrac{1}{\sqrt[n]{|a_n|}} = R > 0$. Then the power series*

$$\sum_{n\geq 0} Z(x, y)^n a_n$$

converges uniformly on compact sets to a θ-holomorphic function, in the interior of the ellipse $x^2 + y^2 - 2xy\cos\theta = R^2$.

Proof. Thanks to the classical Abel's Lemma, it is sufficient to compute the modulus of $Z(x,y)$,

$$\begin{aligned}|Z(x,y)|^2 &= |x + e^{i\theta}(y - 2x\cos\theta)|^2\\ &= |x + \cos\theta(y - 2x\cos\theta) + i\sin\theta(y - 2x\cos\theta)|^2\\ &= x^2 + \cos^2\theta\,(y - 2x\cos\theta)^2 + 2x\cos\theta(y - 2x\cos\theta)\\ &\qquad + \sin^2\theta(y - 2x\cos\theta)^2\\ &= x^2 + y^2 + 4x^2\cos^2\theta - 4xy\cos\theta + 2yx\cos\theta - 4x^2\cos^2\theta\\ &= x^2 + y^2 - 2xy\cos\theta\,.\end{aligned}$$

□

The ellipse $x^2 + y^2 - 2xy\cos\theta = R^2$ that bounds the domain of convergence of a θ-holomorphic power series of radius of convergence R is centered at the origin and it is symmetric with respect to the lines $y = x$ and $y = -x$; its extremal radii are respectively $R/\sqrt{1-\cos\theta}$ and $R/\sqrt{1+\cos\theta}$. If $\theta = \pi/2$, i.e. for holomorphic power series, the domains of convergence are discs centered at the origin.

The two components of any θ-holomorphic function are not harmonic unless $\theta = \pi/2$, or $\theta = 3\pi/2$. Nevertheless they both belong to the kernel of a strongly elliptic partial differential operator.

Proposition 2.6. *Let $f = u + e^{i\theta}v$ be a θ-holomorphic function on a domain Ω. Then*

$$T_\theta(u) := \Delta u + 2u_{xy}\cos\theta = T_\theta(v) = \Delta v + 2v_{xy}\cos\theta = 0$$

on Ω.

Proof. Using Proposition 2.2 we get

$$u_{xx} = v_{yx} = v_{xy} = -u_{yy} - 2v_{yy}\cos\theta = -u_{yy} - 2u_{xy}\cos\theta$$

and

$$v_{yy} = u_{xy} = u_{yx} = -v_{xx} - 2v_{yx}\cos\theta\,.$$

□

We will now generalize what we have done to construct the θ-holomorphic variable Z.

Proposition 2.7. *Let $u : \Omega \to \mathbb{R}$ be a C^2-function on a domain $\Omega \subseteq \mathbb{R}^2$ such that $T_\theta(u) = 0$. Then there exists a function v, belonging to the kernel of T_θ as well, such that $u + e^{i\theta}v$ is θ-holomorphic.*

Proof. We can use Proposition 2.2 to identify v:

$$\begin{cases} v_y = u_x \\ v_x = -u_y - 2u_x \cos\theta \, . \end{cases}$$

Hence $v(x,y) = \int u_x(x,y)dy + c(x)$ with $c'(x) = -\int u_{xx}(x,y)dy - u_y(x,y) - 2u_x(x,y)\cos\theta$. Therefore (up to an additive constant)

$$c(x) = -\int u_x(x,y)dy - \int u_y(x,y)dx - 2u(x,y)\cos\theta$$

and

$$v(x,y) = -\int u_y(x,y)dx - 2u(x,y)\cos\theta \, . \qquad \square$$

Notice that if $\theta = \pi/2$ this procedure corresponds to the one that associates with a harmonic function u (one of) its harmonic conjugates, and hence to construct a holomorphic function with an assigned real part.

In analogy with the factorization of the Laplacian through the Cauchy-Riemann operator, we now see that it is possible to factorize the operator $T_\theta = \Delta + 2\cos\theta \dfrac{\partial^2}{\partial x \partial y}$ as follows:

Proposition 2.8. *For any $\theta \in (0, 2\pi)$, $\theta \neq \pi$, the following equality holds*

$$T_\theta = \Delta + 2\cos\theta \frac{\partial^2}{\partial x \partial y} = \left(\frac{\partial}{\partial x} + e^{i\theta}\frac{\partial}{\partial y}\right)\left(\frac{\partial}{\partial x} + e^{-i\theta}\frac{\partial}{\partial y}\right).$$

Proof. The assertion follows by direct computation. $\square$

This result suggests the possibility of defining a notion of a formal θ-derivative as the operator

$$\frac{\partial}{\partial x} + e^{-i\theta}\frac{\partial}{\partial y} \, .$$

However, in order for this operator to act as a formal derivative, we calculate

$$\left(\frac{\partial}{\partial x} + e^{-i\theta}\frac{\partial}{\partial y}\right)(Z) = \left(\frac{\partial}{\partial x} + e^{-i\theta}\frac{\partial}{\partial y}\right)(x + e^{i\theta}(y - 2x\cos\theta)) = 1 - e^{2i\theta},$$

and therefore we can give the following

Definition 2.9. *Let $f : \Omega \to \mathbb{C}$ be a θ-holomorphic function. The θ-holomorphic function*

$$\frac{\partial_\theta f}{\partial Z} := \frac{1}{1-e^{2i\theta}}\left(\frac{\partial}{\partial x} + e^{-i\theta}\frac{\partial}{\partial y}\right) f(x,y)$$

is called the formal θ-derivative *(or simply* formal derivative*) of f.*

This formal derivative (in the next section it will become apparent why we are insisting on the word *formal*) is normalized in such a way that it equals 1 when applied to the θ-holomorphic variable Z. The same argument applied to the conjugate variable $\overline{Z}(x,y) = x + e^{-i\theta}(y - 2x\cos\theta)$ suggests the correct definition for what we can call the *θ-Cauchy-Riemann operator*:

$$\frac{1}{1-e^{-2i\theta}}\left(\frac{\partial}{\partial x} + e^{i\theta}\frac{\partial}{\partial y}\right).$$

The operator T_θ is an elliptic partial differential operator with constant coefficients, therefore there exists a linear change of coordinates which transforms it into the Laplacian operator (see, e.g., [**4**]). This transformation is represented by the square root of the matrix associated to the symbol of T_θ

$$A = \begin{pmatrix} 1 & \cos\theta \\ \cos\theta & 1 \end{pmatrix},$$

i.e. by the matrix

$$B = \frac{1}{2}\begin{pmatrix} \sqrt{1+\cos\theta} + \sqrt{1-\cos\theta} & \sqrt{1+\cos\theta} - \sqrt{1-\cos\theta} \\ \sqrt{1+\cos\theta} - \sqrt{1-\cos\theta} & \sqrt{1+\cos\theta} + \sqrt{1-\cos\theta} \end{pmatrix}.$$

In terms of A and B it is possible to define a correspondence between harmonic maps and elements of the kernel of T_θ, in fact we have for any C^2-function u on a domain $\Omega \subseteq \mathbb{R}^2$

$$\operatorname{div}(A\nabla u) = \nabla \cdot A\nabla u = B\nabla \cdot B\nabla u.$$

This can be made explicit as follows:

Proposition 2.10. *Let $u(\xi,\eta)$ be a C^2-function on a domain $\Omega \subseteq \mathbb{R}^2$. Then u is harmonic if and only if*

$$\begin{aligned} &T_\theta u(\xi(x,y),\eta(x,y)) \\ &= \frac{\partial^2}{\partial x^2}u(\xi(x,y),\eta(x,y)) + \frac{\partial^2}{\partial y^2}u(\xi(x,y),\eta(x,y)) + \frac{\partial^2}{\partial x\partial y}u(\xi(x,y),\eta(x,y)) = 0 \end{aligned}$$

where

$$\begin{pmatrix} \xi \\ \eta \end{pmatrix} = B^{-1} \begin{pmatrix} x \\ y \end{pmatrix}.$$

Proof. By summing up the following three terms we obtain the assertion:

$$\frac{\partial^2}{\partial x^2} u(\xi(x,y),\eta(x,y)) = u_{\xi\xi}(2 + 2\sqrt{1-\cos^2\theta}) + u_{\xi\eta}(-4\cos\theta) + u_{\eta\eta}(2 - 2\sqrt{1-\cos^2\theta})$$

$$\frac{\partial^2}{\partial y^2} u(\xi(x,y),\eta(x,y)) = u_{\xi\xi}(2 - 2\sqrt{1-\cos^2\theta}) + u_{\xi\eta}(-4\cos\theta) + u_{\eta\eta}(2 + 2\sqrt{1-\cos^2\theta})$$

$$2\cos\theta \frac{\partial^2}{\partial x \partial y} u(\xi(x,y),\eta(x,y)) = u_{\xi\xi}(-4\cos^2\theta) + u_{\xi\eta}(8\cos\theta) + u_{\eta\eta}(-4\cos^2\theta).$$

□

Proposition 2.11. *Let $u + iv$ be a non-constant holomorphic function. Then the function $u \circ B^{-1} + e^{i\theta}(v \circ B^{-1})$ is not θ-holomorphic, (for $\theta \neq \pi/2, 3\pi/2$).*

Proof. Assuming $u(\xi,\eta) + iv(\xi,\eta)$ is holomorphic, we have

$$\begin{cases} u_\xi = v_\eta \\ u_\eta = -v_\xi . \end{cases}$$

As a consequence, the first equation in system

$$\begin{cases} u_x = v_y \\ u_y + v_x + 2v_y\cos\theta = 0 \end{cases}$$

is not fullfilled, where the change of variables is defined by

$$\begin{pmatrix} \xi \\ \eta \end{pmatrix} = \begin{pmatrix} \sqrt{1+\cos\theta} + \sqrt{1-\cos\theta} & \sqrt{1-\cos\theta} - \sqrt{1+\cos\theta} \\ \sqrt{1-\cos\theta} - \sqrt{1+\cos\theta} & \sqrt{1+\cos\theta} + \sqrt{1-\cos\theta} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

(the coefficient 2 can be omitted for this proof). In fact, we have

$$\frac{\partial}{\partial x} u(\xi(x,y),\eta(x,y)) = u_\xi(x,y)(\sqrt{1+\cos\theta} + \sqrt{1-\cos\theta}) + u_\eta(x,y)(\sqrt{1-\cos\theta} - \sqrt{1+\cos\theta})$$

and

$$\frac{\partial}{\partial y}v(\xi(x,y),\eta(x,y)) = v_\xi(x,y)(\sqrt{1-\cos\theta}-\sqrt{1+\cos\theta}) \\ +v_\eta(x,y)(\sqrt{1+\cos\theta}+\sqrt{1-\cos\theta})$$

therefore

$$\begin{aligned}\frac{\partial}{\partial x}u(\xi(x,y),\eta(x,y)) - \frac{\partial}{\partial y}v(\xi(x,y),\eta(x,y))& \\ &= (u_\xi(x,y)-v_\eta(x,y))(\sqrt{1+\cos\theta}+\sqrt{1-\cos\theta}) \\ &+ (u_\eta(x,y)-v_\xi(x,y))(\sqrt{1-\cos\theta}-\sqrt{1+\cos\theta}) \\ &= 2u_\eta(x,y)(\sqrt{1-\cos\theta}-\sqrt{1+\cos\theta})\end{aligned}$$

which does not vanish unless $\cos\theta = 0$, i.e. $\theta = \pi/2, 3\pi/2$, or $u_\eta = 0$. In the latter case, since $u + iv$ is holomorphic, we get that $v_\xi = 0$, and, using the fact that $u_\xi = v_\eta$, we obtain that there exists $a, b \in \mathbb{C}$, $a \neq 0$, such that $u(\xi,\eta) + iv(\xi,\eta) = a(\xi + i\eta) + b$. To conclude, it suffices to show that $\xi(x,y) + i\eta(x,y)$ is not θ-holomorphic. To do that, we compute

$$\begin{aligned}\xi_y + \eta_x + 2\eta_y\cos\theta &= 2(\sqrt{1-\cos\theta}-\sqrt{1+\cos\theta}) \\ &\quad + 2\cos\theta(\sqrt{1+\cos\theta}+\sqrt{1-\cos\theta}) \\ &= 2\sqrt{1+\cos\theta}\sqrt{1-\cos\theta}(\sqrt{1+\cos\theta}-\sqrt{1-\cos\theta})\end{aligned}$$

which does not vanish unless $\theta = \pi, \pi/2, 3\pi/2$. □

It is indeed possible to show that there is no linear transformation of the variables that can transform a θ-holomorphic function into a holomorphic function. Since, a priori, the previous proposition only shows that the transformation B^{-1} is inadequate, we will now show that this is a general fact.

The key lemma is the following

Lemma 2.12. *Let $u(\xi,\eta) : \mathbb{R}^2 \to \mathbb{R}$ be a harmonic function, and suppose that $u(\xi,\eta)$ is not a polynomial function in ξ,η of degree less or equal than 2. If $u(\alpha x+\beta y, \gamma x+\delta y)$ is harmonic for some constants $\alpha,\beta,\gamma,\delta$, then there exist an orthogonal 2×2 matrix U and $r \in \mathbb{R}$ such that*

$$\begin{pmatrix}\alpha & \beta \\ \gamma & \delta\end{pmatrix} = r\,U.$$

Proof. The assertion follows from a direct calculation. Indeed, with obvious notations

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=(\alpha^2+\beta^2)\frac{\partial^2 u}{\partial \xi^2}+(\gamma^2+\delta^2)\frac{\partial^2 u}{\partial \eta^2}+2(\alpha\gamma+\beta\delta)\frac{\partial^2 u}{\partial\eta\partial\xi}$$

and since $u(\xi,\eta)$ is harmonic

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=[(\alpha^2+\beta^2)-(\gamma^2+\delta^2)]\frac{\partial^2 u}{\partial \xi^2}+2(\alpha\gamma+\beta\delta)\frac{\partial^2 u}{\partial\eta\partial\xi}\,.$$

Then if $u(\alpha x+\beta y,\gamma x+\delta y)$ is harmonic, we have:

$$[(\alpha^2+\beta^2)-(\gamma^2+\delta^2)]\frac{\partial^2 u}{\partial \xi^2}+2(\alpha\gamma+\beta\delta)\frac{\partial^2 u}{\partial\eta\partial\xi}=0$$

identically. Similarly

$$-[(\alpha^2+\beta^2)-(\gamma^2+\delta^2)]\frac{\partial^2 u}{\partial \eta^2}+2(\alpha\gamma+\beta\delta)\frac{\partial^2 u}{\partial\eta\partial\xi}=0\,.$$

As a consequence, there exist smooth functions $f,g:\mathbb{R}\to\mathbb{R}$ such that

$$2(\alpha\gamma+\beta\delta)\frac{\partial u}{\partial\xi}(\xi,\eta)-[(\alpha^2+\beta^2)-(\gamma^2+\delta^2)]\frac{\partial u}{\partial\eta}(\xi,\eta)=g(\xi)$$

and

$$[(\alpha^2+\beta^2)-(\gamma^2+\delta^2)]\frac{\partial u}{\partial\xi}(\xi,\eta)+2(\alpha\gamma+\beta\delta)\frac{\partial u}{\partial\eta}(\xi,\eta)=f(\eta)\,.$$

Suppose now that either $2(\alpha\gamma+\beta\delta)$ or $[(\alpha^2+\beta^2)-(\gamma^2+\delta^2)]$ is nonzero; then, up to a common multiplicative nonzero factor,

$$\frac{\partial u}{\partial\xi}(\xi,\eta)=g(\xi)\cos\varphi+f(\eta)\sin\varphi \tag{2}$$

$$\frac{\partial u}{\partial\eta}(\xi,\eta)=-g(\xi)\sin\varphi+f(\eta)\cos\varphi \tag{3}$$

for some $\varphi\in\mathbb{R}$. Therefore,

$$0=\frac{\partial^2 u}{\partial\xi^2}(\xi,\eta)+\frac{\partial^2 u}{\partial\eta^2}(\xi,\eta)=\left(\frac{\partial g}{\partial\xi}(\xi)+\frac{\partial f}{\partial\eta}(\eta)\right)\cos\varphi$$

and

$$0=\frac{\partial^2 u}{\partial\eta\partial\xi}(\xi,\eta)-\frac{\partial^2 u}{\partial\xi\partial\eta}(\xi,\eta)=\left(\frac{\partial f}{\partial\eta}(\eta)+\frac{\partial g}{\partial\xi}(\xi)\right)\sin\varphi\,.$$

This implies that there exists $a \in \mathbb{R}$ such that $\frac{\partial g}{\partial \xi}(\xi) = a$ and $\frac{\partial f}{\partial \eta}(\eta) = -a$ identically. Thus, there exist $b, c \in \mathbb{R}$ such that

$$f(\eta) = a\eta + b, \qquad g(\xi) = -a\xi + c\,.$$

Equations (2) and (3) imply now that

$$\begin{aligned}\frac{\partial u}{\partial \xi}(\xi,\eta) &= -\xi a\cos\varphi + \eta a\sin\varphi + b\sin\varphi + c\cos\varphi\\ \frac{\partial u}{\partial \eta}(\xi,\eta) &= \xi a\sin\varphi + \eta a\cos\varphi - c\sin\varphi + b\cos\varphi\end{aligned}$$

whence we get the two representations

$$u(\xi,\eta) = -\frac{\xi^2}{2}a\cos\varphi + \eta\xi a\sin\varphi + (b\sin\varphi + c\cos\varphi)\xi + h(\eta) \tag{4}$$

$$u(\xi,\eta) = \eta\xi a\sin\varphi + \frac{\eta^2}{2}a\cos\varphi + (-c\sin\varphi + b\cos\varphi)\eta + k(\xi) \tag{5}$$

for suitable smooth functions $h, k : \mathbb{R} \to \mathbb{R}$. From Equations (4) and (5) we get, identically for all $\xi, \eta \in \mathbb{R}$,

$$\begin{aligned}&-\frac{\xi^2}{2}a\cos\varphi + \eta\xi a\sin\varphi + (b\sin\varphi + c\cos\varphi)\xi + h(\eta)\\ &\qquad = \eta\xi a\sin\varphi + \frac{\eta^2}{2}a\cos\varphi + (-c\sin\varphi + b\cos\varphi)\eta + k(\xi)\end{aligned}$$

$$\begin{aligned}&-\frac{\xi^2}{2}a\cos\varphi + (b\sin\varphi + c\cos\varphi)\xi - k(\xi)\\ &\qquad = \frac{\eta^2}{2}a\cos\varphi + (-c\sin\varphi + b\cos\varphi)\eta - h(\eta).\end{aligned}$$

This equality implies the existence of $d \in \mathbb{R}$ such that

$$\begin{aligned}&-\frac{\xi^2}{2}a\cos\varphi + (b\sin\varphi + c\cos\varphi)\xi - k(\xi) = d\\ &\frac{\eta^2}{2}a\cos\varphi + (-c\sin\varphi + b\cos\varphi)\eta - h(\eta) = d\end{aligned}$$

and that

$$\begin{aligned}k(\xi) &= -\frac{\xi^2}{2}a\cos\varphi + (b\sin\varphi + c\cos\varphi)\xi - d\\ h(\eta) &= \frac{\eta^2}{2}a\cos\varphi + (-c\sin\varphi + b\cos\varphi)\eta - d\,.\end{aligned}$$

Now, equation (4) yields that

$$u(\xi,\eta) = -\frac{\xi^2}{2}a\cos\varphi + \eta\xi a\sin\varphi + (b\sin\varphi + c\cos\varphi)\xi + \frac{\eta^2}{2}a\cos\varphi + (-c\sin\varphi + b\cos\varphi)\eta - d$$

is a polynomial function in ξ,η of degree less or equal than 2. By hypothesis this case has to be excluded. Therefore the only possibility is that

$$2(\alpha\gamma + \beta\delta) = [(\alpha^2+\beta^2) - (\gamma^2+\delta^2)] = 0$$

which directly implies the assertion. □

We can therefore prove the following result;

Proposition 2.13. *Let $u + e^{i\theta}v$ be a θ-holomorphic function (for $\theta \neq \pi/2, 3\pi/2$) such that u is not a polynomial function of degree less than or equal to 2. Then, there is no linear variable transformation C such that $u\circ C + iv\circ C$ is holomorphic.*

Proof. Indeed, if such a transformation existed, and using the notation from Proposition 2.11, one could write $u\circ C = u\circ B\circ(B^{-1}\circ C)$. By Proposition 2.10 and Lemma 2.12, this would imply that $B^{-1}\circ C$ corresponds to the multiplication by a non-zero complex number, and hence it is a holomorphic function. Finally, by composition, this would show that $u\circ B + i(v\circ B)$ is holomorphic, which contradicts Proposition 2.11. □

3 - Complex θ-derivatives

In the last section we have introduced an operator which we called formal θ-derivative. We want to show that such an operator is not an actual derivative, as one should suspect since θ-holomorphic functions are not holomorphic in the classical sense (for $\theta \neq \pi/2, 3\pi/2$).

To demonstrate this point, we consider a function $f(x + e^{i\theta}y) = u(x,y) + e^{i\theta}v(x,y) : \Omega \subseteq \mathbb{C} \to \mathbb{C}$ with u and v differentiable in the real sense, and we assume that the limit of the ratio

$$\frac{f(x+h+e^{i\theta}(y+k)) - f(x+e^{i\theta}y)}{h+e^{i\theta}k}$$

exists and is independent of how $h + e^{i\theta}k$ goes to zero. We compute first the limit for the case in which $k = 0$. We have

$$\begin{aligned}
&\lim_{h\to 0} \frac{f(x+h+e^{i\theta}y) - f(x+e^{i\theta}y)}{h} \\
&= \lim_{h\to 0} \frac{u(x+h,y) + e^{i\theta}v(x+h,y) - (u(x,y) + e^{i\theta}v(x,y))}{h} \\
&= u_x + e^{i\theta}v_x\,. \qquad (6)
\end{aligned}$$

We compute now the case in which $h = 0$, and we obtain

$$\begin{aligned}
&\lim_{k\to 0} \frac{f(x+e^{i\theta}(y+k)) - f(x+e^{i\theta}y)}{e^{i\theta}k} \\
&= \lim_{k\to 0} \frac{u(x,y+k) + e^{i\theta}v(x,y+k) - (u(x,y) + e^{i\theta}v(x,y))}{e^{i\theta}k} \\
&= u_y e^{-i\theta} + v_y = u_y(2\cos\theta - e^{i\theta}) + v_y \\
&= 2u_y\cos\theta + v_y - e^{i\theta}u_y\,. \qquad (7)
\end{aligned}$$

By comparing (6) and (7) we obtain the system

$$\begin{cases} v_x = -u_y \\ u_x + 2v_x\cos\theta - v_y = 0 \end{cases}$$

which shows that the function $g(x+e^{i\theta}y) := v(x,y) - e^{i\theta}u(x,y)$ is θ-holomorphic. We therefore conclude the following result:

Theorem 3.1. *Let $f : \Omega \subseteq \mathbb{C} \to \mathbb{C}$ be a θ-holomorphic function which admits complex θ-derivative, $\theta \neq \pi/2, 3\pi/2$. Then f is a constant.*

Proof. Since $f = u + e^{i\theta}v$ is θ-holomorphic and has complex θ-derivative, its components must satisfy simultaneously the two systems

$$\begin{cases} u_x = v_y \\ u_y + 2v_y\cos\theta + v_x = 0 \end{cases}$$

and

$$\begin{cases} v_x = -u_y \\ u_x + 2v_x\cos\theta - v_y = 0\,. \end{cases}$$

Substituting $u_x = v_y$ in the fourth equation, and under the hypothesis on θ, we obtain $v_x = 0$. Similarly by substituting $v_x = -u_y$ in the second equation, we obtain $v_y = 0$. Thus, v is constant. This immediately implies that u is constant as well, which concludes the proof. □

Note that Lemma 2.5 implies that power series are θ-holomorphic in their domain of convergence, and that the coefficients that appear have a nice geometrical meaning. Indeed, if

$$f(Z) = \sum_{n \geq 0} Z^n a_n$$

then $a_0 = f(0), a_1 = \frac{\partial_\theta f}{\partial Z}(0)$, and more generally $n! a_n = \frac{\partial_\theta^n f}{\partial Z^n}(0)$.

Moreover, one can show the following fact:

Proposition 3.2. *If a θ-holomorphic function has n-th formal θ-derivative identically equal to zero, then it is a polynomial of degree at most $n - 1$.*

Proof. We will proceed by induction on n. If $n = 1$ we will show that $\frac{\partial_\theta f}{\partial Z} = 0$ implies that f is constant. Indeed let $f = u + e^{i\theta} v$; then

$$\frac{\partial_\theta f}{\partial Z} = v_x - u_y + e^{i\theta}(u_x + 2u_y \cos\theta + v_y) = 0$$

implies

$$\begin{cases} v_x = u_y \\ u_x + 2u_y \cos\theta + v_y = 0\,. \end{cases}$$

Since f is θ-holomorphic, its component also satisfy

$$\begin{cases} u_x = v_y \\ u_y + 2v_y \cos\theta + v_x = 0\,. \end{cases}$$

Proceeding as in the proof of the last proposition, the comparison of these two systems shows again that f is constant. Assume now that we have proved the statement for n, and we will show that it is true for $n + 1$. Consider f such that $\frac{\partial_\theta^{n+1} f}{\partial Z} = 0$. Then, by what we just proved $\frac{\partial_\theta^n f}{\partial Z}$ is equal to a constant C. Consider now the function $g(Z) := f(Z) - \frac{Z^n C}{n!}$. It is immediate to see that $\frac{\partial_\theta^n g}{\partial Z} = 0$, and therefore by induction g is a polynomial of degree at most $n - 1$, which concludes the proof. □

In conclusion, Theorem 3.1 shows that the formal derivative introduced in the previous section to factorize the operator T_θ is not the complex θ-derivative; more interestingly, it explains the special role that $\theta = \pi/2, 3\pi/2$ plays in order to create a theory (holomorphic and anti-holomorphic functions) where the natural operator actually is the complex derivative. Indeed the formal $\pi/2$-derivative coincides with the complex $\pi/2$-derivative and the same holds for the case of $\theta = 3\pi/2$. In particular, one sees that the θ-holomorphic variable $Z = x + e^{i\theta}(y - 2x\cos\theta)$ does not have complex θ-derivative.

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