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Homotopy theory and complex geometry

Abstract. Abstract homotopy theory provides a framework in which any site $\mathcal{S}$ can be embedded and its objects studied. In this context, objects of the site appear distinguished among more general ones. The purpose of this framework is to create an environment where the site $\mathcal{S}$ and supplementary "combinatorial/homotopical" data can blend together. Such a blending is achieved by a well-established mathematical procedure, the localization. The challenge in using this framework in practice is to find close ties between these general objects and the ones in $\mathcal{S}$. I will describe the results that G. Tomassini and I have obtained on these topics.

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1 - Motivation

Roughly speaking, homotopy theory is the study of topological spaces modulo relations coming from homotopic continuous maps. Since the first half of the past century, it appeared that many questions involving topological spaces were homotopy invariant. This observation motivated the creation of functors from the category Top of topological spaces and continuous maps to abelian categories (mostly modules over rings) to detect finer structures. Such functors are called (co)homology theories. In the second half of the 1950s, mathematicians studied how to transform the category Top in a way that (co)homology theories would loose fewer information, thus effectively making Top more similar to the abelian categories used to study it. It immediately appeared that

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endowing Top of the necessary structures, in a mathematically coherent way, was quite a feat. A new generation of mathematicians, mostly focused on these topics, produced a fair amount of literature and results, but it was not until the first half of the 1990s, when Voevodsky thoroughly employed them, that they assumed a definite and organic form.

The interdisciplinary branch of mathematics identified with the use of abstract homotopy theory on rigid objects, such as algebraic varieties, was effectively created by him in those years. According to Beilinson's foundational ideas, there should exist a cohomology theory defined on algebraic varieties, *motivic cohomology*, very similar to singular cohomology. Few candidates had been proposed few years earlier, but several of Beilinson's conjectural properties were yet to be proven. Among them, was not present one which is fundamental to singular cohomology: the natural action of the Steenrod algebra on the graded cohomology ring. Assuming the existence of such an action in an appropriate form, Voevodsky was able to prove a very deep result in algebraic k-theory: Milnor's conjecture. There is more than one way to construct the well known action of this algebra on singular cohomology. The one chosen by Voevodsky in his application is the longest, required the creation of several auxiliary categories, and the writing of several hundred pages of papers. Some of these foundational results are proved quite in generality, allowing for their use in contexts different from algebraic geometry. One of them is complex geometry.

As it occasionally happens in mathematics, it is better to consider the mathematical object we wish to study as an object of a particular kind among more general ones, created to overcome shortcomings that the particular ones have. The category of complex spaces and holomorphic maps $\mathcal{S}$ is faithfully fully embedded in larger categories $\mathsf{Prsh}_T(\mathcal{S})$ and $\Delta^{op}\mathsf{Prsh}_T(\mathcal{S})$, the category of presheaves of sets and simplicial presheaves of sets over $\mathcal{S}$, respectively. The latter serves as the substrate on which we act with various localizations, eventually obtaining the categories $\mathcal{H}_s$ and $\mathcal{H}$. The category $\mathsf{Prsh}_T(\mathcal{S})$ is closed under limits and colimits (quotients, in particular) and $\Delta^{op}\mathsf{Prsh}_T(\mathcal{S})$ adds combinatorial data that, at first glance, may appear to be solely useful to physically enable to perform localizations. A deeper understanding of such a combinatorial structure, reveals surprising connections with isomorphisms of the category $\mathcal{S}$ (see Section 4). Moreover, we discovered that the structures on $\Delta^{op}\mathsf{Prsh}_T(\mathcal{S})$ allowing to obtain the category $\mathcal{H}$ are intimately related to the notion of hyperbolicity of complex spaces (see Section 3 and 4.1).

2 - The categories

Let $\mathcal{S}$ denote the category of complex spaces and holomorphic maps. The Yoneda embedding $\mathcal{S} \to \mathsf{Prsh}_T(\mathcal{S})$ is a faithfully full embedding, that identifies any complex space X with its representable presheaf $\mathsf{Hol}(-, X)$. The category $\mathsf{Prsh}_T(\mathcal{S})$ is both complete and cocomplete, however the Yoneda embedding is not *well behaved* with existing colimits in $\mathcal{S}$. Indeed, consider the diagram

$$\begin{array}{ccc} \mathbb{CP}^1 & \xrightarrow{i} & \mathbb{CP}^2 \\ \downarrow & & \\ \mathsf{pt} & & \end{array}$$

It has a point pt as colimit in the category of complex spaces, but the colimit presheaf of the Yoneda embedding of the diagram is not the point presheaf $\mathsf{Hol}(-, \mathsf{pt})$[1]. This observation highlights the kind of problems we face when attempting to add general colimits to the category of complex spaces. Adding new objects to a category may result in universal properties that were valid in the initial category are no longer valid in the enlarged one.

In classical homotopy theory, topological spaces and maps are studied "up to homotopy". Categorically, this may be translated in "adding inverses" to all projections $X \times \mathbb{R} \to X$, for any topological space X. A very reasonable question is if it is possible to consider complex spaces "up to *holotopy*", i.e. if inverting all projections $p_X : X \times \mathbb{C} \to X$ for any complex space X would produce a manageable category. As it will appear later, the resulting homotopy theory has some exotic aspects: for example, any hyperbolic complex space exhibits properties making it similar to an individual point in classical homotopy theory. It so happens that even the category $\mathsf{Prsh}_T(\mathcal{S})$ is not suitable to attempt this construction, hence we have to further enlarge it. Let $\Delta^{op}\mathsf{Prsh}_T(\mathcal{S})$ be the category of *simplicial* presheaves. An object of this category is a diagram of presheaves F_i, $i \geq 0$ and morphisms $\partial_h : F_i \to F_{i-1}$, $h = 0, \cdots, i$ (face morphisms), $s_k : F_{i-1} \to F_i$, $k = 0, \cdots, i-1$ (degeneracy morphisms), satisfying certain "simplicial relations". The passage from $\mathsf{Prsh}_T(\mathcal{S})$ to $\Delta^{op}\mathsf{Prsh}_T(\mathcal{S})$ was initially thought to be merely a technical complication necessary to be able to invert all the maps p_X within the Bousfield localization's framework.

One more recent surprising discovery is that $\Delta^{op}\mathsf{Prsh}_T(\mathcal{S})$ is intimately related to the category $\mathsf{Grpd}_{\mathcal{S}}$ of $\mathcal{S}$-groupoids, if we endow the former category by a certain localizing *model structure*. Moreover, stacks in $\mathsf{Grpd}_{\mathcal{S}}$ correspond

[1]for instance, the preshaf $F(X) = \{\text{continuous functions } X \to \mathbb{C}\}$ has sections on $\mathbb{CP}^2$ that are constant on $\mathbb{CP}^1$ but are not themselves constant, hence the sheaf F negates the universal property that $\mathsf{Hol}(-, \mathsf{pt})$ must satisfy in order to be the colimit in $\mathsf{Prsh}_T(\mathcal{S})$.

to very specific objects in the model structure on $\Delta^{op}\mathsf{Prsh}_T(\mathcal{S})$ (see [**4**]). Since passing from $\mathcal{S}$-groupoids to stacks, and eventually to *(complex) analytic* stacks, keeps adding classical "complex spaces flavour" to these objects, we realize that under the thick layer of abstract nonsense inherent to simplicial presheaves still hides some complex analytical content. The purpose of model structures on the category of simplicial preshaves is to represent the morphisms we wish to invert as weak equivalences. When this is possible, a well-established machinery is used to make the localized categories more manegeable. In our applications we considered two weak equivalences: the *simplicial* and the $\mathbb{C}$-weak equivalence, the former being a stronger condition than the latter. Their associated homotopy (localized) categories are denoted by $\mathcal{H}_s$ and $\mathcal{H}$, respectively.

3 - Hyperbolicity

One observation started our interest in these techniques applied to complex spaces. A particular family of simplicial presheaves is fundamental in the localization process: $\mathcal{X} \in \Delta^{op}\mathsf{Prsh}_T(\mathcal{S})$ is called $\mathbb{C}$-*local* if it is simplicially fibrant and the projections $p : \mathcal{Y} \times \mathbb{C} \to \mathcal{Y}$ induce bijection of sets $\mathsf{Hom}_{\mathcal{H}_s}(\mathcal{Y}, \mathcal{X}) \to \mathsf{Hom}_{\mathcal{H}_s}(\mathcal{Y} \times \mathbb{C}, \mathcal{X})$ for all $\mathcal{Y} \in \Delta^{op}\mathsf{Prsh}_T(\mathcal{S})$. We will come back later on the first condition. The second one reminds very much of a hyperbolicity-type condition. Indeed, let me recall Brody's Theorem:

a compact complex space X is Kobayashi hyperbolic if and only if any holomorphic map $f : \mathbb{C} \to X$ is constant.

This is equivalent to require that all projections $Y \times \mathbb{C} \to Y$ induce bijections $\mathsf{Hol}(Y, X) \to \mathsf{Hol}(Y \times \mathbb{C}, X)$ for all complex spaces Y. There is an evident similarity between the definitions of $\mathbb{C}$-locality and this condition, which I will call *Brody hyperbolicity*. In fact, the relation between these two concepts is stronger (Theorem 3.1, p.10, [**1**]): a complex space X is $\mathbb{C}$-local if and only if for any complex space Y, the projection $Y \times \mathbb{C} \to Y$ induces bijections $\mathsf{Hol}(Y, X) \to \mathsf{Hol}(Y \times \mathbb{C}, X)$. Since any complex space is simplicially fibrant in $\Delta^{op}\mathsf{Prsh}_T(\mathcal{S})$ (ref. Theorem 2.3, [**3**]), we get a **generalization** of complex space hyperbolicity by defining a simplicial presheaf to be Brody hyperbolic if it is $\mathbb{C}$-local. A strictly related notion is the one of Brody hyperbolic complex space X *modulo* a closed subspace $C \subset X$. This is a complex space such that any holomorphic map $\mathbb{C} \to X$ has image contained in C. We define a simplicial presheaf $\mathcal{X}$ to be hyperbolic *modulo a subsimplicial presheaf* $\mathcal{C}$ if $\mathcal{X}/\mathcal{C}$ is a hyperbolic simplicial presheaf. Even in this case, the notion for simplicial presheaves extends the one for complex spaces (see Corollary 3.3, p.11, [**1**]).

The remark on hyperbolicity alone allows mathematicians to apply a number of general model categories results to complex spaces. Among them, I would like to mention the existence of a *hyperbolic resolution functor*. An interesting question is if it makes sense to determine the "closest" hyperbolic object to a complex space X, in a functorial way. One answer is provided by the following

Theorem 3.1 (cfr. page 9, [**1**]). *There exists an* hyperbolic resolution functor $(\mathfrak{H}, \mathfrak{r})$ *with the following properties:*

1. $\mathfrak{H} : \Delta^{op}\mathsf{Prsh}_T(\mathcal{S}) \to \Delta^{op}\mathsf{Prsh}_T(\mathcal{S})$ *is a functor and* $\mathfrak{r} : \mathsf{Id} \to \mathfrak{H}$ *is a natural transformation;*
2. $\mathfrak{H}(\mathcal{X})$ *is simplicially fibrant and hyperbolic for any* $\mathcal{X} \in \Delta^{op}\mathsf{Prsh}_T(\mathcal{S})$*;*
3. $\mathfrak{r}_{\mathcal{X}} : \mathcal{X} \to \mathfrak{H}(\mathcal{X})$ *is injective and a* $\mathbb{C}$*-weak equivalence;*
4. *if* $\mathcal{Y}$ *is hyperbolic, then* $r : \mathcal{Y} \to \mathfrak{H}(\mathcal{Y})$ *is a simplicial weak equivalence;*
5. *if* $f : \mathcal{X} \to \mathcal{Y}$ *is a* $\mathbb{C}$*-weak equivalence,* $\mathfrak{H}(f) : \mathfrak{H}(\mathcal{X}) \to \mathfrak{H}(\mathcal{Y})$ *is a simplical weak equivalence.*

A consequence of this theorem is that $\mathfrak{H}(\mathcal{X})$ is indeed the "closest" hyperbolic simplicial sheaf to $\mathcal{X}$: let $\mathcal{Y}$ be a hyperbolic simplicial presheaf and $f : \mathcal{X} \to \mathcal{Y}$ any morphism of simplicial presheaves, then there exists a unique morphism (in $\mathcal{H}_s$) $\tilde{f} : \mathfrak{H}(\mathcal{X}) \to \mathcal{Y}$ such that f factors as $\mathcal{X} \to \mathfrak{H}(\mathcal{X}) \to \mathcal{Y}$. The hyperbolic model is essentially unique in the sense that any hyperbolic object in the class of $\mathcal{X}$ in $\mathcal{H}$ is its hyperbolic model; more precisely, two such hyperbolic simplicial presheaves are simplicially weak equivalent.

Of course, there no such a thing as free lunch and the price to pay for this general result (there are no assumptions on the simplicial presheaves considered) is that $\tilde{f}$ is only a morphism in $\mathcal{H}_s$, i.e. a *fraction* in $\Delta^{op}\mathsf{Prsh}_T(\mathcal{S})$. At a set theoretic level, one would imagine to obtain a candidate for a hyperbolic model by "taking quotients" of $\mathcal{X}$ by all the images of maps $\mathbb{C} \to \mathcal{X}$. It is therefore quite surprising that the canonical morphism $\mathfrak{r}_{\mathcal{X}} : \mathcal{X} \to \mathfrak{H}(\mathcal{X})$ is injective. To understand better the passage from $\mathcal{X}$ to $\mathfrak{H}(\mathcal{X})$, imagine to take the *homotopy quotient* of $\mathcal{X}$ in the category $\mathcal{H}_s$, instead of the quotient in $\Delta^{op}\mathsf{Prsh}_T(\mathcal{S})$: that involves attaching a simplicial cone over $\mathbb{C}$ to $\mathcal{X}$ (*mapping cone*). A model for the simplicial cone over $\mathbb{C}$ is $(\mathbb{C} \times \Delta^1)/\mathbb{C} \times \Delta^0_0$, where Δ^0_0 is the subsimplicial set of Δ^1 generated by the point $\partial_0(p) \in (\Delta^1)_0$, p being the only non-degenerate point in $(\Delta^1)_1$. The trivialization of maps is done by attaching new combinatorial data to $\mathcal{X}$ to make those maps *simplicially homotopic* to a constant and not constant themselves.

These conclusions hold true for complex spaces X, being a particular case of simplicial presheaves, however, even considering these very special simplicial spaces, $\mathfrak{H}(X)$ will not be representable by a complex space in general. If it does exist a complex space Y representing the hyperbolic model of a simplicial presheaf, it is unique up to biholomorphism[2]. $\mathfrak{H}(\mathbb{C})$ is represented by a point since it is isomorphic to $\mathbb{C}$ in $\mathcal{H}$ and is hyperbolic itself. $p_* : \mathfrak{H}(V) \to \mathfrak{H}(X)$ is an isomorphism in $\mathcal{H}_s$[3], if $p : V \to X$ is a vector bundle, because p is a $\mathbb{C}$-weak equivalence; if X is a Brody hyperbolic complex space, then it is a representative of the hyperbolic model of V, since $\mathfrak{H}(V) \cong X$ in $\mathcal{H}_s$[4], thus in $\mathcal{H}$ a fortiori. Intuitively, a non-Brody hyperbolic complex space X whose hyperbolic model is representable by a Brody hyperbolic complex space is not very far from being hyperbolic itself. Because of this, we introduced the concept of *weak Brody hyperbolicity* for complex spaces: X is weakly Brody hyperbolic if its hyperbolic model can be represented by a complex space. By using certain functors, we were able to prove that complex projective spaces $\mathbb{CP}^n$ are **not** weakly Brody hyperbolic, for $n \geq 1$ (Section 6, [**1**]).

3.1 - *Holotopy groups*

Arguably, the most important functors to study CW complexes in homotopy theory are the *homotopy groups*. They are defined as pointed homotopy classes of pointed continuous maps from n dimensional spheres to the CW-complex. In the case of complex spaces, they can be used to study the underlying topological structure, but completely ignore complex structures, should they exist. It would be nice to have some similar construction, sensitive to the complex structures of the spaces, although it is impossible to expect such hypothetical groups to have the striking relationship with homotopy equivalences of homotopy groups. The naive approach to define homotopy groups, which involves maps from spheres, halts at the very beginning. Not only odd dimensional spheres cannot have an almost complex structure, but a classical theorem of Borel and Serre states that only S^2 and S^6 can (and do) have one. Thus, the category of complex spaces and holomorphic maps is not the correct one to consider complex spaces, should we wish to study maps from *spheres* to them. In the paper [**6**], the authors did define presheaves involving appropriate homotopy classes of maps from certain models of spheres in an algebraic context . Motivated by those constructions, we considered the complex analytic case (Section 4, [**1**]); the existence of bounded topologically-contractible objects like the unit Poincaré disc $\mathbb{D}$ made the objects involved even more exotic. Models of *circles* we used

[2] Lemma 3.3, [**1**].
[3] because of property 5 of Theorem 3.1.
[4] because of property 4 of Theorem 3.1.

are: the *parabolic* $\mathbb{C}/(0 \amalg 1)$, the *hyperbolic* $\mathbb{D}/(z_1 \amalg z_2)$ and the complex space $\mathbb{C} \setminus 0$. The quotients are taken in the category of **presheaves**, as opposed to complex spaces. Notice that the hyperbolic disk is, in fact, a family of circles parametrized by the complex numbers z_1 e z_2. Morphisms in the pointed category $\mathcal{H}_\bullet$ from appropriate smash products of these circles to a simplicial presheaf $\mathcal{X}$ define *holotopy groups (and sets)* of $\mathcal{X}$. To give an idea of the subtleties involved, I mention that one can prove that the parabolic circle is isomorphic to the simplicial circle $S^1_s = \Delta^1/\partial\Delta^1$, in the category $\mathcal{H}$, but **not** in $\mathcal{H}_{s\bullet}$. S^1_s is a simplicial presheaf comprising just a finite number of points in any simplicial level, hence the presheaf $\mathbb{C}/(0 \amalg 1)$ loses any complex analytic content, when considered in $\mathcal{H}_\bullet$. Similarly, we will mention in a moment that the complex spaces $\mathbb{CP}^n$ have a non homotopy-trivial map from S^1_s to them, despite complex spaces being simplicial presheaves with constant (i.e. trivial) simplicial structure. Theorem 4.1 of [**1**] links complex spaces to these functors: it states that all the holotopy groups of a Brody hyperbolic complex space vanish, except possibly in dimension 0. This result can be used to show that a complex space is not weakly Brody hyperbolic, if we are able to provide a nontrivial holotopy class in positive dimension. This is precisely what we do in Section 6 of [**1**]: by means of the *topological realization functor*, we prove that the canonical, closed embedding $\mathbb{CP}^1 \hookrightarrow \mathbb{CP}^n$ represents a nontrivial holotopy class in dimension $(2, 1)$.

4 - Stacks, the geometrization of simplicial sheaves

The algebraic analogue of the category $\mathcal{H}$ had been created with the purpose of determining an appropriate action of the Steenrod operations Q_t on motivic cohomolgy groups of simplicial presheaves. The passage from schemes over a field to simplicial presheaves over the site of schemes with the Nisnevich topology, appeared to simply be a complication necessary to make the homotopy machinery applicable. No algebraic content was apparent on these abstract gadgets, except what implied by working on a site, hence simplicial dimension 0 data. In the first decade of the years 2000s, abstract homotopy theorists were able to unravel some of the obscurity inherent in simplicial sheaves, by linking them with $\mathcal{S}$-*groupoids* and *stacks* ([**4**], [**5**]). These abstract mathematical objects are categories statisfying an increasing number of conditions, aiming at making them behave more and more like objects of the base site, algebraic schemes, in the original applications. Historically, their founding ideas and creation go back to Grothendieck and Deligne-Mumford. They are motivated by the interest mathematicians have in endowing moduli “spaces” by geometrical structures, allowing them to be studied through classical techniques. The following are the key results relating groupoids and stacks to simplicial presheaves.

Theorem 4.1. *There exists a Quillen equivalence* (π_{oid}, N) *of functors*

$$\pi_{oid} : (S^2_s)^{-1}\Delta^{op}\mathsf{Prsh}_T(\mathcal{S})_J \to (\mathsf{Grp}/\mathcal{S})_L$$
$$\text{and} \quad N : (\mathsf{Grp}/\mathcal{S})_L \to (S^2_s)^{-1}\Delta^{op}\mathsf{Prsh}_T(\mathcal{S})_J.$$

Where $(S^2_s)^{-1}\Delta^{op}\mathsf{Prsh}_T(\mathcal{S})_J$ is the localized category with respect of morphisms generated by the trivial map fron the simplicial sphere $S^2_s \to pt$, J is the model structure we endowed the category of simplicial presheaves in order to create $\mathcal{H}_s$. $\mathcal{S}$ is the base site (compex spaces with the strong topology in our case) and L is a model structure whose weak equivalences strictly relate to the condition for an $\mathcal{S}$ groupoid to be a stack.

Theorem 4.2. *The pair of functors* (π_{oid}, N) *induce an equivalence between the full subcategories of fibrant simplicial presheaves and stacks.*

The immediate consequence is that simplicial presheaves with trivial holotopy in simplicial degrees 2 and higher maybe thought of as groupoids, via the explicit functor π_{oid} and viceversa, via the functor N. Moreover, the obscure condition making a simplicial presheaf fibrant for the Joyal model structure, is, in fact, a "gluing" condition making a stack out of a groupoid. Although a stack structure is still not sufficient to perform constructions that are usual for complex spaces, it constitutes a huge leap forward towards simplicial sheaves geometrization. It is the supplementary *complex analytic* condition on the stack which makes them manageable: for a stack $\mathcal{Y}$, this amounts to have an *atlas* $p : X \to \mathcal{Y}$, smooth and surjective map, and assume that the diagonal morphism $\mathcal{Y} \to \mathcal{Y} \times \mathcal{Y}$ is representable. At times, we had to assume p étale; these stacks are usually called *Deligne-Mumford* analytic stacks. Analytic stacks can be represented by simplicial complex spaces. Moreover, some questions regarding analytic stacks can be rephrased in terms of complex spaces by means of the atlas.

4.1 - *Stack hyperbolicity*

After the introduction of hyperbolicity for simplicial presheaves, we wondered if that was the "right" definition. One of the most celebrated results in hyperbolic geometry is the Brody's Theorem that states the equivalence between two properties of complex spaces: the Kobayashi pseudodistance of a complex space X is a distance and the fact that any holomorphic map $\mathbb{C} \to X$

is constant. The *Kobayashi pseudodistance* between two points p and q in X is defined to be the infimum of all the lenghts of the *chains of holomorphic discs* from p to q. A chain of holomorphic discs from p to q is a sequence $f_1, \ldots, f_k$ of holomorphic maps $f_i : \mathbb{D} \to X$ such that $f_{j-1}(b_j) = f_j(a_j)$, with $a_1, \ldots, a_k, b_1, \ldots, b_k \in \mathbb{D}$, $1 \leq j \leq k$. The *length* of a chain is the number $\sum_{i=0}^{n-1} d_{\mathbb{D}}(a_i, b_i)$ where $d_{\mathbb{D}}$ is the Poincaré distance on $\mathbb{D}$. The *Kobayashi pseudodistance* $d_X(p,q)$ between p and q is then defined as

$$\inf \sum_{i=0}^{n-1} d_{\mathbb{D}}(a_i, b_i)$$

where the infimum is taken over all chains of holomorphic discs from p to q.

The aim of the paper [**1**] is to provide further evidence that the notion of simplicial presheaf hyperbolicity we have given in [**3**] is useful; in particular, we prove Brody's Theorem for Deligne-Mumford analytic stacks. The proof effectively begins by providing a definition of Kobayashi hyperbolicity for groupoids, which extends the one for complex spaces and is very similar in spirit (ref. Section 4.2.1.). The notion is given for arbitrary presheaves and then extended to groupoids by making parabolic holotopy groups in presheaves and requiring Kobayashi hyperbolicity for all the holotopy presheaves of a groupoid, which effectively reduces to hyperbolicity just of the presheaves π_0 and π_1. Giving a definition of (relative) analytic chains of a presheaf was in itself nontrivial and constitutes the foundation for the Kobayashi pseudodistance in the sections of a presheaf and consequently for the notion of Kobayashi hyperbolicity for presheaves. The strategy of replacing groupoids with holotopy groups is motivated by the fact that the latter completely determine global simplicial homotopy equivalences of groupoids and therefore of (local) stack equivalences. Proposition 4.3 of [**1**] shows that, for stacks, Brody hyperbolicity as well is equivalent to Brody hyperbolicity for holotopy presheaves. Therefore, to prove the main theorem it suffices to consider holotopy presheaves. The proof involves manipulation of the sections of such presheaves. By progressively adding structures on groupoids up to obtaining analytic stacks $X \to \mathcal{Y}$, we could express these sections over a complex space U in terms of certain holomorphic maps $U_i \to X$ and $U_i \to X \times_{\mathcal{Y}} X$, for open subspaces U_i of X.

Another key idea is to use the *coarse moduli space* $X \to Q(\mathcal{X})$ associated to a flat analytic groupoid $\mathcal{X} = R \rightrightarrows X$, fact proved in [**2**]. The coarse moduli space we will consider is the one associated to $X \times_{\mathcal{Y}} X \rightrightarrows X$ for a Deligne-Mumford analytic stack $X \to \mathcal{Y}$. A purposedly defined metric on $Q(\mathcal{Y})$ endows the sets of sections of the holotopy presheaves of metric structures and compactness of the analytic stack $\mathcal{Y}$ is crucially used to establish the Brody's Theorem.

In the last section of [**3**], we provide two applications. En route to prove Brody's theorem, we remarked that Brody hyperbolicity of the coarse moduli space $Q(\mathcal{Y})$ is a stronger condition than Brody hyperbolicity for the Deligne-Mumford analytic stack $\mathcal{Y}$. However, the concept of stack hyperbolicity cannot be reduced to the hyperbolicity of a complex space. We construct an example of a Brody hyperbolic stack with a non hyperbolic coarse moduli space: the stack is the Galois quotient $[X/\mathrm{Gal}(\mathbb{C}(X)/\mathbb{C}(T))]$ where T is a one dimensional torus and $X \to T$ is a degree 4 Galois covering branched in two points, each with ramification index 4. The coarse moduli space of this stack is T.

On complex spaces, hyperbolicity implies rather peculiar properties: one is that a hyperbolic space has finitely many biholomorphisms. In section 9, we prove that, under the stronger condition of hyperbolicity of the coarse moduli space, the 2-group of automorphisms of a compact Deligne-Mumford analytic stack has only finitely many isomorphisms classes.

I wish to express my gratitude to Pino for so many mathematical discussions during these past years; his interdisciplinary interests and open-mindedness in mathematics have ignited the fuel moving us across the study of this exceptionally heterogeneous subject.

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