Riv. Mat. Univ. Parma, Vol. **11** (2020), 99-122

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Prolongation of diffeomorphisms and smoothness of invariant submanifolds

Abstract. We study various questions related to the smoothness of a real submanifold M which is invariant under a family of real-analytic or holomorphic diffeomorphisms. We show that in various situations it is possible to conclude that M is necessarily real-analytic (or the same smoothness of the diffeomorphisms involved if these are not analytic). The prolongation method we use also allows to recover some known results by employing relatively simple tools.

Keywords. Local diffeomorphism, invariant submanifolds, smoothness and analyticity.

Mathematics Subject Classification (2010): 37F99, 32V40.

1 - Introduction

Let M be a real submanifold of $\mathbb{R}^n$ or $\mathbb{C}^n$. The general problem we want to consider is the following: suppose that M is invariant under a family of (complex or real) analytic diffeomorphisms of the ambient space. Does it follow that M is itself real-analytic, except possibly on a "thin" subset? Analogous questions can be posed when the diffeomorphisms are just of class C^∞ or C^k for some $k \in \mathbb{N}$.

One version of this problem can be formulated more precisely as follows: let $M \subset \mathbb{R}^n$ be a smooth m-dimensional submanifold, $0 \in M$, and let $\psi \in \mathrm{Diff}^\omega(\mathbb{R}^n, 0)$ be a germ of real-analytic diffeomorphism fixing 0. Under which conditions does the ψ-invariance of M force it to be of class C^ω around 0?

Received: January 14, 2019; accepted: June 25, 2019.

The author was supported by the grant "Complex Geometry of Real Manifolds" from the University Research Board at AUB.

Clearly, this needs not be the case in general: for instance the graph $M = \{y = f(x)\} \subset \mathbb{R}^2$ of any smooth even function $f : \mathbb{R} \to \mathbb{R}$ is invariant under the reflection $\psi(x, y) = (-x, y)$. Another simple example is given by any $\psi \in \mathrm{Diff}^\omega(\mathbb{R}^n, 0)$ admitting a large-dimensional set of fixed points, where we can choose as M any nowhere analytic submanifold $M \subset \{\psi(x) = x\}$.

In both cases the differential of ψ at 0 admits eigenvalues of modulus 1. However, nowhere analytic smooth invariant submanifolds may exist even if ψ is hyperbolic (i.e. $d\psi(0)$ admits no such eigenvalues):

Example 1.1. *In $\mathbb{R}^3$, let $\psi(x, y, z) = (\frac{1}{2}x, 2y, \frac{1}{2}z)$ and define $M = \{z = x\phi(xy)\}$ for an arbitrary, nowhere analytic function $\phi \in C^\infty(\mathbb{R})$ with $\phi(0) = 0$. Then M is a nowhere analytic ψ-invariant surface of class C^∞.*

One of the aims of this paper is to show that this is not possible when the eigenvalues of the restriction of $d\psi(0)$ to $T_0(M)$ all have modulus strictly smaller (or, equivalently, strictly larger) than 1. We call such an invariant submanifold *contracting.*

Theorem 1.2. *Let M be a smooth contracting submanifold for $\psi \in \mathrm{Diff}^\omega(\mathbb{R}^n, 0)$: then M is of class C^ω around* 0.

Though the statement above might be well-known (see related results in [**3**, Theorem 3.1]; see also [**5**] for the case of CR manifolds invariant under complex-analytic contractions), we were not be able to find it explicitly presented in this form in the literature. Additionally, the approach we follow might itself be of some interest in that it allows to reduce the proof to relatively elementary tools (the stable manifold theorem) without relying on advanced dynamical techniques.

In Theorem 1.2 it is in fact sufficient to require that M is of class C^k for a certain k which depends on the eigenvalues of $d\psi(0)$; the smoothness assumption is stated more precisely in Theorem 4.1. We also remark that no assumption is made on the resonances of the eigenvalues of $d\psi(0)$, hence situations in which the invariant submanifold M is non-unique are also included in the previous result.

The proof is achieved in sections 3, 4 by considering the prolongation of the action of ψ to a suitable jet bundle: the behavior of the eigenvalues of the lift allows then to apply the stable manifold theorem, and thus show that the defining equations of M locally satisfy a system of real-analytic PDEs, forcing it to be of class C^ω.

The study of the prolongation of the action of local diffeomorphisms turns out to be also useful in the context of some different but related problems. A set $K \subset \mathbb{R}^n$ is called *homogeneous* under local diffeomorphisms of class C^k

($k \in \mathbb{N} \cup \{\infty, \omega\}$) if for all $p, q \in K$ there exists a neighborhood U of p in $\mathbb{R}^n$ and a diffeomorphism $\phi : U \to \phi(U)$ such that $\phi(K \cap U) = K \cap \phi(U)$ and $\phi(p) = \phi(q)$. It was proved in [**6**] that a locally closed subset of $\mathbb{R}^n$ is C^1-homogeneous if and only if it is a submanifold of class C^1. In [**8**] this result was extended to the C^∞ class. As an application of the prolongation method, in section 5 we show that the C^∞ case can be recovered in a relatively straightforward way from the C^1 case by applying an inductive procedure involving the prolongation to the space of 1-jets. It is worth pointing out that the case $k = \omega$ is still an open problem; indeed some of the results in the present paper are linked to certain approaches to the proof of the real-analytic version of this result.

The situation when the manifold M is invariant but not contracting is much more subtle, and whether it is possible to conclude that M is real-analytic might depend on invariants of higher order than the eigenvalues of ψ (cf. [**4**] for a treatment of the case of curves in the plane). It is interesting, then, that for some special (but still large) subgroups of diffeomorphisms one can give a partially positive answer - in the sense that analyticity holds in general only outside of the origin. This is the case for the group of conformal diffeomorphism of $\mathbb{R}^2$ or the group of analytic shears of $\mathbb{R}^2$. The proof is based on a somewhat deeper analysis of the dynamics of the diffeomorphism germs involved and is given in section 6. For instance, the study of the conformal case relies on the coordinates provided by the Leau-Fatou flower theorem, but as it turns out the crucial ingredient is again the analysis of the action of the prolongation of the diffeomorphism to the space of 1-jets.

2 - Preliminaries

2.1 - *Tensor notation*

Let $j \in \mathbb{N}$. We will denote by I the "asymmetric" multiindices, that is the elements of the set $\{1, \ldots, m\}^j$. We will denote by α the usual multiindices of length j, i.e. $\alpha = (\alpha_1, \ldots, \alpha_m)$ with $|\alpha| = \sum_i \alpha_i = j$. To any $I = (I_1, \ldots, I_j) \in \{1, \ldots, m\}^j$ we associate a multiindex $\alpha(I)$ by defining $\alpha(I) = (\alpha_1, \ldots, \alpha_m)$ with $\alpha_\kappa = |\{i : I_i = \kappa\}|$.

We will use coordinates $(\eta_I)_{I \in \{1,\ldots,m\}^j}$ on the vector space $(\mathbb{R}^m)^{\otimes j}$ of tensors of order j over $\mathbb{R}^m$, and coordinates $(\xi_\alpha)_{|\alpha|=j}$ for the subspace $\mathrm{Sym}^j(\mathbb{R}^m)$ of symmetric tensors. In these coordinates, the inclusion $\mathrm{Sym}^j(\mathbb{R}^m) \hookrightarrow (\mathbb{R}^m)^{\otimes j}$ corresponds to the linear map

$$\mathrm{Sym}^j(\mathbb{R}^m) \ni \xi = (\xi_\alpha)_{|\alpha|=j} \to \iota(\xi) = (\xi_{\alpha(I)})_{I \in \{1,\ldots,m\}^j} \in (\mathbb{R}^m)^{\otimes j}.$$

In a similar fashion, we will use coordinates $(\eta_{I,i})_{1 \le i \le d, I \in \{1,\ldots,m\}^j}$,

$(\xi_{\alpha,i})_{1\le i\le d,|\alpha|=j}$ for the spaces $\mathbb{R}^d\otimes(\mathbb{R}^m)^{\otimes j}$ and $\mathbb{R}^d\otimes \mathrm{Sym}^j(\mathbb{R}^m)$ respectively. More in general, we will use coordinates $\eta=(\eta^1,\dots,\eta^\ell)$ on the space $\prod_{\kappa=1}^{\ell}\mathbb{R}^d\otimes(\mathbb{R}^m)^{\otimes\kappa}$ and $\xi=(\xi^1,\dots,\xi^\ell)$ on the space $\prod_{\kappa=1}^{\ell}\mathbb{R}^d\otimes\mathrm{Sym}^\kappa(\mathbb{R}^m)$, where $\eta^j=(\eta^j_{I,i})_{1\le i\le d, I\in\{1,\dots,m\}^j}$ and $\xi^j=(\xi^j_{\alpha,i})_{1\le i\le d,|\alpha|=j}$.

For $j\in\mathbb{N}$ and any $I=(I_1,\dots,I_j)\in\{1,\dots,m\}^j$, we use the notation

$$\partial_I=\frac{\partial}{\partial x_{I_j}}\frac{\partial}{\partial x_{I_{j-1}}}\cdots\frac{\partial}{\partial x_{I_1}},$$

while for any multiindex $\alpha=(\alpha_1,\dots,\alpha_m)$ with $|\alpha|=\sum_i\alpha_i=j$ we write as customary

$$\partial_\alpha=\frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1}\dots\partial x_m^{\alpha_m}}.$$

Of course, we have $\partial_I=\partial_{\alpha(I)}$ for any $I\in\{1,\dots,m\}^j$; in the following, though, it be will useful to keep separate notations for the same operator.

Let V,W be vector spaces. Since $V^*\otimes W\cong\mathrm{Hom}(V,W)$, for any $T\in V^*\otimes W$ we will still denote by T the corresponding homomorphism $T:V\to W$ (and vice versa). For vector spaces V,W_1,W_2,W_3 and any pair of elements $T_1\in W_1\otimes V\otimes W_3$, $T_2\in V^*\otimes W_2$, we use this identification to define the contraction of T_1 and T_2 as $(Id\otimes T_2\otimes Id)T_1\in W_1\otimes W_2\otimes W_3$. If $W_3\cong\mathbb{R}$ (i.e. the third factor is not there) we write the contraction as

$$\cdot:(W_1\otimes V)\times(V^*\otimes W_2)\to W_1\otimes W_2$$

where $T_1\cdot T_2=(Id\otimes T_2)T_1$ for any $T_1\in W_1\otimes V$, $T_2\in V^*\otimes W_2$. If $V=\mathbb{R}^i$, we identify $\mathbb{R}^i$ with $(\mathbb{R}^i)^*$ via the standard Euclidean scalar product, and we consider $\cdot$ as a product $(W_1\otimes\mathbb{R}^i)\times(\mathbb{R}^i\otimes W_2)\to W_1\otimes W_2$. We remark that, using the isomorphism $W_1\otimes\mathbb{R}^d\cong\mathrm{Hom}(\mathbb{R}^d,W_1)$, we have $T_1\cdot T_2=(T_1\otimes Id)T_2=(Id\otimes T_2)T_1$. We also define the "inverse" map $G\to G$ (where $G\cong GL(i,\mathbb{R})$ is an open dense subset of $\mathbb{R}^i\otimes\mathbb{R}^i$) by requiring $T\cdot T^{-1}=Id$ for all $T\in G$.

Given a real vector space V and a smooth map $f:\mathbb{R}^m\to V$ we denote by f_x the map $\mathbb{R}^m\to V\otimes\mathbb{R}^m$ defined by

$$\mathbb{R}^m\ni x\to f_x(x)=\sum_{i=1}^m\left(\frac{\partial f}{\partial x_i}(x)\otimes\frac{\partial}{\partial x_i}\right)\in V\otimes\mathbb{R}^m.$$

For $j\in\mathbb{N}$, we define the map $f_{x^j}:\mathbb{R}^m\to V\otimes(\mathbb{R}^m)^{\otimes j}$ recursively by $f_{x^j}(x)=(f_{x^{j-1}})_x(x)$. With this convention, if $V=\mathbb{R}^d$, $f=(f_1,\dots,f_d)$ and $x_0\in\mathbb{R}^m$, we have that $f_{x^j}(x_0)$ is the element $\eta=(\eta_{I,i})_{1\le i\le d,I\in\{1,\dots,m\}^j}$ of $\mathbb{R}^d\otimes(\mathbb{R}^m)^{\otimes j}$ defined by $\eta_{I,i}=\partial_I f_i(x_0)$. In particular, $f_x(x_0)$ is just the differential of f at the point x_0. It is also clear (since $\partial_I = \partial_{\alpha(I)}$) that $f_{x^j}(x_0)$ is in fact an element of $\mathbb{R}^d \otimes \mathrm{Sym}^j(\mathbb{R}^m)$ for all $j \geq 1$.

Furthermore, given any smooth map $\rho : \mathbb{R}^m \to \mathbb{R}^m$ and putting $g(x) = f_{x^j}(\rho(x))$, the chain rule can be written as $g_x(x) = f_{x^{j+1}}(\rho(x)) \cdot \rho_x(x)$, where $f_{x^{j+1}}(\rho(x)) \in (V \otimes (\mathbb{R}^m)^j) \otimes \mathbb{R}^m$, $\rho_x(x) \in \mathbb{R}^m \otimes \mathbb{R}^m$ and $\cdot$ is the product defined above.

Suppose $f : \mathbb{R}^m \to V$ and $g : \mathbb{R}^m \to V^*$, $h = gf : \mathbb{R}^m \to \mathbb{R}$. Then the Leibniz rule implies $h_x = g_x f + (g \otimes Id) f_x \in \mathbb{R} \otimes \mathbb{R}^m \cong \mathbb{R}^m$, where $(g \otimes Id) : \mathbb{R}^m \to \mathrm{Hom}(V \otimes \mathbb{R}^m, \mathbb{R}^m)$. If now $f : \mathbb{R}^m \to W_1 \otimes V$, $g : \mathbb{R}^m \to V^* \otimes W_2$, $f \cdot g = (Id \otimes g) f : \mathbb{R}^m \to W_1 \otimes W_2$, we deduce $(f \cdot g)_x = (Id \otimes g_x) f + (Id \otimes g \otimes Id) f_x = f \cdot g_x + (Id \otimes g \otimes Id) f_x$, where $Id \otimes g \otimes Id$ is a map $\mathbb{R}^m \to \mathrm{Hom}(W_1 \otimes V \otimes \mathbb{R}^m \to W_1 \otimes W_2 \otimes \mathbb{R}^m)$.

2.2 - *Stable manifold theorem*

Given a linear operator A and an eigenvalue λ of A, the *generalized eigenspace* associated to λ is defined as the set of the vectors v such that $(A-\lambda I)^k v = 0$ for some $k \in \mathbb{N}$. Let $\psi \in \mathrm{Diff}^\omega(\mathbb{R}^n, 0)$ (the group of germs of real-analytic diffeomorphisms fixing the origin in $\mathbb{R}^n$), and let $\lambda_1, \dots, \lambda_r \in \mathbb{C}^*$ $(1 \leq r \leq n)$ be the eigenvalues of $d\psi(0)$. For $\lambda \in \{\lambda_1, \dots, \lambda_r\}$, $\lambda \in \mathbb{R}$, we denote by $E_\lambda \subset \mathbb{R}^n$ the generalized eigenspace associated to λ. We still denote by $d\psi(0)$ the extension of the differential of ψ to $\mathbb{C}^n$ and for any $\lambda \in \{\lambda_1, \dots, \lambda_r\}$, $\lambda \in \mathbb{C} \setminus \mathbb{R}$, we denote by $E_\lambda \subset \mathbb{C}^n$ the (complex) generalized eigenspace associated to λ, and put $E_{\lambda,\overline{\lambda}} = (E_\lambda \oplus E_{\overline{\lambda}}) \cap \mathbb{R}^n$. The (real) stable space $E^s \subset \mathbb{R}^n$ for $d\psi(0)$ is defined as follows:

$$E^s = \bigoplus_{\lambda \in \{\lambda_1,\dots,\lambda_r\}, |\lambda|<1, \lambda \in \mathbb{R}} E_\lambda \oplus \bigoplus_{\lambda \in \{\lambda_1,\dots,\lambda_r\}, |\lambda|<1, \lambda \in \mathbb{C}\setminus\mathbb{R}} E_{\lambda,\overline{\lambda}}.$$

We will need the following version of the well-known stable manifold theorem (see [**1**]):

Theorem 2.1. *Let $\psi \in \mathrm{Diff}^\omega(\mathbb{R}^n, 0)$ and let E^s be the stable subspace of $d\psi(0)$. Then there exists a ψ-invariant local embedded submanifold W^s_{loc} of class C^ω, whose tangent space at 0 is E^s, and $\delta > 0$ such that the set of the $p \in B(0,\delta)$ whose orbit is exponentially convergent to 0 coincides with $W^s_{loc} \cap B(0,\delta)$. W^s_{loc} is called the* stable manifold *of ψ through 0.*

2.3 - *Bundles of manifold jets*

Let $p \in \mathbb{R}^n$ and fix $m, k \in \mathbb{N}$, $m \leq n$. We denote by $M(m,k,p)$ the set of germs at p of m-dimensional submanifolds $M \subset \mathbb{R}^n$, $p \in M$, which are of class C^k in a neighborhood of p. Let $\sim_k$ be the equivalence relation in $M(m,k,p)$ given by $(M,p) \sim_k (M',p) \Leftrightarrow$ the order of contact between M and M' at p is equal to k. We denote by $J^k(m,p)$ the quotient of $M(m,k,p)$ with respect to $\sim_k$, and by $j^k_p : M(m,k,p) \to J^k(m,p)$ the relative projection. Furthermore, for any domain $U \subset \mathbb{R}^n$ we define

$$\mathcal{J}^k(m,U) = \bigcup_{p \in U} J^k(m,p).$$

The set $\mathcal{J}^k(m,U)$ can be endowed with a natural structure of (real-analytic) manifold. As a matter of fact,

$$\mathcal{J}^k(m,U) \cong U \times Gr(m,\mathbb{R}^n) \times \prod_{2 \leq j \leq k} \mathbb{R}^d \otimes \mathrm{Sym}^j(\mathbb{R}^m)$$

where $d = n - m$ and $\mathrm{Sym}^j(\mathbb{R}^m) \subset (\mathbb{R}^m)^{\otimes j}$ is the space of symmetric tensors of order j over $\mathbb{R}^m$. Note that $\mathrm{Sym}^j(\mathbb{R}^m)$ is diffeomorphic to $\mathbb{R}^{N(m,j)}$, where $N(m,j) = \binom{m+j-1}{m-1}$ is the number of multiindices in m variables of order j. A chart

$$\varphi^{(k)} : U \times \prod_{1 \leq j \leq k} \left(\mathbb{R}^{N(m,j)}\right)^d \to \mathcal{J}^k(m,U),$$

corresponding to the splitting $\mathbb{R}^n(x,y) = \mathbb{R}^m(x) \times \mathbb{R}^d(y)$, can be described explicitly as follows:

$$\varphi^{(k)}(x_0, y_0, \xi^1, \ldots, \xi^k) = k\text{–jet at } (x_0,y_0) \text{ of } M = \{\rho_1 = \ldots = \rho_d = 0\}, \text{where}$$

$$\rho_j(x,y) = y_j - (y_0)_j - \sum_{1 \leq |\alpha| \leq k} \xi^{|\alpha|}_{\alpha,j}(x - x_0)^\alpha \ \forall 1 \leq j \leq d.$$

It is straightforward to check that the $\binom{n}{m}$ charts constructed in a way analogous to $\varphi^{(k)}$ (corresponding to the $\binom{n}{m}$ charts of $Gr(m,\mathbb{R}^n)$) define a real-analytic atlas for $\mathcal{J}^k(m,U)$. In the following, we are always going to consider the coordinates given by $\varphi^{(k)}$ in a neighborhood of the point $(x_0,y_0) = (0,0) \in U$, $\xi^\ell = 0$.

Moreover, with this structure the map $\mathcal{J}^k(m,U) \ni \eta \to \pi(\eta) \in U$ defined by $\pi(\eta) = p \Leftrightarrow \eta \in J^k(m,p)$ is clearly of class C^ω, and corresponds to the natural projection on the U-factor in the coordinates induced by $\varphi^{(k)}$. Hence $\mathcal{J}^k(m,U)$

can be regarded as a (trivial) real-analytic fiber bundle $\pi : \mathcal{J}^k(m,U) \to U$ with fibers diffeomorphic to $J^k(m,0)$.

Let $M \subset U$ be an embedded m-dimensional submanifold of class $C^{k'}$, $k' \geq k$. We define a (uniquely determined) subset $\mathcal{M}^{(k)} \subset \mathcal{J}^k(m,U)$ by requiring $\pi(\mathcal{M}^{(k)}) = M$ and $\mathcal{M}^{(k)} \cap \pi^{-1}(p) = \{j^k_p((M,p))\}$ for all $p \in M$. We call $\mathcal{M}^{(k)}$ the *prolongation* of M to the k-jet bundle. If M is defined as $\{y_j = f_j(x)\}_{1\leq j\leq d}$ in a neighborhood of $0 \in U$, passing through the chart $\varphi^{(k)}$ one can see that $\mathcal{M}^{(k)}$ is locally parametrized as

$$M \ni p \to \left(p, \alpha! \frac{\partial^{|\alpha|} f_j(p)}{\partial x^\alpha}_{1\leq|\alpha|\leq k, 1\leq j\leq d}\right) \in U \times \prod_{1\leq j\leq k} \left(\mathbb{R}^{N(m,j)}\right)^d,$$

which shows that $\mathcal{M}^{(k)}$ is a m-dimensional $C^{k'-k}$-submanifold of $\mathcal{J}^k(m,U)$, and that the map $\pi : \mathcal{M}^{(k)} \to M$ is a diffeomorphism of class $C^{k'-k}$.

3 - Prolongation of a local diffeomorphism

Let $U \subset \mathbb{R}^n$ be a domain, and let $\psi : U \to \psi(U) \subset \mathbb{R}^n$ be a diffeomorphism of class $C^{k'}$, $k' \geq k$. For any $p \in U$, ψ, we consider the map

$$M(m,k,p) \ni (M,p) \to (\psi(M), \psi(p)) \in M(m,k,\psi(p));$$

it is immediate to check that

$$(\psi(M), \psi(p)) \sim_k (\psi(M'), \psi(p)) \Leftrightarrow (M,p) \sim_k (M',p),$$

hence the map above induces a (real-analytic) bijection $\widetilde{\psi}^{(k)}_p : J^k(m,p) \to J^k(m,\psi(p))$ between the respective quotient spaces, defined by $\widetilde{\psi}^{(k)}_p(j^k_p(M,p)) = j^k_{\psi(p)}(\psi(M),\psi(p))$. Since by definition $\mathcal{J}^k(m,U) = \cup_{p\in U} J^k(m,k,p)$, the collection $\{\widetilde{\psi}^{(k)}_p\}_{p\in U}$ defines in turn a map

$$\widetilde{\psi}^{(k)} : \mathcal{J}^k(m,U) \to \mathcal{J}^k(m,\psi(U))$$

satisfying $\pi \circ \widetilde{\psi}^{(k)} = \psi$. We call $\widetilde{\psi}^{(k)}$ the *prolongation* of ψ to $\mathcal{J}^k(m,U)$. In fact, $\widetilde{\psi}^{(k)}$ is a bundle isomorphism of class $C^{k'-k}$ between $\mathcal{J}^k(m,U)$ and $\mathcal{J}^k(m,\psi(U))$, which is actually real-analytic along the fibers.

Furthermore, for any m-dimensional submanifold $M \subset U$ of class C^k, let $N = \psi(M) \subset \psi(U)$ and let $\mathcal{M}^{(k)} \subset \mathcal{J}^k(m,U)$, $\mathcal{N}^{(k)} \subset \mathcal{J}^k(m,\psi(U)$ be the respective prolongations: then one has $\widetilde{\psi}^{(k)}(\mathcal{M}^{(k)}) = \mathcal{N}^{(k)}$. In particular, M is ψ-invariant if and only if $\mathcal{M}^{(k)}$ is $\widetilde{\psi}^{(k)}$-invariant.

From now on we will assume that $0 \in U$ and work in the coordinates induced on $\mathcal{J}^k(m,U)$ by the chart $\varphi^{(k)}$ defined in the previous section. We are interested in the case in which $\psi(0)=0$ and the differential $d\psi(0)$ admits an invariant m-dimensional subspace, which we can assume to coincide with $\mathbb{R}^m(x)$ in suitable coordinates. In our notation, this amounts to $\widetilde{\psi}^{(1)}$ fixing the origin in the chart $\varphi^{(1)}$ for $\mathcal{J}^1(m,U)$. We want to compute the eigenvalues of the differential $d(\widetilde{\psi}^{(k)})$ at the point $\{(x_0,y_0)=(0,0),\xi^\ell=0\}$.

In order to do so, let $p \in U$, and let $(M,p) \in M(m,k,p)$. If p is close enough to 0 and $T_p(M)$ is close enough to $\mathbb{R}^m(x)$, in a neighborhood of $p=(x_0,y_0)$ we can write $M=\{y=f(x)\}$, where f is a $\mathbb{R}^d$-valued mapping of class C^k defined in a neighborhood of x_0 in $\mathbb{R}^m$, and furthermore the germ $(\psi(M),\psi(p))$ can be expressed in a similar way as $\psi(M)=\{y=\hat{f}(x)\}$ around $\psi(p)=q=(x_1,y_1)$. Writing the components of ψ as $\psi(x,y)=(g(x,y),h(x,y))$, then $\hat{f}$ must satisfy the mapping equation

$$\hat{f}(g(x,f(x))) = h(x,f(x)) \tag{1}$$

for x in a neighborhood of x_0 in $\mathbb{R}^m$. The linear map given by $g_x(x_0,y_0)+g_y(x_0,y_0)f'(x_0)$ is invertible if f' is small enough. This implies that there exists an inverse map $\rho(x)=\rho^f(x)$ of class C^k, defined in a neighborhood of x_1 in $\mathbb{R}^m$, such that $g(\rho(x),f(\rho(x)))\equiv x$. Thus we can rewrite (1) as

$$\hat{f}(x) = h(\rho(x),f(\rho(x))), \tag{2}$$

valid for x in a neighborhood of x_1 in $\mathbb{R}^m$. Computing the first derivatives, we get

$$\hat{f}_x(x) = h_x(\rho(x),f(\rho(x)))\cdot\rho_x(x) + h_y(\rho(x),f(\rho(x)))\cdot f_x(\rho(x))\cdot\rho_x(x), \tag{3}$$

$$\hat{f}_x(x_1) = h_x(x_0,y_0)\cdot\rho_x(x_1) + h_y(x_0,y_0)\cdot f_x(x_0)\cdot\rho_x(x_1).$$

Differentiating the relation $g(\rho(x),f(\rho(x)))\equiv Id$, we have

$$(g_x(\rho(x),f(\rho(x))) + g_y(\rho(x),f(\rho(x)))\cdot f_x(\rho(x)))\cdot\rho_x(x) = Id, \text{ i.e.}$$

$$\rho_x(x_1) = (g_x(x_0,y_0)+g_y(x_0,y_0)\cdot f_x(x_0))^{-1}$$

hence

$$\hat{f}_x(x_1) = (h_x(x_0,y_0)+h_y(x_0,y_0)\cdot f_x(x_0))\cdot(g_x(x_0,y_0)+g_y(x_0,y_0)\cdot f_x(x_0))^{-1}. \tag{4}$$

We will interpret the expression (4) in the following way. Let $\eta=(\eta_{j,i})$, $1\le i\le d$, $1\le j\le m$, be coordinates for the space $\mathbb{R}^d\otimes\mathbb{R}^m$. Using the notation of section 2.1, we define a map $P^{p,1}:\mathbb{R}^d\otimes\mathbb{R}^m\to\mathbb{R}^d\otimes\mathbb{R}^m$ by

$$P^{p,1}(\eta) = (h_x(x_0,y_0)+(h_y(x_0,y_0)\otimes Id)\eta)\cdot(g_x(x_0,y_0)+(g_y(x_0,y_0)\otimes Id)\eta)^{-1}.$$

Of course, the map $P^{p,1}$ is actually defined for η in a neighborhood of 0 in $\mathbb{R}^d \otimes \mathbb{R}^m$, and it is rational in the components of η. By (4), the map $P^{p,1}$ is the expression of the map $\widetilde{\psi}_p^{(1)} : J_1(m,p) \to J_1(m,q)$ in the coordinates induced by $\varphi^{(1)}$.

Let us now specialize to $(x_0, y_0) = (0,0)$. By assumption, $\mathbb{R}^m(x)$ is an invariant space for $d\psi(0)$, which means $h_x(0,0) = 0$. Hence we can write the map $P^{0,1}$ as

$$P^{0,1}(\eta) = ((h_y(0,0) \otimes Id)\eta) \cdot (g_x(0,0) + (g_y(0,0) \otimes Id)\eta)^{-1}.$$

Note that $(g_x(0,0) + (g_y(0,0) \otimes Id)\eta)^{-1} = g_x(0,0)^{-1} + O(|\eta|)$, which in turns implies

$$P^{0,1}(\eta) = (h_y(0,0) \otimes g_x^{-1}(0,0))\eta + O(|\eta^2|),$$

where now we interpret $g_x^{-1}(0,0)$ as a linear map $\mathbb{R}^m \to \mathbb{R}^m$.

This computation can be extended to higher order jets:

Lemma 3.1. *Fixed any $p = (x_0, y_0) \in U$, for all $1 \leq \ell \leq k$, there exists a rational map*

$$P^{p,\ell} : \prod_{j=1}^{\ell} \mathbb{R}^d \otimes (\mathbb{R}^m)^{\otimes j} \to \mathbb{R}^d \otimes (\mathbb{R}^m)^{\otimes \ell}$$

satisfying the following properties:

- *$P^{p,\ell}$ restricts to a map $P^{p,\ell} : \prod_{j=1}^{\ell} \mathbb{R}^d \otimes \mathrm{Sym}^j(\mathbb{R}^m) \to \mathbb{R}^d \otimes \mathrm{Sym}^\ell(\mathbb{R}^m)$;*
- *$\partial_\alpha \hat{f}_i(x_1) = P^{p,\ell}_{i,\alpha}\left(\partial_\beta f_\iota(x_0)_{|\beta| \leq \ell, 1 \leq \iota \leq d}\right)$ for any multiindex α with $|\alpha| = \ell$ and any $1 \leq i \leq d$;*
- *if $p = p_0 = (0,0)$, we can write $P^{p_0,\ell} = P_1^{p_0,\ell} + P_2^{p_0,\ell}$, where $P_1^{p_0,\ell}$ is a rational map having second-order dependence on the factor $\mathbb{R}^d \otimes (\mathbb{R}^m)^{\otimes \ell}$ and $P_2^{p_0,\ell}$ is a linear map only depending on the factor $\mathbb{R}^d \otimes (\mathbb{R}^m)^{\otimes \ell}$, induced by*

$$h_y(0,0) \otimes g_x^{-1}(0,0)^{\otimes \ell} : \mathbb{R}^d \otimes (\mathbb{R}^m)^{\otimes \ell} \to \mathbb{R}^d \otimes (\mathbb{R}^m)^{\otimes \ell}.$$

Moreover, the map $P^{p,\ell}$ depends on p in a $C^{k'-\ell}$ smooth way.

Proof. We start with the following observation: for any $1 \leq j \leq k$, there exists a rational map

$$Q^{p,j} : \prod_{\kappa=1}^{j} \mathbb{R}^d \otimes (\mathbb{R}^m)^{\otimes \kappa} \to \mathbb{R}^m \otimes (\mathbb{R}^m)^{\otimes j}$$

such that for any $I \in \{1,\dots,m\}^j$, $1 \le i \le m$,

$$\partial_I \rho_i(x_1) = Q^{p,j}_{I,i}(\partial_J f_\iota(x_0) : 1 \le \iota \le d, J \in \{1,\dots,m\}^\kappa, 1 \le \kappa \le j); \tag{5}$$

furthermore, $Q^{p,j}$ restricts to a map $Q^{p,j} : \prod_{\kappa=1}^{j} \mathrm{Sym}^\kappa(\mathbb{R}^m) \to \mathrm{Sym}^j(\mathbb{R}^m)$. Indeed, for $j = 1$ we have

$$\rho_x(x) = (g_x(\rho(x), f(\rho(x))) + g_y(\rho(x), f(\rho(x))) \cdot f_x(\rho(x)))^{-1}$$

and it is straightforward to prove the claim inductively by differentiating the previous expression and evaluating at $x = x_1$. Note that in particular $\rho_{x^j}(x_1) = Q^{p,j}(\partial_J f_\iota(x_0))$.

Next, we will prove inductively that, for any $j \ge 2$, we can write

$$\begin{aligned} \hat{f}_{x^j}(x) &= (h_y(\rho(x), f(\rho(x))) \otimes (\rho_x(x))^{\otimes j}) f_{x^j}(\rho(x)) \\ &\quad + h_y(\rho(x), f(\rho(x))) \cdot f_x(\rho(x)) \cdot \rho_{x^j}(x) \\ &\quad + h_x(\rho(x), f(\rho(x))) \cdot \rho_{x^j}(x) + R^j(x) \end{aligned} \tag{6}$$

where R^j is a map whose components are polynomials in $\partial_I h(\rho(x), f(\rho(x)))$ (with $I \in \{1,\dots,n\}^\kappa$, $1 \le \kappa \le j$), $\partial_J f(\rho(x))$ (with $J \in \{1,\dots,m\}^\kappa$, $1 \le \kappa < j$) and $\partial_L \rho(x)$ (with $L \in \{1,\dots,m\}^\kappa$, $1 \le \kappa < j$). More explicitly, there is a polynomial map

$$\widetilde{R}^j : \prod_{\kappa=1}^{j} \mathbb{R}^d \otimes (\mathbb{R}^n)^{\otimes\kappa} \times \prod_{\kappa=1}^{j-1} \mathbb{R}^d \otimes (\mathbb{R}^m)^{\otimes\kappa} \times \prod_{\kappa=1}^{j-1} \mathbb{R}^m \otimes (\mathbb{R}^m)^{\otimes\kappa} \to \mathbb{R}^d \otimes (\mathbb{R}^m)^{\otimes j}$$

such that $R^j(x) = \widetilde{R}^j(\partial_I h_{i_1}(\rho(x), f(\rho(x))), \partial_J f_{i_2}(\rho(x)), \partial_L \rho_{i_3}(x))$ for all $x \in U$. For $j = 1$ we have in fact (from (3))

$$\begin{aligned} \hat{f}_x(x) &= h_y(\rho(x), f(\rho(x))) \cdot f_x(\rho(x)) \cdot \rho_x(x) + h_x(\rho(x), f(\rho(x))) \cdot \rho_x(x) \\ &= (h_y(\rho(x), f(\rho(x))) \otimes \rho_x(x)) f_x(\rho(x)) + h_x(\rho(x), f(\rho(x))) \cdot \rho_x(x), \end{aligned}$$

therefore we can set $R^1 = 0$. Assuming that (6) holds for a certain j, we verify that it holds for $j + 1$ by differentiating in it. By the Leibniz and the chain rules we can compute for $j = 1$

$$\begin{aligned} [(h_y(\rho(x), f(\rho(x))) \otimes \rho_x(x)) f_x(\rho(x))]_x &= (h_y(\rho(x), f(\rho(x))) \otimes (\rho_x(x))^{\otimes 2}) f_{x^2}(\rho(x)) \\ &\quad + h_y(\rho(x), f(\rho(x))) \cdot f_x(\rho(x)) \cdot \rho_{x^2}(x) + S_2 \end{aligned}$$

and for $j \geq 2$

$$[(h_y(\rho(x), f(\rho(x))) \otimes (\rho_x(x))^{\otimes j}) f_{x^j}(\rho(x)))]_x = (h_y(\rho(x), f(\rho(x))) \otimes (\rho_x(x))^{\otimes (j+1)}) f_{x^{j+1}}(\rho(x))) + S_{j+1}$$

where in both cases S_{j+1} is a map whose components are polynomials in $\partial_I h(\rho(x), f(\rho(x)))$, $\partial_J f(\rho(x))$, $\partial_L \rho(x)$ and do not contain derivatives of order $j+1$ of either ρ at x or of f at $\rho(x)$. Analogously,

$$[h_y(\rho(x), f(\rho(x)))\cdot f_x(\rho(x))\cdot \rho_{x^j}(x)]_x = h_y(\rho(x), f(\rho(x)))\cdot f_x(\rho(x))\cdot \rho_{x^{j+1}}(x) + S'_{j+1},$$

$$[h_x(\rho(x), f(\rho(x))) \cdot \rho_{x^j}(x)]_x = h_x(\rho(x), f(\rho(x))) \cdot \rho_{x^{j+1}}(x) + S''_{j+1},$$

where again S'_{j+1} and S''_{j+1} are polynomials that do not contain $\rho_{x^{j+1}}(x)$ or $f_{x^{j+1}}(\rho(x))$. Since it is immediately seen that the first derivatives of the terms in R^j yield terms in R^{j+1}, differentiating (6) we thus get

$$\begin{aligned} \hat{f}_{x^{j+1}}(x) &= (h_y(\rho(x), f(\rho(x))) \otimes (\rho_x(x))^{\otimes (j+1)}) f_{x^{j+1}}(\rho(x)) \\ &+ h_y(\rho(x), f(\rho(x))) \cdot f_x(\rho(x)) \cdot \rho_{x^{j+1}}(x) \\ &+ h_x(\rho(x), f(\rho(x))) \cdot \rho_{x^{j+1}}(x) + [R^j]_x + S_{j+1} + S'_{j+1} + S''_{j+1} \end{aligned}$$

which gives the inductive step with $R^{j+1} = [R^j]_x + S_{j+1} + S'_{j+1} + S''_{j+1}$.

Evaluating (6) at $x = x_1$ we have

$$\begin{aligned} \hat{f}_{x^j}(x_1) &= (h_y(x_0, y_0) \otimes (\rho_x(x_1))^{\otimes j}) f_{x^j}(x_0) + h_y(x_0, y_0) \cdot f_x(x_0) \cdot \rho_{x^j}(x_1) \\ &+ h_x(x_0, y_0) \cdot \rho_{x^j}(x_1) + R^j(x_1); \end{aligned}$$

define now, using coordinates $\eta = (\eta^1, \ldots, \eta^\ell)$ on the space $\prod_{\kappa=1}^{\ell} \mathbb{R}^d \otimes (\mathbb{R}^m)^{\otimes \kappa}$, the map $P^{p,\ell} : \prod_{\kappa=1}^{\ell} \mathbb{R}^d \otimes (\mathbb{R}^m)^{\otimes \kappa} \to \mathbb{R}^d \otimes (\mathbb{R}^m)^{\otimes \ell}$ as

$$\begin{aligned} P^{p,\ell}(\eta) &= (h_y(x_0, y_0) \otimes (Q^{p,1}(\eta))^{\otimes \ell}) \eta^\ell + h_y(x_0, y_0) \cdot \eta^1 \cdot Q^{p,\ell}(\eta) \\ &+ h_x(x_0, y_0) \cdot Q^{p,\ell}(\eta) \\ &+ \widetilde{R}^\ell(\partial_I h_i(x_0, y_0), \eta^\kappa_{(1 \leq \kappa \leq \ell-1)}, Q^{p,\ell}_{L,l}(\eta)_{(1 \leq l \leq m, L \in \{1,\ldots,m\}^\kappa, 1 \leq \kappa \leq \ell-1)}), \end{aligned}$$

so that $\hat{f}_{x^\ell}(x_1) = P^{p,\ell}(\partial_I f_i(x_0))$. This expression verifies the first two statements of the lemma because of (5) and the polynomial form of $\widetilde{R}^\ell$. It is also clear that the dependence of $P^{p,\ell}$ on p is of class $C^{k'-\ell}$.

Let us now choose $(x_0, y_0) = (0, 0)$. We define $P_2^{p_0,\ell}(\eta) = (h_y(0,0) \otimes (g_x^{-1}(0,0))^{\otimes \ell})\eta^\ell$ and $P_1^{p_0,\ell}(\eta) = P^{p_0,\ell}(\eta) - P_2^{p_0,\ell}(\eta)$. Since, as computed before,

$$Q^{p_0,1}(\eta) = (g_x(0,0) + g_y(0,0) \cdot \eta^1)^{-1} = g_x^{-1}(0,0) + O(|\eta|),$$

we have
$$(h_y(0,0)\otimes(Q^{p,1}(\eta))^{\otimes\ell})\eta^\ell = P_2^{p_0,\ell}(\eta)+O(|\eta|^2);$$
furthermore we have $h_x(0,0)=0$ since $\mathbb{R}^m(x)$ is an invariant space for $d\psi(p_0)$. The proof is then concluded by observing that the summand $h_y(0,0)\cdot\eta^1\cdot Q^{p_0,\ell}(\eta)$ in the expression of $P^{p_0,\ell}(\eta)$ does not contain any linear term in η^ℓ, while $\widetilde{R}^\ell$ does not depend on η^ℓ at all. □

4 - Analiticity of invariant manifolds

Assume that M is a m-dimensional invariant submanifold for $\psi \in \mathrm{Diff}^\omega(\mathbb{R}^n,0)$, $0\in M$. We say that M is *contracting* if $T_0(M)\subset E^s$. In such a case we always choose coordinates in which $T_0(M)=\mathbb{R}^m(x)$, and writing $\psi=(g,h)$ it follows that $h_x(0)=0$. Since $\mathbb{R}^m$ is an invariant subspace for $d\psi(0)$, we can reorder its eigenvalues in such a way that $\lambda_1,\dots,\lambda_s$ $(s\le r)$ are the eigenvalues of $g_x(0)$ – in particular we have $|\lambda_j|<1$ for $1\le j\le s$. Furthermore, we order the eigenvalues in such a way that $|\lambda_1|$ is the maximum of $\{|\lambda_j| : 1\le j\le s\}$ and $|\lambda_r|$ is the minimum of $\{|\lambda_j| : s+1\le j\le r\}$. Put $\tau=\tau(\psi)=\frac{\log|\lambda_r|}{\log|\lambda_1|}$ (note that $\log|\lambda_1|\neq 0$).

Theorem 4.1. *Let M be an invariant contracting submanifold for $\psi\in \mathrm{Diff}^\omega(\mathbb{R}^n,0)$, $0\in M$. Suppose that M is of class C^∞: then it is of class C^ω.*

More precisely, let $\tau=\tau(\psi)$ be defined as above. If M is of class C^{k+1} with $k>\tau$, then it is real-analytic.

Lemma 4.2. *With the assumptions above, the orbit of every point $p\in M$ sufficiently close to 0 is exponentially convergent to the origin, i.e. $|\psi^{\circ j}(p)|\le\mu^j$ for some $0<\mu<1$.*

Proof. Since the spectral radius of the linear map $A=g_x(0,0):\mathbb{R}^m\to\mathbb{R}^m$ is by assumption smaller than 1, we can choose a suitable norm $\|\cdot\|$ on $\mathbb{R}^m$ in such a way that $\|A\|=\nu<1$ (indeed, we can take ν arbitrarily close to $|\lambda_1|$). Assume that M is locally defined by $\{y=f(x)\}$ with f of class C^1, $f(0)=0$, $f_x(0)=0$; for x close enough to 0, we can write $g(x,f(x))=Ax+\|x\|^2\theta(x)$, where the map $\theta:\mathbb{R}^m\setminus\{0\}\to\mathbb{R}^m$ is bounded. It follows that $\|g(x,f(x))\|\le\|A\|\|x\|+C\|x\|^2$ for some $C>0$, hence, fixed $\epsilon>0$ such that $\nu+\epsilon<1$, we have $\|g(x,f(x))\|\le(\nu+\epsilon)\|x\|$ for all $\|x\|\le\epsilon/C$. We put $\mu=\nu+\epsilon$.

If now $p_0=(x_0,f(x_0))\in M$, we define $(x_j,y_j)=p_j=\psi^{\circ j}(p_0)$. By the ψ-invariance of M we have $y_j=f(x_j)$, $x_{j+1}=g(x_j,f(x_j))$ for any $j\in\mathbb{N}$. From the previous arguments we thus get $\|x_j\|\le\mu^j$ for all $j\in\mathbb{N}$ if x_0 is small enough.

Since f is of class C^1 and $f_x(0) = 0$, for small x_0 we have $|f(x_j)| \leq \|x_j\| \leq \mu^j$ for all $j \in N$. The conclusion of the lemma is then verified with respect to the norm $\|x\| + |y|$ on $\mathbb{R}^n(x, y)$, and thus with respect to the Euclidean norm. □

Corollary 4.3. *If M is of class C^k and $j < k$, the orbit under $\widetilde{\psi}^{(j)}$ of any point $p \in \mathcal{M}^{(j)}$ such that $\pi^{(j)}(p)$ is sufficiently close to 0 is exponentially convergent to $\widetilde{p} = (\pi^{(j)})^{-1}(0) \cap \mathcal{M}^{(j)}$.*

Proof. Up to a polynomial change of coordinates, we can assume that $M = \{y = f(x)\}$ with $\partial_\alpha f(0) = 0$ for all multiindices α with $|\alpha| \leq j$. As remarked earlier, a local parametrization for $\mathcal{M}^{(j)}$ around $p_0 \cong 0$ in the chart $\varphi^{(j)}$ is given by

$$\mathbb{R}^m \ni x \to \big(x, f(x), \alpha! \partial_\alpha f_\kappa(x)_{1 \leq |\alpha| \leq j, 1 \leq \kappa \leq d}\big) \in U \times \prod_{1 \leq i \leq j} \Big(\mathbb{R}^{N(m,i)}\Big)^d .$$

The conclusion then follows in the same way as in Lemma 4.2, using the facts that for any $(x_0, y_0) \in M$, $(x_i, y_i) = \psi^{\circ i}(x_0, y_0)$ we have $|x_i| \leq \mu^i$ and that $\partial_\alpha f_\kappa(x)$ is a function of class C^1 vanishing at 0 for all $|\alpha| \leq j$, $1 \leq \kappa \leq d$. □

Proof [Theorem 4.1]. Let the coordinates (x, y) on $\mathbb{R}^n$ be chosen in such a way that $M = \{y = f(x)\}$ with $\partial_\alpha f(0) = 0$ for all $|\alpha| \leq k$. In the chart $\varphi^{(k)}$, it follows that the origin is a fixed point for $\widetilde{\psi}^{(k)}$. Using coordinates $(x, y, \xi^1, \dots, \xi^k)$ for $U \times \prod_{1 \leq j \leq k} \big(\mathbb{R}^{N(m,j)}\big)^d \cong U \times \prod_{1 \leq j \leq k} \mathbb{R}^d \otimes \mathrm{Sym}^j(\mathbb{R}^m)$ and writing $p = (x, y)$, we can locally express $\widetilde{\psi}^{(k)}$ as

$$\widetilde{\psi}^{(k)}(p, \xi^1, \dots, \xi^k) = (g(p), h(p), P^{p,1}(\xi^1), P^{p,2}(\xi^1, \xi^2), \dots, P^{p,k}(\xi^1, \dots, \xi^k))$$

where the maps $P^{p,j}$, $1 \leq j \leq k$, are defined in Lemma 3.1. Furthermore, the differential of $\widetilde{\psi}^{(k)}$ at 0 is given by the following block-triangular matrix:

$$d\widetilde{\psi}^{(k)}(0) = \begin{pmatrix} d\psi(0) & 0 & 0 & \cdots & 0 \\ * & \hat{P}_2^{p_0,1} & 0 & \cdots & 0 \\ * & * & \hat{P}_2^{p_0,2} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ * & * & * & \cdots & \hat{P}_2^{p_0,k} \end{pmatrix}$$

where for all $1 \leq j \leq k$ we denote by $\hat{P}_2^{p_0,j} : \mathbb{R}^d \otimes \mathrm{Sym}^j(\mathbb{R}^m) \to \mathbb{R}^d \otimes \mathrm{Sym}^j(\mathbb{R}^m)$ the restriction of the linear map $P_2^{p_0,j} : \mathbb{R}^d \otimes (\mathbb{R}^m)^{\otimes j} \to \mathbb{R}^d \otimes (\mathbb{R}^m)^{\otimes j}$ defined in Lemma 3.1.

In particular, the eigenvalues of $\hat{P}_2^{p_0,j}$ are a subset of the eigenvalues of $P_2^{p_0,j}$. Since $P_2^{p_0,j} = h_y(0,0) \otimes g_x^{-1}(0,0)^{\otimes j}$, the latter ones are given by the set

$$\Lambda_j = \{\lambda_i \lambda_{\ell_1}^{-1} \lambda_{\ell_2}^{-1} \cdots \lambda_{\ell_j}^{-1} : s+1 \leq i \leq r, 1 \leq \ell_\kappa \leq s\}.$$

The assumption $k > \tau$ implies in particular that $\Lambda_k \subset \{z \in \mathbb{C} : |z| > 1\}$, so that the eigenvalues of $\hat{P}_2^{p_0,k}$ have all modulus bigger than 1.

It follows that the span over $\mathbb{C}$ of the vectors $\{\frac{\partial}{\partial \xi^k_{\alpha,\ell}}\}_{|\alpha|=k, 1\leq \ell \leq d}$ (which is an invariant subspace for the complex linear extension of $d\widetilde{\psi}^{(k)}(0)$ due to the triangular form of the matrix above) is generated by the union of generalized eigenspaces relative to eigenvalues of modulus bigger than 1. In particular, none of the $\frac{\partial}{\partial \xi^k_{\alpha,\ell}}$ belong to $E^{(k),s}$, the stable space for $d\widetilde{\psi}^{(k)}(0)$, and $E^{(k),s}$ projects injectively into the factor $\mathbb{R}^n \times \prod_{1\leq j\leq k-1} \mathbb{R}^d \otimes \mathrm{Sym}^j(\mathbb{R}^m)$.

Let $\mathcal{S}^{(k)}$ be the (real-analytic) stable manifold for $\widetilde{\psi}^{(k)}$ through 0. From the fact that $T_0(\mathcal{S}^{(k)}) = E^{(k),s}$ and the arguments above follows that there exist functions $\{F_{\alpha,\ell}\}_{|\alpha|=k,1\leq\ell\leq d}$, locally defined on a neighborhood of 0 in $\mathbb{R}^n \times \prod_{1\leq j\leq k-1} \mathbb{R}^d \otimes \mathrm{Sym}^j(\mathbb{R}^m)$ and of class C^ω, such that

$$\mathcal{S}^{(k)} \subset \mathcal{Z} = \bigcap_{|\alpha|=k, 1\leq\ell\leq d} \{\xi^k_{\alpha,\ell} = F_{\alpha,\ell}(p, \xi^1, \ldots, \xi^{k-1})\}$$

around the origin in $\mathbb{R}^n \times \prod_{1\leq j\leq k} \mathbb{R}^d \otimes \mathrm{Sym}^j(\mathbb{R}^m)$.

Our next observation is that the prolongation $\mathcal{M}^{(k)}$ of M is locally contained in $\mathcal{S}^{(k)}$, and thus in $\mathcal{Z}$. Indeed since M is of class C^{k+1} we have from Corollary 4.3 that the orbit under $\widetilde{\psi}^{(k)}$ of any point of $\mathcal{M}^{(k)}$ close enough to 0 is exponentially convergent to the origin, thus Theorem 2.1 implies that locally $\mathcal{M}^{(k)} \subset \mathcal{S}^{(k)}$.

It follows that the map $f = (f_1, \ldots, f_d)$ locally satisfies the (overdetermined) system of partial differential equations with real-analytic coefficients

$$\partial_\alpha f_i(x) = F_{\alpha,i}(x, f(x), (\partial_\beta f_\kappa(x))_{|\beta|<k, 1\leq\kappa\leq d}) \text{ for all } |\alpha| = k, 1 \leq i \leq d, \tag{7}$$

$$\partial_\alpha f_i(0) = 0 \text{ for all } |\alpha| < k, 1 \leq i \leq d.$$

The fact that f is of class C^ω around 0 follows then from (7) via, for instance, an iterative application of the Cauchy-Kowalevski theorem.

Indeed, putting $x = (x_1, \ldots, x_m)$ we can first show that $f(x_1, 0)$, $(\partial_\beta f_\kappa(x_1, 0))_{|\beta|<k,1\leq\kappa\leq d}$ are of class C^ω. Defining

$$\mathcal{V} : \mathbb{R} \to W = \mathbb{R}^d \times \prod_{1\leq j\leq k-1} \mathbb{R}^d \otimes \mathrm{Sym}^j(\mathbb{R}^m)$$

as $\mathcal{V}(x_1) = (f(x_1, 0), \partial_\beta f_\kappa(x_1, 0))$ we have by (7) that $\mathcal{V}(0) = 0$ and $\frac{d\mathcal{V}}{dx_1}(x_1) = (\mathcal{L}\mathcal{V}(x_1), \mathcal{H}(x_1, \mathcal{V}(x_1)))$ where $\mathcal{L}$ is a linear operator acting on W and $\mathcal{H}(t, \mathcal{V})$ is a map of class C^ω defined on $\mathbb{R} \times W$ whose components are given by

$$\mathcal{H}(t, \mathcal{V}) = F_{\alpha,i}(t, 0, \mathcal{V}_{\beta,\kappa}), 1 \le i \le d, |\alpha| = k, \alpha_1 \ge 1.$$

The unique analytic solution of this ODE is given by $\mathcal{V}(x_1)$.

For the inductive step, assume that $f(x_1, \ldots, x_{\ell-1}, 0)$, $(\partial_\beta f_\kappa(x_1, \ldots, x_{\ell-1}, 0))_{|\beta|<k, 1\le\kappa\le d}$ are analytic on $\mathbb{R}^{\ell-1} \subset \mathbb{R}^m$ and let $\mathcal{V}(x', x_\ell) = (f(x', x_\ell, 0), \partial_\beta f_\kappa(x', x_\ell, 0))$ where $x' = (x_1, \ldots, x_{\ell-1})$. $\mathcal{V}$ satisfies a system of the kind $\frac{\partial\mathcal{V}}{\partial x_\ell}(x', x_\ell) = (\mathcal{L}\mathcal{V}(x', x_\ell), \mathcal{H}(x', x_\ell, \mathcal{V}(x', x_\ell)))$ with real analytic boundary conditions $\mathcal{V}(x', 0) = (f(x', 0), \partial_\beta f_\kappa(x', 0))$, where again $\mathcal{L}$ is a suitable linear operator and $\mathcal{H}$ is a real-analytic map whose components are a certain subfamily of the $F_{\alpha,i}$. By the Cauchy-Kowalevski theorem, we have that $\mathcal{V}(x', x_\ell)$ is the unique analytic solution of this system. □

R e m a r k 1. If we just assume ψ to be a diffeomorphism of class $C^{k'}$ with $k' \ge k+2$, the same proof can still be applied to show that M is in fact of class $C^{k'}$ as soon as it is of class C^{k+1}. Indeed, in this case we have that $\mathcal{S}^{(k)}$, and hence each $F_{\alpha,i}$, is of class $C^{k'-k}$. Then (7) implies that f is of class $C^{k'}$ by the standard bootstrap argument: we first have that each $f_{\alpha,i}$ is of class C^2 since both $F_{\alpha,i}$ and its arguments $(\partial_\beta f_\kappa(x))_{|\beta|<k, 1\le\kappa\le d}$ are of class at least C^2. This implies that f is of class C^{k+2}, and applying recursively the same argument we conclude that f is of class $C^{k'}$.

5 - Locally closed subsets

Let $K \subset \mathbb{R}^n$ be a locally closed subset; from [**6**], [**7**] follows that if K is C^1-homogeneous then it is locally a submanifold of class C^1. We wish to give a simple argument showing that this result implies that the same is true for C^k-homogeneity, $k \in \mathbb{N}$ (and thus also for C^∞-homogeneity). The inductive procedure we use is analogous to the one employed in [**8**]; however, our argument relies on the results in [**6**], [**7**] as the basis for the induction (while the one in [**8**] does not) and thus is somewhat simpler.

Let $M \subset \mathbb{R}^n$ be a m-dimensional submanifold of class C^1, and let $\mathcal{M} = \{(p, T_pM) : p \in M\} \subset \mathbb{R}^n \times Gr(m, \mathbb{R}^n)$ be the 1st-jet prolongation of M. Choosing coordinates (x, y) for $\mathbb{R}^n \cong \mathbb{R}^m(x) \times \mathbb{R}^{n-m}(y)$, $x = (x_1, \ldots, x_m)$, $y = (y_1, \ldots, y_{n-m})$, and assuming that $0 \in M$, $T_0(M) = \mathbb{R}^m(x)$, we can express M locally as a graph $M = \{y = f(x)\}$ for a suitable $f \in C^1(\mathbb{R}^m, \mathbb{R}^{n-m})$. Then $\mathcal{M}$ can be locally written as $\mathcal{M} = \{y = f(x), \xi = df(x)\}$ where we denote by $\xi = (\xi_{j\ell})_{1\le j\le m}^{1\le\ell\le n-m}$ the local coordinates for a chart of $Gr(m,\mathbb{R}^n)$ and by df the differential matrix of f (since the arguments are all local, from now on we restrict to a neighborhood of 0 and we identify $Gr(m,\mathbb{R}^n) \cong \mathbb{R}^{m(n-m)}$). It is then clear that $\mathcal{M}$ is a locally closed subset of $\mathbb{R}^n \times Gr(m,\mathbb{R}^n)$ – being the graph of a continuous map – and it is a (m-dimensional) submanifold of class C^k if M is a submanifold of class C^{k+1}, $k \ge 1$.

The converse is not true: $\mathcal{M}$ can be a submanifold of class C^ω even if M is not more than C^1-smooth (for example $M = \{y = x^{4/3}\} \subset \mathbb{R}^2$, see the remark after Lemma 7.1 in [**8**]). Nevertheless

Lemma 5.1. *Suppose that $\mathcal{M}$ is a submanifold of class C^k, $k \ge 1$. Then there is a non-empty open set $U \subset M$ such that $M \cap U$ is of class C^{k+1}.*

Proof. Let π be the projection $\pi : \mathbb{R}^n \times Gr(m,\mathbb{R}^n) \to \mathbb{R}^m$, $\pi(x,y,\xi) = x$. By construction, $\pi|_{\mathcal{M}}$ is one to one; furthermore $\mathcal{M}$ is m-dimensional, since $\pi|_{\mathcal{M}} : \mathcal{M} \to \mathbb{R}^m$ is a local homeomorphism. It follows that $\min\{\dim\ker\pi|_{T_q\mathcal{M}} : q \in \mathcal{M}\} = 0$, otherwise by the rank theorem there would exist $\tilde{x} \in \mathbb{R}^m$ such that $\pi^{-1}(\tilde{x}) \cap \mathcal{M}$ is a positive dimensional manifold. Choose $q_0 \in \mathcal{M}$ such that $\ker\pi|_{T_{q_0}\mathcal{M}} = \{0\}$, and let $x_0 = \pi(q_0)$. By the implicit function theorem we have that, around q_0, $\mathcal{M}$ can be written as the graph of a map F of class C^k defined on a neighborhood V of x_0 in $\mathbb{R}^m$. By construction we must have $F(x) = (f(x), df(x))$ for $x \in V$, hence f is of class C^{k+1} around x_0. □

Remark 2. In fact, the open set U in Lemma 5.1 is dense in M.

We claim that, for any $m, n, k \in \mathbb{N}$, an m-dimensional submanifold of $\mathbb{R}^n$ of class C^1 which is C^k-homogeneous is of class C^k. Assume that this is true for all $k \le k_0$ ($k_0 \ge 1$) and let $M \subset \mathbb{R}^n$ be an m-dimensional C^{k_0+1}-homogeneous submanifold of class C^1.

As discussed in section 3, any C^{k_0+1}-smooth local diffeomorphism ψ of $\mathbb{R}^n$ prolongs to a C^{k_0}-smooth local diffeomorphism $\widetilde{\psi}$ of $\mathbb{R}^n \times Gr(m,\mathbb{R}^n)$ (by using the action of $d\psi$ on the m-planes), and $\widetilde{\psi}(\mathcal{M}) \subset \mathcal{M}$ if $\psi(M) \subset M$. Hence from the C^{k_0+1}-homogeneity of M follows that $\mathcal{M}$ is C^{k_0}-homogeneous. Since $\mathcal{M}$ is locally closed, by [**6**], [**7**] we have that $\mathcal{M}$ is a submanifold of class C^1. The inductive assumption then implies that $\mathcal{M}$ is of class C^{k_0}. By Lemma 5.1 there is an open set $U \ne \emptyset$ such that $M \cap U$ is of class C^{k_0+1}: by homogeneity, then, M is of class C^{k_0+1} everywhere.

6 - Planar curves

In this section, we prove some more precise analyticity statements that can be given in low dimension. We will show that curves in $\mathbb{R}^2$ which are invariant under a diffeomorphism germ ψ must be analytic (at least outside 0) even if they are not contracting, provided that ψ is not an involution and belongs to certain special subgroups of germs. In the proof, for the contracting case, we also recover in a more elementary way the statement of Theorem 1.2.

We start with the group $\mathrm{Hol}(\mathbb{C}, 0)$ of germs of holomorphic diffeomorphisms defined around the origin in $\mathbb{C} \cong \mathbb{R}^2$ and fixing the origin.

Proposition 6.1. *Let $f \in \mathrm{Hol}(\mathbb{C}, 0)$, f not an involution.Let $\gamma \subset \mathbb{C}$ be a (real) embedded curve of class C^1. Suppose that $0 \in \gamma$ and γ is invariant under f. Then there exists a neighborhood U of 0 in $\mathbb{C}$ such that $U \cap \gamma \setminus \{0\}$ is real-analytic.*

Proof. Write the expansion of f around 0 as $f(z) = \lambda z + O(z^2)$, $\lambda \in \mathbb{C}$. Since the differential of f at 0 must preserve the direction $T_0(\gamma)$, it follows that $\lambda \in \mathbb{R} \setminus \{0\}$. Up to replacing f with $f^{\circ 2} = f \circ f$ (which is not the identity since f is not an involution), we can assume $\lambda > 0$, and possibly considering f^{-1} instead of f we can further suppose $\lambda \leq 1$.

If $0 < \lambda < 1$, then f is holomorphically conjugated to its linearization $\widetilde{f}(z) = \lambda z$ (see for example [**2**, Theorem 2.1]); let $\widetilde{\gamma}$ be the image of γ under the linearizing change of coordinates. Up to a rotation, we can assume that $T_0(\widetilde{\gamma})$ is horizontal (i.e. generated by $\partial/\partial x$). The analyticity of γ will follow from the following

Claim. $\widetilde{\gamma}$ coincides with the x-axis.

Indeed, otherwise, choose $p \in \widetilde{\gamma}$ such that $\arg T_p(\widetilde{\gamma}) \neq 0$ (here and in the rest of the proof, we are going to improperly apply the function arg to linear subspaces $T \subset \mathbb{C}$, defined modulo multiples of π as the argument of a vector generating T). By invariance of $\widetilde{\gamma}$ under $\widetilde{f}$ we have that $\arg T_{p_j}(\widetilde{\gamma}) = \arg T_p(\widetilde{\gamma}) \neq 0$ for all $j \in \mathbb{N}$, where $p_j = \widetilde{f}^{\circ j}(p) = \lambda^j p$: it follows that $\widetilde{\gamma}$ is not of class C^1 at 0, a contradiction. Therefore, in this case the curve $\widetilde{\gamma}$ (and hence γ) is actually real-analytic around 0.

To treat the case when $\lambda = 1$ (the *parabolic* case) we will use the Leau-Fatou flower theorem, which provides a description of the dynamics of such a germ f. Since we need to examine the proof of this result rather than its statement alone, we shall refer to the proof which is contained in [**2**, Theorem 2.12], and employ the notation set up in there.

We first recall the essential features of the theorem. We can write the Taylor expansion of f as $f(z) = z + a_{k+1}z^{k+1} + O(z^{k+2})$ with $k \geq 1$, $a_{k+1} \in \mathbb{C}\setminus\{0\}$. The k directions $v_1^+, \ldots, v_k^+ \in bD \cong S^1$ which solve the equation $\frac{a_{k+1}}{|a_{k+1}|}v^k = -1$ are called *attracting directions* for f. Then there exist simply connected domains $P_{v_1^+}, \ldots, P_{v_k^+} \subset \mathbb{C}$ with the following properties:

(1) $0 \in bP_{v_j^+}$ and $f(P_{v_j^+}) \subset P_{v_j^+}$;

(2) $\lim_{n\to\infty} f^{\circ n}(z) = 0$ and $\lim_{n\to\infty} \dfrac{f^{\circ n}(z)}{|f^{\circ n}(z)|} = v_j^+$ for all $z \in P_{v_j^+}$.

The domain $P_{v_j^+}$ is called *attracting petal* centered at the direction v_j^+. The *repelling directions* v_j^- and the *repelling petals* $P_{v_j^-}$ are the attracting directions/petals associated to the germ f^{-1}. The attracting and repelling petals can be so chosen that their union (plus the point 0) is a neighborhood of 0 in $\mathbb{C}$; also, from the proof follows that each petal locally contains an open sector centered at 0. Moreover, for any $j = 1, \ldots, k$ the action $f_{|P_{v_j^+}}$ is holomorphically conjugated to the map $\zeta \to \zeta + 1$, defined on a half-plane of the form $\{\zeta \in \mathbb{C} : \mathrm{Re}\zeta > C\}$ for some $C > 0$.

Let P be a petal which intersects γ; without loss of generality (possibly considering f^{-1} in place of f and conjugating with a complex linear transformation) we can suppose that P is an attracting petal, centered at $v = 1$. From the property (2) above we deduce that $T_0(\gamma)$ is the x-axis; it also follows that $\gamma \cap \{x > 0\}$ is locally contained in P.

Let Ψ be a map conjugating f to $\zeta \to \zeta + 1$ (such a Ψ is called *Fatou coordinate*), and let $\widetilde{\gamma}$ be the image of γ under Ψ. Our aim is to show that $\widetilde{\gamma}$ is of the form $\{y = y_0\}$ for some $y_0 \in \mathbb{R}$. If $\widetilde{\gamma}$ does not coincide with a horizontal line, there exists $p \in \widetilde{\gamma}$ such that $\arg T_p(\widetilde{\gamma}) = \alpha \neq 0$. This of course also implies $\arg T_{p+n}(\widetilde{\gamma}) = \alpha$ for all $n \in \mathbb{N}$.

We are thus lead to computing the differential $d(\Psi^{-1})$ at the point $p + n$, which is given by the multiplication by a certain $\xi_n \in \mathbb{C} \setminus \{0\}$. We are going to show that $\arg \xi_n \to \pi$ as $n \to \infty$: posing $q_n = \Psi^{-1}(p + n)$, this would imply that $q_n \to 0$ and $\arg T_{q_n}(\gamma) = \arg \xi_n + \alpha \to \pi + \alpha \neq \{0, \pi\}$ as $n \to \infty$, which would contradict the fact that γ is of class C^1.

We turn then to the construction of the Fatou coordinate Ψ; as mentioned above, we follow the one given in [**2**]. The map Ψ is obtained in two steps. First of all, the restriction of f to a domain of the kind $\{|z^k - \delta| < \delta\}$ (for a small $\delta > 0$) is conjugated, through the map $\psi(z) = 1/(kz^k)$, to a function $\varphi : H_\delta \to H_\delta$ of the kind

$$\varphi(z) = z + 1 + b/z + R(z) \tag{8}$$

where $R(z) = O(1/z^2)$ and $H_\delta = \{\mathrm{Re} w > 1/2k\delta\}$. Afterwards, φ is conjugated to the translation $\zeta \to \zeta + 1$ through a holomorphic mapping $\sigma : H_\delta \to \mathbb{C}$ (that is to say, $\sigma \circ \varphi(z) = \sigma(z) + 1$), so that $\Psi = \sigma \circ \psi$.

Let $p \in \widetilde{\gamma}$ be the point chosen above. We set $r = \sigma^{-1}(p)$ and, for $n \in \mathbb{N}$, $r_n = \sigma^{-1}(p + n)$. It also follows that $r_n = \psi(q_n)$ (where the points $q_n = \Psi^{-1}(p+n)$ are defined above) and that $r_n = \varphi^{\circ n}(r)$. We have that $\arg r_n \to 0$ as $n \to \infty$. Indeed, for every $z \in H_\delta$ one has (see [**2**, Th. 2.12, Eq. (2.18)]) $|\varphi^{\circ n}(z)| = O(n)$: by (8), this implies $r_{n+1} - r_n = 1 + O(1/n)$. It follows that for any $\epsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that $|\arg(r_n - r_{n_0})| < \epsilon$ for all $n > n_0$ (in fact, for all large n, r_{n+1} is contained in a sector centered at r_n with opening angle less than ϵ), which in turn implies that $|\arg r_n| < 2\epsilon$ for all large enough n, as claimed.

Computing, now, the derivative of $\psi^{-1}(z) = 1/(kz)^{1/k}$ gives $\frac{\partial}{\partial z}\psi^{-1}(z) = -1/(kz)^{\frac{k+1}{k}}$. It follows that $\arg(\frac{\partial}{\partial z}\psi^{-1}(r_n)) = \pi - \frac{k+1}{k}\arg r_n \to \pi$ as $n \to \infty$. Since $\Psi^{-1} = \psi^{-1} \circ \sigma^{-1}$, with ξ_n as previously defined we get $\xi_n = \frac{\partial}{\partial z}\psi^{-1}(r_n) \cdot \frac{\partial}{\partial z}\sigma^{-1}(p+n)$, thus to show that $\arg \xi_n \to \pi$ it is sufficient to show that $\frac{\partial}{\partial z}\sigma^{-1}(p+n) \to 1$ or, equivalently, that $\frac{\partial}{\partial z}\sigma(r_n) \to 1$ as $n \to \infty$. We will concentrate on the latter.

The mapping σ is constructed as the limit of the functions $\sigma_n(z) = \varphi^{\circ n}(z) - n - b\log n$; it can be shown that the sequence $\{\sigma_n\}_{n\in\mathbb{N}}$ is uniformly convergent on compact subsets of H_δ. We will prove that, for any $\epsilon > 0$, we can fix a sufficient large n_0 such that $|\frac{\partial}{\partial z}\sigma_j(r_n) - 1| = |\frac{\partial}{\partial z}\varphi^{\circ j}(r_n) - 1| < \epsilon$ for all $n \geq n_0$ and $j \in \mathbb{N}$. This will imply that $|\frac{\partial}{\partial z}\sigma(r_n) - 1| < \epsilon$ for $n \geq n_0$, which is the desired conclusion.

In order to do this, we need to estimate the derivatives of $\varphi^{\circ j}$. Let $R(z)$ be the function appearing in the expression (8); then R is not obtained as a convergent series in $1/z$ (indeed, one should not expect φ to extend meromorphically to a neighborhood of ∞). However, from the computation performed in the proof of the Leau-Fatou theorem follows that there is a convergent series $S \in \mathbb{C}\{x\}$, $S(x) = \sum_{i\geq 2k} s_i x^i$, such that $R(z) = S(1/z^{\frac{1}{k}})$. Since $S'(x) \leq C_0|x|^{2k-1}$ for some $C_0 > 0$ and

$$\frac{\partial}{\partial z}R(z) = S'\left(\frac{1}{z^{\frac{1}{k}}}\right) \cdot \left(-\frac{1}{z^{\frac{k+1}{k}}}\right),$$

we get $|\frac{\partial}{\partial z}R(z)| \leq C_0/|z|^{\frac{2k-1}{k}+\frac{k+1}{k}} = C_0/|z|^3$. Posing $T(z) = -b/z^2 + \frac{\partial}{\partial z}R(z)$, we deduce that $|T(z)| \leq C_1/|z|^2$ for some $C_1 > 0$ and $\frac{\partial}{\partial z}\varphi(z) = 1 + T(z)$. Now from (8) we get, for all $z \in H_\delta$,

$$\varphi^{\circ(j+1)}(z) = \varphi^{\circ j}(z) + 1 + \frac{b}{\varphi^{\circ j}(z)} + R(\varphi^{\circ j}(z))$$

differentiating which we obtain

$$\frac{\partial}{\partial z}\varphi^{\circ(j+1)}(z) - \frac{\partial}{\partial z}\varphi^{\circ j}(z) = T(\varphi^{\circ j}(z)) \cdot \frac{\partial}{\partial z}\varphi^{\circ j}(z)$$

so that

$$\left|\frac{\partial}{\partial z}\varphi^{\circ(j+1)}(z) - \frac{\partial}{\partial z}\varphi^{\circ j}(z)\right| \leq \frac{C_1}{|\varphi^{\circ j}(z)|^2} \cdot \left|\frac{\partial}{\partial z}\varphi^{\circ j}(z)\right|. \tag{9}$$

Fix any small $\epsilon > 0$. Then we can select $n_1 \in \mathbb{N}$ such that $\sum_{n=n_1}^{\infty} 1/n^2 < \epsilon/8C_1$. Choose a point $z_1 \in H_\delta$ such that $\operatorname{Re} z_1 \geq n_1/2$. Again from the proof of the Leau-Fatou theorem in [**2**] (see Th. 2.12, Eq. (2.15)), we get that

$$\operatorname{Re}\varphi^{\circ j}(z) > \operatorname{Re} z + j/2 \quad (\Rightarrow |\varphi^{\circ j}(z)| > \operatorname{Re} z + j/2) \tag{10}$$

for all $z \in H_\delta$, $j \in \mathbb{N}$. We will now prove by induction that $|\frac{\partial}{\partial z}\varphi^{\circ j}(z_1) - 1| < \epsilon$ for all $j \in \mathbb{N}$. For $j = 1$, by definition of T we have $\frac{\partial}{\partial z}\varphi(z_1) - 1 = T(z_1)$, hence $|\frac{\partial}{\partial z}\varphi(z_1) - 1| = |T(z_1)| \leq C_1/|z_1|^2 \leq 4C_1/(n_1)^2 < \epsilon/2$. Suppose, then, that for some $j \in \mathbb{N}$ we have $|\frac{\partial}{\partial z}\varphi^{\circ j}(z_1) - 1| \leq 8C_1 \sum_{i=n_1}^{n_1+j-1} 1/i^2 < \epsilon$; in particular, this implies $|\frac{\partial}{\partial z}\varphi^{\circ j}(z_1)| < 2$. Using (9) and (10), we get

$$\left|\frac{\partial}{\partial z}\varphi^{\circ(j+1)}(z_1) - 1\right| \leq \left|\frac{\partial}{\partial z}\varphi^{\circ(j+1)}(z_1) - \frac{\partial}{\partial z}\varphi^{\circ j}(z_1)\right| + \left|\frac{\partial}{\partial z}\varphi^{\circ j}(z_1) - 1\right|$$

$$\leq \frac{4C_1}{(n_1+j)^2} \cdot 2 + 8C_1 \sum_{i=n_1}^{n_1+j-1} \frac{1}{i^2} = 8C_1 \sum_{i=n_1}^{n_1+j} \frac{1}{i^2} < \epsilon,$$

which gives the inductive step. Summing up, the previous argument provides the estimate $|\frac{\partial}{\partial z}\sigma(z_1) - 1| < \epsilon$ for all z_1 satisfying $\operatorname{Re} z_1 \geq n_1/2$. On the other hand, by (10) follows immediately that $\operatorname{Re} r_n = \operatorname{Re}\varphi^{\circ n}(r) \to \infty$ as $n \to \infty$. Together with the previous statement, this implies $\frac{\partial}{\partial z}\sigma(r_n) \to 1$ as $n \to \infty$, which concludes the proof of the lemma. □

Next, we are interested in the case of the group $\mathrm{Shr}^\omega(\mathbb{R}^2, 0)$ of germs of *analytic shears*, that is diffeomorphisms of the kind $\phi(x, y) = (h(x), y + g(x))$ where $h, g : \mathbb{R} \to \mathbb{R}$ are real-analytic, $h(0) = g(0) = 0$ and $h'(0) = 0$.

Proposition 6.2. *Let $\phi \in \mathrm{Shr}^\omega(\mathbb{R}^2, 0)$, ϕ not an involution, and let $\gamma \subset \mathbb{R}^2$ be a ϕ-invariant germ of curve around 0, which is of the form $\gamma = \{y = f(x)\}$ for some continuous function f. Then $\gamma \setminus \{0\}$ is of class C^ω; moreover, γ is uniquely determined.*

Proof. The shear ϕ has the following form:

$$\phi(x,y) = (h(x), y + g(x))$$

where h, g are germs of functions real-analytic around 0 in $\mathbb{R}$ and $dh(0)/dx \neq 0$. We can write $h(x) = \lambda x + O(x^2)$, where $\lambda \in \mathbb{R} \setminus \{0\}$. Replacing ϕ by $\phi^{\circ 2}$ or $\phi^{\circ(-2)}$ (none of them is the identity since ϕ is not an involution) if necessary, we can further assume that $0 < \lambda \leq 1$.

Suppose, first, that $0 < \lambda < 1$. The local holomorphic extension $h(z)$ of h to a neighborhood of 0 in $\mathbb{C}$ has the form $h(z) = \lambda z + O(z^2)$, hence it is linearizable by a local holomorphic change of coordinates $\eta(z)$. Since the restriction $h(x)$ of $h(z)$ to the real axis is real-valued, the same holds for the restriction $\eta(x)$ of $\eta(z)$ (cfr. [**2**, Proposition 1.9], where the coefficients of the series defining η are explicitly computed). Conjugating ϕ by the map $(x,y) \to (\eta(x), y)$, it follows that in the new coordinates we can assume it to take the form $\phi(x,y) = (\lambda x, y + g_1(x))$ for a certain $g_1 \in C^\omega(\mathbb{R}, 0)$. Let $\{y = f_1(x)\}$ be the expression of γ in these coordinates. The invariance of γ under ϕ, then, translates into the following identity

$$f_1(x) + g_1(x) = f_1(\lambda x), \tag{11}$$

holding for x in a small enough neighborhood of 0 in $\mathbb{R}$. We can show by a direct power series computation that (11) admits, locally, a real-analytic solution $\widetilde{f}_1$. Indeed, if $g_1(x) = \sum_{j=1}^\infty a_j x^j$, looking for $\widetilde{f}_1$ of the form $\widetilde{f}_1(x) = \sum_{j=1}^\infty b_j x^j$ we get

$$\sum_{j=1}^\infty (1 - \lambda^j) b_j x^j = -\sum_{j=1}^\infty a_j x^j$$

which has the unique solution $b_j = -a_j/(1-\lambda^j)$, $j \in \mathbb{N}$. From $0 < \lambda < 1$ follows that the factor $(1-\lambda^j)^{-1}$ is uniformly bounded in j, thus the series $\widetilde{f}_1$ has a positive radius of convergence. Let $\widetilde{\gamma}$ be the germ of real-analytic curve defined by $\{y = \widetilde{f}_1(x)\}$; we claim that $\gamma = \widetilde{\gamma}$.

In order to verify the claim, fix $C > 0$ such that $|g_1(x)| \leq C|x|$ in a neighborhood of 0, and let $p_0 \in \gamma$, $p_0 = (x_0, y_0)$. Then, if $\{p_j\}_{j\in\mathbb{N}}$ is the orbit of p_0 under ϕ (i.e. $p_j = \phi^{\circ j}(p_0)$) we also have $\{p_j\} \subset \gamma$ by invariance. One verifies by induction that

$$p_j = (x_j, y_j) = (\lambda^j x_0, y_0 + \sum_{k=0}^{j-1} g_1(\lambda^k x_0))$$

for all $j \geq 1$. Now, since $p_j \in \gamma$ and $x_j \to 0$ as $j \to \infty$, from the fact that $0 \in \gamma$ follows that we must also have $y_j \to 0$ as $j \to \infty$. It follows that $\Sigma(x_0) = \sum_{k=0}^{\infty} g_1(\lambda^k x_0)$ converges and that $y_0 = -\Sigma(x_0)$ is uniquely determined by the abscissa x_0 and by the components of the shear ϕ. We deduce that $f_1(x) = -\Sigma(x)$ so that γ is in turn uniquely determined, hence $\gamma = \widetilde{\gamma}$ is real-analytic. We also observe that, since $\sum_k |g_1(\lambda^k x)| \leq C|x|(1-\lambda)^{-1}$, the series defining $\Sigma(x)$ is in fact absolutely convergent; one can also check by a straightforward power series computation that $\sum_k g_1(\lambda^k x) = \sum_j b_j x^j$ (with b_j as above).

We turn now to the case $\lambda = 1$. We note that we cannot have $h(x) \equiv x$; otherwise, in view of (11) (with $\lambda = 1$) we would also get $g(x) \equiv 0$, against the assumption that ϕ is a non-trivial germ. After a linear change of coordinates, and possibly taking ϕ^{-1} in place of ϕ, we can thus assume that h has the expression $h(x) = x - x^{k+1} + O(x^{k+2})$ for some $k \geq 1$. A further real-analytic conjugation of h allows to put it in the form $h(x) = x - x^{k+1} + ax^{2k+1} + O(x^{2k+2})$ (see [**2**, Remark 1.14]). The equation expressing the invariance of γ under ϕ now reads

$$f(x) + g(x) = f(h(x)) = f(x + O(x^{k+1})). \tag{12}$$

Denote by $g(w)$ the holomorphic extension of g to a neighborhood U of 0 in $\mathbb{C}_w$, and let $g(w) = g_\ell w^\ell + O(w^{\ell+1})$, $\ell \geq 1$ be the Taylor expansion of g. Note that, in the case when f is of class C^∞, taking the k-order Taylor expansion about 0 of both sides in (12) we get that $\ell \geq k+1$; we will show that the same conclusion can be drawn if f is just assumed to be continuous. For a large enough $C > 0$, we have $\frac{1}{C}|w|^\ell \leq |g(w)| \leq C|w|^\ell$ for all $w \in U$.

As before, let $h(z) = z - z^{k+1} + az^{2k+1} + O(z^{2k+2})$ be the local holomorphic extension of h to a neighborhood of 0 in $\mathbb{C}_z$, $z = x + iu$. In what follows we recycle the terminology and the notation employed in Lemma 6.1. Since the coefficient of z^{k+1} is -1 we have that $v = 1 \in S^1$ is an attracting direction for the parabolic germ h. By the Leau-Fatou flower theorem, the positive x-axis is the center of (hence locally contained in) an attracting petal $P \subset \mathbb{C}$. Consider the map $\psi(z) = 1/kz^k$, conjugating $h|_P$ to a function $\varphi : H_\delta \to H_\delta$ of the kind $\varphi(z) = z + 1 + b/z + R(z)$. We recall, from equation (10), that $|\varphi^{\circ j}(z)| > \operatorname{Re} z + j/2 > j/2$ for all $z \in H_\delta$, $j \in \mathbb{N}$.

We can choose a small enough $R > 0$ such that $h^{\circ j}(z) \in U$ for all $z \in B_R(0) \cap P$ and $j \in \mathbb{N}$. Let $B_r(x)$ (for small $x, r > 0$) be a ball contained in $B_R(0) \cap P$; then $\psi(B_r(x)) \subset H_\delta$, which in view of the previous paragraph implies $|\varphi^{\circ j}(\psi(z))| > j/2$ for all $z \in B_r(x)$, $j \in \mathbb{N}$. Composing with the inverse of ψ we get

$$|h^{\circ j}(z)| = |\psi^{-1} \circ \varphi^{\circ j} \circ \psi(z)| = \frac{1}{(k|\varphi^{\circ j}(\psi(z))|)^{\frac{1}{k}}} \leq \frac{D}{j^{\frac{1}{k}}} \tag{13}$$

for all $z \in B_r(x)$, $j \in \mathbb{N}$, where $D = (2/k)^{1/k}$. On the other hand, for any $z \in H_\delta$ we have $|\varphi^{\circ j}(z)| = O(j)$ (see Lemma 6.1), so that with the same argument we get $h^{\circ j}(x') \geq D'/j^{1/k}$ for all small enough $x' \in \mathbb{R}^+$ and $j \in \mathbb{N}$, where $D' > 0$ is a constant depending on x' – note that $h^{\circ j}(x') > 0$ for small $x' > 0$.

Suppose first that $\ell \leq k$; by the choice of the constant C above, we get for any small $x' > 0$ $|g(h^{\circ j}(x'))| \geq D'^{\ell}/Cj^{\frac{\ell}{k}} \geq D''/j$. Moreover, the sign of $g(h^{\circ j}(x'))$ is constant for $j \in \mathbb{N}$, depending only on the sign of g_ℓ. It follows that in the case $\ell \leq k$ the series $\sum_{j=0}^{\infty} g(h^{\circ j}(x'))$ is divergent for any small $x' > 0$.

If instead $\ell \geq k+1$, by (13) and the choice of C we have that $|g(h^{\circ j}(z))| \leq CD^{k+1}/j^{\frac{k+1}{k}}$ for all $z \in B_r(x)$, $j \in \mathbb{N}$. It follows that the series $\sum_{j=0}^{\infty} g(h^{\circ j}(z))$ converges uniformly over $B_r(x)$ to a holomorphic function $\Sigma(z)$. Since for any $x > 0$ small enough there exists $r > 0$ such that $B_r(x) \subset B_R(0) \cap P$, we conclude that in the case $\ell \geq k+1$ the series $\Sigma(x) = \sum_{j=0}^{\infty} g(h^{\circ j}(x))$ converges and defines a real-analytic function on a neighborhood of 0 in $\mathbb{R}^+$.

We will now show that $\gamma \cap \{x > 0\}$ is real-analytic in a neighborhood of 0 (the treatment of $\gamma \cap \{x < 0\}$ is similar). Fix $p_0 = (x_0, y_0) \in \gamma$ with $x_0 > 0$ small enough, and let $\{p_j = \phi^{\circ j}(p_0)\}$ be the orbit of p_0 under the shear map ϕ. In the same way as before, we can inductively compute

$$p_j = (x_j, y_j) = (h^{\circ j}(x_0), y_0 + \sum_{k=0}^{j-1} g(h^{\circ k}(x_0))$$

for all $j \in \mathbb{N}$. By the Leau-Fatou theorem, since x_0 belongs to the attracting petal P for $h(z)$, we have $x_j \to 0$ as $j \to \infty$; since $\{p_j\} \subset \gamma$ and $0 \in \gamma$, we must again have $y_j \to 0$. It follows that the series $\sum_{k=0}^{\infty} g(h^{\circ k}(x_0))$ converges – hence, by the discussion above, $\ell \geq k+1$ – and that $y_0 = -\sum_{k=0}^{\infty} g(h^{\circ k}(x_0)) = -\Sigma(x_0)$. In conclusion, we have that $f(x) = -\Sigma(x)$ is real-analytic for $x > 0$, hence $\gamma \cap \{x > 0\}$ is real-analytic and, furthermore, it is univocally determined by the germ ϕ (since the series defining $\Sigma(x)$ only depends on g and h). □

Remark 3. In general, even when a shear $\psi \in \mathrm{Shr}^{\omega}(\mathbb{R}^2, 0)$ admits a (unique) invariant curve, this needs not be real-analytic around 0. For instance, defining

$$\psi(x, y) = \left(\frac{x}{1-x}, y + x^2\right)$$

then one can verify that ψ admits an (at least continuous) invariant curve γ, but γ is not real-analytic – although $\gamma \setminus \{0\}$ is. Indeed, following the proof of Lemma 6.2 we can define $f(0) = 0$ and

$$f(x) = -\sum_{k=0}^{\infty} \left(\frac{x}{1-kx}\right)^2 \text{ for } x < 0, \ f(x) = \sum_{k=1}^{\infty} \left(\frac{x}{1+kx}\right)^2 \text{ for } x > 0$$

so that $\gamma = \{y = f(x)\}$ is the unique invariant curve for ψ. Clearly $\gamma \setminus \{0\}$ is real-analytic, but a straightforward computation shows that f is not of class C^2 around 0. One can also check that ψ admits a unique formal invariant curve of the form $\{y = \widehat{f}(x)\}$ with $\widehat{f} \in \mathbb{R}[[x]]$: it follows that $\widehat{f}$ cannot be convergent, otherwise by Lemma 6.2 its sum would locally coincide with $f(x)$. As it turns out, the coefficients of $\widehat{f}$ are in fact given by the Bernoulli numbers (I thank H.C. Herbig for this observation).

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