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Elisabetta Barletta and Sorin Dragomir

Robinson-Sparling construction of CR structures associated to shearfree null geodesic congruences

Abstract. We review the construction of Lorentzian metrics, such as Fefferman type metrics, associated to a given 3-dimensional nondegenerate CR manifold M, and admitting shearfree null geodesic congruences N. This class of metrics is obtained by a lifting procedure from M to $M \times \mathbb{R}$ devised by I. Robinson and A. Trautman (cf. [**71**]–[**72**]) and notably radiative gravitational fields are searched for (cf. e.g. R.K. Sachs, [**74**]) within the class. Conversely, nondegenerate CR structures arise (by the Robinson-Trautmann construction, [**71**]) on leaf spaces $\mathfrak{M}/N$ associated to space-times $\mathfrak{M}$ adapted to given optical structures $\big((K, L), J\big)$. The Graham-Sparling construction (cf. [**40**], [**77**]) is shown to be a particular case of Robinson-Trautman construction where the complex structure on the complex line bundle $\mathrm{Ker}(L)/K \to \mathfrak{M}$ is induced by an f-structure with two complemented frames obtained as a covariant derivative of the given null Killing vector field N.

Keywords. CR structure, tangential Cauchy-Riemann equations, Fefferman metric, flag structure, optical structure, Lorentzian metric, space-time, Maxwell field, Dirac equation.

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Contents

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1 - CR *versus* Lorentzian geometry and CR embedding problem

The purpose of this expository paper[1] is to review the geometric ingredients needed to describe the interaction between CR and pseudohermitian geometry (such as discovered by S. M. Webster, [**84**], and N. Tanaka, [**78**]) on one hand, and space-time physics on the other. Said interaction has two sources i.e. it has been discovered independently within two distinct areas of scientific investigation, which are mathematical analysis of functions of several complex variables, and general relativity and gravitation theory coupled with electromagnetism. Given a smoothly bounded strictly pseudoconvex domain $\Omega \subset \mathbb{C}^n$, let $K(z,w)$ be the Bergman kernel of Ω and let us set

$$U(z,\zeta) = \left|\zeta\right|^{2/(n+1)} K(z,z)^{-1/(n+1)}, \quad z \in \Omega, \ \ \zeta \in \mathbb{C} \setminus \{0\}.$$

[1]Extended version of the lecture given at the meeting *Complex Analysis and Geometry in Pisa*, October 5-6, 2018.

By a result of C. Fefferman (cf. [**33**])

$$G=\sum_{j,k=0}^{n}\frac{\partial^2 U}{\partial z^j\,\partial \overline{k}}\,dz^j\odot d\overline{z}^k\,,\quad z^0=\zeta,$$

is a semi-Kählerian metric on $\Omega\times\big(\mathbb{C}\setminus\{0\}\big)$. If $\mathbf{j}:\Omega\times S^1\hookrightarrow\Omega\times\big(\mathbb{C}\setminus\{0\}\big)$ is the inclusion then, in the limit as $z\to\partial\Omega$, the metric $\mathbf{j}^*G$ tends to a Lorentzian metric g on $\partial\Omega\times S^1$ whose restricted conformal class $\big[g\big]=\{e^{u\circ\pi}g\,:\,u\in C^\infty(\partial\Omega,\,\mathbb{R})\}$ is a biholomorphic invariant (here $\pi:\partial\Omega\times S^1\to\partial\Omega$ is the projection). This is *Fefferman's metric* as first discovered in [**33**]. An open problem left by C. Fefferman at the time [**33**] was written, was whether one may build a Lorentzian metric g on $M\times S^1$ for any strictly pseudoconvex real hypersurface $M\subset\mathbb{C}^n$ in such a manner that the restricted conformal class $\big[g\big]$ be a CR invariant. The problem was solved by J. M. Lee (cf. [**53**]) who built a Lorentzian metric $F_\eta\in\mathrm{Lor}\big(C(M)\big)$ on the total space $C(M)$ of the canonical circle bundle $S^1\to C(M)\xrightarrow{\pi}M$ over a strictly pseudoconvex CR manifold M, not necessarily embedded, endowed with a positively oriented contact form η. When M is a real hypersurface in $\mathbb{C}^n$ the canonical circle bundle is trivial [i.e. $C(M)\approx M\times S^1$, a principal bundle isomorphism]. The construction in [**53**] is such that there is a conformal isometry of $\big(C(\partial\Omega),\,F_\eta\big)$ onto $\big(\partial\Omega\times S^1\,,\,g\big)$.

Another scheme for associating a Lorentzian metric $g\in\mathrm{Lor}(M\times\mathbb{R})$ to any 3-dimensional CR manifold M was devised by I. Robinson and A. Trautman (cf. [**71**]– [**72**]) and A. Trautman (cf. [**82**]), cf. (15) in Section § 2.6.1 below, producing Lorentzian metrics $g=g\big(\eta,\,\sigma,\,P,\,H,W\big)\in\mathrm{Lor}(M\times\mathbb{R})$ associated to a given pseudohermitian structure η, a globally defined C^∞ section $\sigma:M\to T_{1,0}(M)$, and a parameter (C^∞ function)

$$\big(P,\,H,\,W\big):M\times\mathbb{R}\to\big(\mathbb{R}\setminus\{0\}\big)\times\mathbb{R}\times\mathbb{C}\,.$$

The construction of the assignment $\big(\eta,\,\sigma,\,P,\,H,W\big)\mapsto g\big(\eta,\,\sigma,\,P,\,H,W\big)$ doesn't require additional assumptions on the Levi form G_η (i.e. it is general enough to include the case of a Levi flat CR manifold M) and it is explained in detail by C.D. Hill and J. Lewandowski and P. Nurowski (cf. [**42**]), although stripped of the flag and optical geometry background in [**71**]– [**72**], to the purpose of ingeniously applying the construction to the CR embedding problem.

The *embedding problem* for 3-dimensional CR manifolds is to build two functionally independent CR functions $u_a:U\to\mathbb{C}$, $a\in\{1,2\}$, defined on some neighbourhood $U\subset M$ of any given point $x_0\in M$, such that $\big(u_1\,,\,u_2\big):U\to\mathbb{C}^2$ be a CR immersion. The problem admits an obvious global version, besides from the local version just stated. Any real analytic CR manifold is locally embeddable, essentially by applying the Cauchy-Kowalewski theorem to the tangential Cauchy-Riemann equations $\overline{\partial}_b u = 0$. This is a celebrated result by A. Andreotti and C.D. Hill (cf. [**2**]). By a result of A. Andreotti and G.A. Fredricks (cf. [**3**]) real analytic CR manifolds are also globally realizable, as CR submanifolds of some complex manifold, perhaps other than $\mathbb{C}^2$. In the C^∞ category however, not even the local problem is always solvable. Indeed, by a result of L. Nirenberg (cf. [**61**]) a certain perturbation $\overline{L} + \varphi\, \partial/\partial t$ of the Lewy operator $\overline{L}$ (cf. e.g. [**45**], p. 235) on $\mathbb{R}^3$ furnishes a 3-dimensional CR manifold which is not embeddable in any neighbourhood of the origin. Our discussion is confined to the case of 3-dimensional CR manifolds so we only mention briefly that positive results for higher dimensional CR manifolds were obtained by M. Kuranishi (any strictly pseudoconvex CR manifold of dimension $2n+1 \geq 9$ is locally embeddable in $\mathbb{C}^{n+1}$, cf. [**52**]) while T. Akahori solved (cf. [**1**]) the problem in dimension $2n+1 = 7$, and the CR embedding problem is open in dimension $2n+1 = 5$. Going back to the 3-dimensional case, one may not over emphasise the importance of the work by C.D. Hill and J. Lewandowski and P. Nurowski (cf. [**42**]) who focused on the close relationship between local embeddability of 3-dimensional CR manifolds M and the existence of Lorentzian metrics on $M \times \mathbb{R}$ satisfying Einstein's equations $R_{\mu\nu} = \Lambda\, g_{\mu\nu}$. The subtle approach in [**42**] is to write the Cartan structure equations (for the Einstein field g at hand)

$$d\Gamma^\mu{}_\nu + \Gamma^\mu{}_\alpha \wedge \Gamma^\alpha{}_\nu = \frac{1}{2}\, R^\mu{}_{\nu\alpha\beta}\, \Theta^\alpha \wedge \Theta^\beta$$

with respect to a special coframe $\{\Theta^\mu \ : \ 0 \leq \mu \leq 3\}$ [where Θ^a, $a \in \{1,2\}$, are complex 1-forms and Θ^b, $b \in \{3,4\}$, are real 1-forms such that g admits the particularly simple representation $g_{12} = g_{21} = 1$ and $g_{03} = g_{30} = 1$ and $g_{\mu\nu} = 0$ otherwise] and find indices $(\mu_0\, , \, \nu_0)$ such that the complex Pfaffian system

$$\Gamma_{\mu_0\, \nu_0} = 0$$

is (as a consequence of Einstein's equations) involutive on (an open subset of) $M \times \mathbb{R}$. Then (by the real Frobenius theorem together with existence of isothermal coordinates on a Riemann surface) one may show that

$$\Gamma_{\mu_0\, \nu_0} = h\, d\zeta$$

for some real function h and some complex function ζ such that $d\zeta \wedge d\overline{\zeta} \neq 0$. The projection of ζ on M gives a first CR function u_1. To obtain a second CR function u_2 (such that $du_1 \wedge du_2 \neq 0$) one assumes (again cf. [**42**]) that, besides from the existence of a Lorentzian metric g satisfying Einstein's equations

$$R_{22} = 0, \quad R_{24} = 0, \quad R_{44} = 0, \tag{1}$$

[where g is derived, from the given strictly pseudoconvex CR structure $T_{1,0}(M)$ together with a vector field $N \in \mathfrak{X}(M \times \mathbb{R})$ tangent to the $\mathbb{R}$ term, by a lifting procedure that we describe in § 2.6.1] there also exits a nonvanishing Maxwell field $\mathcal{F} = F + i * F$ i.e.

$$dF = 0, \quad d * F = 0, \tag{2}$$

which is both null (i.e. $\mathcal{F} \wedge \mathcal{F} = 0$) and aligned with the congruence N (i.e. $\mathcal{F} = f\,\eta \wedge \theta^1$). One uses in an essential manner 1) the fact that the field equations $\mathrm{Ric}(g) = \Lambda\, g$ split (as a consequence of the choice of coframe $\{\Theta^\mu\}$) into three types of equations, which are (1) above and (3)-(4) below

$$R_{12} = \Lambda, \quad R_{34} = \Lambda, \tag{3}$$

$$R_{33} = R_{23} = 0, \tag{4}$$

and 2) a beautiful result by I. Robinson (cf. [**69**]) producing null anti-selfdual 2-forms $\mathcal{F} = f\,\eta \wedge \theta^1$ which satisfy Maxwell's equations (2) if and only if $\overline{L} f = 0$ (where $\overline{L}$ is Lewy's operator). See also the survey by A. Trautman, [**83**].

By a result of J. M. Lee (cf. [**53**]) the Fefferman metric F_η is never Einstein, hence it may not be used to produce CR functions along the scheme in [**42**]. However the same choice of special coframe $\{\Theta^\mu\}$ as in [**42**] proved effective in the mathematical analysis of the gravitational field equations on $C(\mathbb{H}_1)$ in the presence of the matter distribution described by the energy-momentum tensor $\mathrm{Ric}(F_{\eta_0}) - \frac{1}{2}\,\mathrm{Scal}(F_{\eta_0})\, F_{\eta_0}$ (cf. E. Barletta et al., [**14**]). On the other hand the properties of the Fefferman metric F_η are certainly related to the embeddability of the given CR manifold M, since for M to embed so should Fefferman's space $(C(M),\, F_\eta)$, yielding constrains on the Pontrjagin forms of F_η. Indeed, by a result of the authors (cf. [**10**]) if $\eta_0 = \frac{i}{2}(\overline{\partial} - \partial)|z|^2$ then $(C(S^{2n+1}),\ F_{\eta_0})$ admits a conformal embedding in the semi-Euclidean space $\mathbb{R}^{2n+4}_2$ (e.g. implying the vanishing of the first Pontrjagin form of F_{η_0}).

The paper is organized as follows. In Sections § 2.1 to § 2.5 we collect a few basic notions and facts in CR and pseudohermitian geometry. We follow the exposition in [**31**] and [**13**]. Section § 2.6 is devoted to a lifting procedure, due to I. Robinson and A. Trautman (cf. [**71**]) and A. Trautman (cf. [**83**]), of a strictly pseudoconvex CR structure $T_{1,0}(M)$ to a Lorentzian metric g on $M \times \mathbb{R}$ and to showing that the Fefferman metric of the Heisenberg group $(\mathbb{H}_1\,,\, \eta_0)$ is locally a particular instance of such g. In Sections § 2 up to § 4 we review elements of flag and optical geometry. In Sections § 5.1 to § 5.2 we exhibit the main finding by G. Sparling (cf. [**77**]) and C. R. Graham ([**40**]) that every Lorentzian manifold $(\mathfrak{M}, g)$ admitting a null Killing vector field $N \in \mathfrak{X}(\mathfrak{M})$ with

$N \rfloor W = 0$ and $N \rfloor C = 0$ carries a natural f-structure with two complemented frames $(J, N, V, \theta, \sigma)$ where $J = \nabla N$.

The concept of an f-structure (here $J^3 + J = 0$) is the classical one in K. Yano (cf. [**85**]) yet none of the results by D. E. Blair and G. D. Ludden and K. Yano (cf. [**20**]) apply, in part because of the lack of compactness, but mainly because the work [**20**] is *a priori* confined to Riemannian geometry. Recovering results from [**17**]–[**19**], [**20**], [**24**], [**39**], from the realm of Riemannian geometry to that of Lorentzian geometry is certainly desirable, and perhaps feasible, but an open problem as yet. Nonetheless we believe that the cultural identification of the studies in [**77**] and [**40**] as work on the geometry of f-structures on Lorentzian manifolds will prompt new developments of the field. For instance, the very explicit calculations in classical tensor notation, as performed in Sections § 5.1 to § 5.2 (by following [**40**]), may provide an answer to the question "what is a Finslerian CR structure?" For previous attempts in this direction one may see [**29**].

Graham-Sparling construction is paralleled to Robinson-Trautman construction in Section § 5.3.3 and recognized as a particular instance of the latter. In Section § 7 we follow work by M. Arminjon and F. Reifler ([**7**]) and N. Kamran and R. G. McLenaghan (cf. [**46**]) to respectively show how Dirac's equation in quantum mechanics may be formulated on a curved space-time (as generalizing the classical formulation on Minkowski space) and indicate an instance where the occurrence of two independent null Killing vector fields N_a, $a \in \{1, 2\}$, allows for separation of variables in the massive charged Dirac equation on a curved space-time $\mathfrak{M}$. The background Lorentzian metric needed in [**46**] belongs to the class defined by formula (15) (in our Section § 2.6.1) and, under appropriate assumptions, the shear free null geodesic congruences determined by N_a give rise to two strictly pseudoconvex CR structures $T_{1,0}(M_a)$ on the orbit spaces $M_a = \mathfrak{M}/N_a$. While the role played by CR geometry *versus* physics of radiative Einstein and Maxwell fields is well explained in [**42**], in a language accessible to the more mathematics oriented reader, the relevance of the CR structures $T_{1,0}(M_a)$ in devising separation of variables for Dirac's equation is not fully understood, and their further investigation is proposed as an open problem.

2 - CR structures

2.1 - *Tangential Cauchy-Riemann equations*

Let M be a $(2n+1)$-dimensional C^∞ manifold. An *almost CR structure* of *CR dimension* n is a complex rank n subbundle $T_{1,0}(M) \subset T(M) \otimes \mathbb{C}$ such that

$$T_{1,0}(M) \cap T_{0,1}(M) = (0), \tag{5}$$

where $T_{0,1}(M) = \overline{T_{1,0}(M)}$ and an overbar denotes complex conjugation. An almost CR structure is (*formally*) *integrable* if

$$Z, W \in C^\infty\big(U, T_{1,0}(M)\big) \Longrightarrow [Z, W] \in C^\infty\big(U, T_{1,0}(M)\big) \tag{6}$$

for any open subset $U \subset M$. An integrable almost CR structure $T_{1,0}(M)$ is a *CR structure* and a pair $(M, T_{1,0}(M))$ consisting of a manifold M and an (almost) CR structure of CR dimension n is an (*almost*) *CR manifold*, of *CR dimension* n. The *tangential Cauchy-Riemann operator* is the first order differential operator

$$\overline{\partial}_b : C^1\big(M, \mathbb{C}\big) \to C\big(T_{0,1}(M)^*\big), \quad \big(\overline{\partial}_b u\big)\overline{Z} = \overline{Z}(u),$$

for any C^1 function $u : M \to \mathbb{C}$ and any $Z \in T_{1,0}(M)$. The *tangential C-R equations* are

$$\overline{\partial}_b u = 0 \tag{7}$$

and a C^1 solution u to (7) is a *CR function*. Let $\mathrm{CR}^k(M)$ be the space of all CR functions of class C^k ($k \in \mathbb{N} \cup \{\infty, \omega\}$). Given an (almost) CR manifold $(M,\, T_{1,0}(M))$ its *Levi distribution* is

$$H(M) = \mathrm{Re}\big\{T_{1,0}(M) \oplus T_{0,1}(M)\big\}.$$

The Levi distribution carries the complex structure

$$J : H(M) \to H(M), \;\; J\big(Z + \overline{Z}\big) = i\big(Z - \overline{Z}\big), \;\; Z \in T_{1,0}(M).$$

The *conormal bundle* is the real line subbundle $H(M)^\perp \subset T^*(M)$ given by

$$H(M)^\perp_x = \{\omega \in T^*_x(M) : \mathrm{Ker}(\omega) \supset H(M)_x\}, \quad x \in M.$$

A *pseudohermitian structure* on M is a globally defined, nowhere zero, cross section $\eta \in C^\infty\big(H(M)^\perp\big)$. Pseudohermitian structures are merely differential 1-forms $\eta \in \Omega^1(M)$ such that $\mathrm{Ker}(\eta) = H(M)$. When M is orientable (which will be assumed through the paper) the conormal bundle is trivial (i.e. $H(M)^\perp \approx M \times \mathbb{R}$, a vector bundle isomorphism) hence pseudohermitian structures do exist. The *Levi form* is

$$G_\eta(X, Y) = (d\eta)(X, JY), \quad X, Y \in H(M).$$

A CR manifold $(M, T_{1,0}(M))$ is *nondegenerate* (respectively *strictly pseudoconvex*) if the Levi form G_η is nondegenerate (respectively positive-definite) for some pseudohermitian structure η. If M is nondegenerate then every pseudohermitian structure η is a contact form i.e. $\eta \wedge (d\eta)^n$ is a volume form on M. A CR manifold is *Levi flat* if $G_\eta = 0$ for some η, and thus for all. Equivalently the Levi distribution is completely integrable.

Theorem 2.1 (T. Levi-Civita, [**55**]). *Every Levi flat CR manifold $(M, T_{1,0}(M))$ of CR dimension n carries a codimension one foliation $\mathcal{F}$ by complex n-dimensional manifolds such that $T(\mathcal{F}) = H(M)$.*

Theorem 2.1 is an immediate corollary of the classical Frobenius theorem [on the integrability of involutive Pffafian systems, such as $H(M)$] together with Newlander-Nirenberg theorem (on the integrability of almost complex structures with vanishing Nijenhuis torsion). Newlander-Nirenberg theorem was unknown at the time [**55**] was written (a direct proof to Theorem 2.1 is provided in [**55**]). The local defining submersions (cf. e.g. [**59**], p. 15) of the foliation $\mathcal{F}$ in Theorem 2.1 are real valued CR functions. Therefore the (obvious) analogy between holomorphic functions (on complex manifolds) and CR functions (on CR manifolds) is confined to the nondegenerate case (where any real valued CR function may be shown to be constant).

Given two (almost) CR manifolds $(M, T_{1,0}(M))$ and $(N, T_{1,0}(N))$ a *CR isomorphism* is a C^∞ diffeomorphism $F : M \to N$ such that $(d_x F) T_{1,0}(M)_x = T_{F(x)}(N)$ for any $x \in M$. This is equivalent to F preserving the Levi distributions $(d_x F) H(M)_x = H(N)_{F(x)}$ and commuting with the complex structures $(d_x F) \circ J_{M,x} = J_{N,F(x)} \circ (d_x F)$ for any $x \in M$. A C^∞ diffeomorphism of M preserving $H(M)$ is customarily referred to as a *contact transfomation* (even though M may not be a contact manifold).

2.2 - *Real hypersurfaces*

Examples of (almost) CR manifolds appear as real hypersurfaces of (almost) complex manifolds. For instance let $M \subset \mathbb{C}^{n+1}$ be a real hypersurfce and let $T_{1,0}(M)$ be defined by

$$T_{1,0}(M)_x = \left[T_x(M) \otimes_{\mathbb{R}} \mathbb{C}\right] \cap T^{1,0}\left(\mathbb{C}^{n+1}\right)_x, \quad x \in M, \tag{8}$$

where $T^{1,0}(\mathbb{C}^{n+1}) = \left\{X - i\, JX \; : \; X \in T(\mathbb{C}^{n+1})\right\}$ $(i = \sqrt{-1})$ is the holomorphic tangent bundle over $\mathbb{C}^{n+1}$ (the span of $\{\partial/\partial z^j \; : \; 1 \leq j \leq n+1\}$) and J denotes the complex structure on $\mathbb{C}^{n+1}$. Then $T_{1,0}(M)$ is a CR structure on M, of CR dimension n, *induced* on M by the complex structure of $\mathbb{C}^{n+1}$. The integrability of $T_{1,0}(M)$ is inherited by (8) from the integrability property of the complex structure J. If $\Omega \subset \mathbb{C}^{n+1}$ is an open set, $f : \Omega \to \mathbb{C}$ is a holomorphic function, and $u = f \circ \iota$ is the trace of f on $U = \Omega \cap M$, then $u \in \mathrm{CR}^\infty(U)$. In particular smooth boundaries of domains $\Omega \subset \mathbb{C}^{n+1}$ are real hypersurfaces and hence CR manifolds, with the induced CR structure (8). For instance the sphere $S^{2n+1} = \partial \mathbb{B}^{n+1}$, as the boundary of the unit ball $\mathbb{B}^{n+1} = \{z \in \mathbb{C}^{n+1} : |z| < 1\}$, and the Heisenberg group $\mathbb{H}_n = \mathbb{C}^n \times \mathbb{R}$ as $F(\zeta, t) = \big(\zeta,\ t + i|\zeta|^2\big)$ is a C^∞ diffeomorphism $\mathbb{H}_n \approx \partial \mathcal{S}^{n+1}$ onto the boundary of the Siegel domain $\mathcal{S}^{n+1} = \big\{z \in \mathbb{C}^{n+1} : \mathrm{Im}(z^{n+1}) > \sum_{\alpha=1}^{n} \big|z^\alpha\big|^2\big\}$.

Let $\Omega \subset \mathbb{C}^{n+1}$ be a bounded domain. By a classical result of C. Fefferman (cf. [**32**]) if $M = \partial\Omega$ is a C^∞ real hypersurface in $\mathbb{C}^{n+1}$ whose induced CR structure (8) is strictly pseudoconvex, then every biholomorphic transformation $\Phi \in \mathrm{Hol}(\Omega)$ extends smoothly to the boundary and its boundary values $F = \Phi|_{\partial\Omega} : M \to M$ is a CR isomorphism, and in particular a contact transformation. Let $K(z, \zeta)$ be the Bergman kernel of Ω (cf. e.g. S. Bergman, [**16**]). The *Bergman metric* is the Kählerian metric on Ω given by

$$g_{j\overline{k}} = \frac{\partial^2 \log K(z,z)}{\partial z^j \, \partial \overline{z}^k}, \quad 1 \leq j, k \leq n+1.$$

Let $\omega = i\, \partial\overline{\partial} \log K(z,z)$ be the corresponding symplectic structure of the Kählerian manifold (Ω, g). Every biholomorphic transformation of Ω is an isometry of (Ω, g) i.e. $\mathrm{Hol}(\Omega) \subset \mathrm{Isom}(\Omega, g)$ (cf. e.g. S. Helgason, [**41**]). As each $\Phi \in \mathrm{Hol}(\Omega)$ preserves the complex structure on Ω to start with, one also has $\Phi^*\omega = \omega$ i.e. Φ is a symplectomorphism. Conversely, if $\Phi : \Omega \to \Omega$ is a symplectomorphism of $(\Omega,\ \omega)$ which is not a biholomorphism (so that Fefferman's theorem doesn't apply) yet Φ extends smoothly to the boundary, then, by a result of A. Korány and H. M. Reimann (cf. [**51**]) the boundary values F of Φ is a contact transformation of $(\partial\Omega,\ H(\partial\Omega))$ in itself. The proof in [**51**] relies on Fefferman's asymptotic expansion of the Bergman kernel (cf. [**32**]) as well as the proof of Fefferman's theorem, so the assumption of strict pseudoconvexity on $M = \partial\Omega$ should be in force.

As a typical approach in complex analysis of functions of several complex variables, when building the induced CR structure $T_{1,0}(M)$ one disregards all geometric structures on $\mathbb{C}^{n+1}$, except for its complex structure J. An alternative "metric" approach to the same CR structure is however available, and typical in differential geometry [of submanifolds of (almost) Hermitian manifolds]. Precisely let g_0 be the natural flat Kähler metric of $\mathbb{C}^{n+1}$ and let $g = \iota^* g_0$ be the induced metric, or first fundamental form of the given immersion $\iota : M \subset \mathbb{C}^{n+1}$. Let us assume that M is orientable and choose a globally defined unit normal field ν on M

$$g_0\big(\nu\,,\,\nu\big) = 1, \quad g_0\big(X\,,\,\nu\big) = 0, \quad X \in \mathfrak{X}(M).$$

Then $\xi = -J\nu$ is tangent to M and $\eta(X) = g(X, \xi)$ is a differential 1-form on M. Moreover $H(M) = \mathrm{Ker}(\eta) \subset T(M)$ is a distribution of hyperplanes and the restriction of J to $H(M)$ is $H(M)$-valued. Consequently $J : H(M) \to H(M)$ is a complex structure on $H(M)$ i.e. $J^2 X = -X$ for any $X \in H(M)$. Let $J^{\mathbb{C}}$ be the $\mathbb{C}$-linear extension of J to $H(M) \otimes \mathbb{C}$. Then $\big(J^{\mathbb{C}}\big)^2 V = -V$ for any $V \in H(M) \otimes \mathbb{C}$ hence $\mathrm{Spec}\big(J^{\mathbb{C}}\big) = \{\pm i\}$. Finally $T_{1,0}(M) = \mathrm{Eigen}\big(J^{\mathbb{C}}, i\big)$.

It should be observed, within the same class of examples, that $J : H(M) \to H(M)$ may be extended to a rank $2n$ tensor field ϕ of type $(1,1)$ on M given by

$$\phi(\xi) = 0, \quad \phi(X) = JX, \quad X \in H(M)$$

followed by linear extension to the whole of $T(M) = H(M) \oplus \mathbb{R}\xi$. Then the synthetic object (ϕ, ξ, η, g) is an almost contact metric structure on M (in the sense of D. E. Blair, [**17**]).

2.3 - *Contact Riemannian manifolds*

The last observation in Section § 2.2 suggests that the same construction may be performed on any contact Riemannian manifold $\big(M, (\phi, \xi, \eta, g)\big)$. Here M is a $(2n+1)$-dimensional manifold, not necessarily embedded in some complex manifold, $\phi : T(M) \to T(M)$ is a vector bundle endomorphism, $\xi \in \mathfrak{X}(M)$ is a tangent vector field, $\eta \in \Omega^1(M)$ is a differential 1-form, and g is a Riemannian metric on M, obeying to the following axioms

$$\phi^2 = -I + \eta \otimes \xi, \quad \phi(\xi) = 0, \quad \eta \circ \phi = 0,$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\,\eta(Y),$$

$$d\eta = \Phi, \quad \Phi(X, Y) = g(X, \phi Y).$$

Then ϕ descends to a complex structure $J : \mathrm{Ker}(\eta) \to \mathrm{Ker}(\eta)$ and

$$T_{1,0}(M) = \mathrm{Eigen}\big(J^{\mathbb{C}}, i\big) \subset \mathrm{Ker}(\eta) \otimes \mathbb{C} \tag{9}$$

is an almost CR structure on M yet $T_{1,0}(M)$ is integrable if and only if $Q = 0$ where Q is the *Tanno tensor* i.e.

$$Q_{jk}{}^{i} = \nabla_k \phi^i{}_j + \eta_j\, \phi^i{}_k - \eta_k\, \nabla_j \xi^i + \xi^i\, \nabla_j \eta_k\,.$$

Cf. S. Tanno, [**79**]. The first to consider the almost CR structure (9) was S. Ianuş (cf. [**44**]) who also gave a sufficient condition for its integrability: if the contact Riemannian structure (ϕ, ξ, η, g) is normal i.e. $[\phi, \phi] + 2(d\eta) \otimes \xi = 0$, then (9) is integrable. The characterisation above (in terms of the vanishing of the Tanno tensor) is however a finding by the authors (cf. [**12**]).

2.4 - *Tangent sphere bundle*

In this section we give another example of an (almost) CR structure, related to the geometry of the tangent bundle of a given m-dimensional Riemannian manifold (N, h). Let $S^{m-1} \to U(N) \to N$ be the *tangent sphere bundle* i.e. $U(N)_p = \{v \in T_p(N) : h_p(v, v) = 1\}$ for any $p \in N$. Let $\Pi : T(N) \to N$ be the projection. For every local coordinate system (U, x^i) on N let $\big(\Pi^{-1}(U),\, x^i,\, y^i\big)$ be the induced local coordinates on $T(N)$ and $\mathfrak{L} = y^i\,(\partial/\partial x^i)^\Pi$ the *Liouville vector* (a section in the pullback bundle $\Pi^{-1}T(N) \to T(N)$). Here $X^\Pi \in \mathfrak{X}(T(M))$ [the *natural lift* of $X \in \mathfrak{X}(N)$] is given by $X^\Pi_v = X_{\Pi(v)}$ for any $v \in T(N)$. Let $h^\Pi = \Pi^{-1}h$ and $\nabla^\Pi = \Pi^{-1}\nabla$ be respectively the pullbacks by Π of the Riemannian metric h and of its Levi-Civita connection ∇ [respectively a Riemannian bundle metric and a connection in the pullback bundle $\Pi^{-1}T(N)$]. For each $v \in T(N)$ let $\mathfrak{N}_v$ consist of all $w \in T_v(T(M))$ such that $\big(\nabla^\Pi_W \mathfrak{L}\big)_v = 0$ where $W \in \mathfrak{X}(T(N))$ is an arbitrary C^∞ extension of w i.e. $W_v = w$. Then $\mathfrak{N}$ is a *nonlinear connection* on N i.e. a C^∞ distribution on $T(N)$ such that

$$T_v(T(N)) = \mathfrak{N}_v \oplus \mathrm{Ker}(d_v\Pi), \quad v \in T(N). \tag{10}$$

If $W \in \mathfrak{X}(T(N))$ then $\Pi_* W \in C^\infty\big(\Pi^{-1}TN\big)$ is given by $\big(\Pi_* W\big)_v = (d_v\Pi)W_v$ for any $v \in T(N)$. Let $\gamma : \Pi^{-1}T(N) \to T(T(N))$ be the *vertical lift* i.e. locally $\gamma_v\big(\partial/\partial x^i\big)_{\Pi(v)} = \big(\partial/\partial y^i\big)_v$ for any $v \in \Pi^{-1}(U)$. Then γ is $\mathrm{Ker}(d\Pi)$-valued and gives a vector bundle isomorphism $\Pi^{-1}T(N) \approx \mathrm{Ker}(d\Pi)$. Let $K : T(T(N)) \to \Pi^{-1}T(N)$ be the *Dombowski map* i.e. $K = \gamma^{-1} \circ V$ where $V : T(T(N)) \to \mathrm{Ker}(d\Pi)$ is the natural projection [associated to the decomposition (10)]. The total space $T(N)$ of the tangent bundle carries the Riemannian metric g given by

$$g(A, B) = h^\Pi\big(\Pi_* A\,,\, \Pi_* B\big) + h^\Pi\big(KA\,,\, KB\big), \quad A, B \in \mathfrak{X}(T(N)),$$

[the *Sasaki metric* of (N, h)]. If

$$N^i_j : \Pi^{-1}(U) \to \mathbb{R}, \quad N^i_j = \left\{ \begin{array}{c} i \\ jk \end{array} \right\} y^k, \quad \frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} + N^j_i \frac{\partial}{\partial y^j},$$

then $\big\{\delta/\delta x^i \;:\; 1 \le i \le m\big\}$ is a local frame in $\mathfrak{N} \to T(N)$, defined on the open set $\Pi^{-1}(U)$. Let $\beta : \Pi^{-1}T(N) \to \mathfrak{N}$ be the *horizontal lift* i.e. the vector bundle isomorphism locally given by

$$\beta_v \left(\frac{\partial}{\partial x^i}\right)_{\Pi(v)} = \left(\frac{\delta}{\delta x^i}\right)_v, \quad v \in \Pi^{-1}(U).$$

The total space $T(N)$ of the tangent bundle also carries the natural almost complex structure J given by

$$J\gamma X = \beta X, \quad J\beta X = -\gamma X, \quad X \in C^\infty\big(\Pi^{-1}TN\big),$$

compatible to the Sasaki metric g so that $\big(T(N),\, g,\, J\big)$ is an almost Hermitian manifold. The almost complex structure J is integrable if and only if (N, h) is locally Euclidean (cf. P. Dombrowski, [**28**]). The total space $M = U(N)$ of the tangent sphere bundle is an orientable real hypersurface in $T(N)$. Let $T_{1,0}(M)$ be the almost CR structure induced by $(g,\, J)$ on $M = U(N)$. While in general $T_{1,0}(M)$ fails to be integrable, for any space form $N = N^m(1)$ [any Riemannian manifold N of constant sectional curvature 1] $T_{1,0}\big(U\big(N^m(1)\big)\big)$ is a strictly pseudoconvex CR structure (cf. the authors, [**9**]).

2.5 - *Pseudohermitian geometry*

Let M be a $(2n+1)$-dimensional orientable nondegenerate CR manifold M, of CR dimension n, and let η be a pseudohermitian structure on M, and hence a contact form. Let $\xi \in \mathfrak{X}(M)$ be the *Reeb vector* of (M, η) i.e. the nowhere zero tangent vector field ξ, transverse to the Levi distribution, determined by $\eta(\xi) = 1$ and $\xi \rfloor d\eta = 0$. The *Webster metric* is the semi-Riemannian metric g_η on M given by

$$g_\eta(X, Y) = G_\eta(X, Y), \quad g_\eta(X, \xi) = 0, \quad g_\eta(\xi, \xi) = 1,$$

for any $X, Y \in H(M)$. By a result of S.M. Webster (cf. [**84**]) and N. Tanaka (cf. [**78**]) there is a unique linear connection ∇ on M [the *Tanaka-Webster connection* of (M, η)] such that i) the Levi distribution $H(M)$ is parallel with respect to ∇, ii) $\nabla g_\eta = 0$, $\nabla J = 0$, and iii) the torsion T_∇ of ∇ satisfies

$$T_\nabla(Z, W) = 0, \quad T_\nabla(Z, \overline{W}) = 2\,i\,G_\eta(Z, \overline{W})\,\xi, \quad \tau \circ J + J \circ \tau = 0,$$

for any $Z, W \in T_{1,0}(M)$, where $\tau(X) = T_\nabla(\xi, X)$ for any $X \in \mathfrak{X}(M)$ [τ is the *pseudohermitian torsion* of ∇]. Given a local frame $\{T_\mu : 1 \le \mu \le n\} \subset C^\infty(U, T_{1,0}(M))$ we set

$$g_{\alpha\overline{\beta}} = G_\eta\big(T_\alpha\,,\, T_{\overline{\beta}}\big), \quad \big[g^{\mu\overline{\nu}}\big] = \big[g_{\mu\overline{\nu}}\big]^{-1}.$$

Then

$$d\eta = 2\,i\,g_{\alpha\overline{\beta}}\,\theta^\alpha \wedge \theta^{\overline{\beta}}\,. \tag{11}$$

Let R^∇ be the curvature tensor field of ∇ and let us set

$$R_C{}^D{}_{AB} T_D = R^\nabla\big(T_A\,,\, T_B\big) T_C\,.$$

As to the range of indices the conventions are

$$A, B, C, \cdots \in \{0,\, 1, \cdots,\, n,\, \overline{1},\, \cdots,\, \overline{n}\}, \quad T_0 = \xi, \quad \alpha,\, \beta,\, \gamma, \cdots \in \{1,\, \cdots,\, n\}.$$

The *pseudohermitian Ricci tensor* and *pseudohermitian scalar curvature* are

$$R_{\mu\overline{\nu}} = R_\alpha{}^\alpha{}_{\mu\overline{\nu}}\,, \quad R = g^{\mu\overline{\nu}} R_{\mu\overline{\nu}}\,.$$

An *adapted coframe* consists of complex valued 1-forms $\{\theta^\mu : 1 \le \mu \le n\}$ such that

$$\theta^\mu(T_\nu) = \delta^\mu_\nu\,, \quad \theta^\mu(T_{\overline{\nu}}) = 0, \quad \theta^\mu(\xi) = 0.$$

For all local calculations one also sets

$$\nabla T_\beta = \omega^\alpha{}_\beta\, T_\alpha\,, \quad \omega^\alpha{}_\beta = \Gamma^\alpha_{A\beta}\, \theta^A\,, \quad \theta^0 = \eta, \quad \theta^{\overline{\alpha}} = \overline{\theta^\alpha}\,,$$

so that $\omega^\alpha{}_\beta$ and Γ^A_{BC} are respectively the connection 1-forms and the connection coefficients [with respect to the local frame $\{T_A\}$ in $T(M)\otimes\mathbb{C}$]. Let us consider the (locally defined) differential 2-forms

$$\Pi^\beta{}_\alpha = d\omega^\beta{}_\alpha - \omega^\gamma{}_\alpha \wedge \omega^\beta{}_\gamma\,, \quad \Omega^\beta{}_\alpha = \Pi^\beta{}_\alpha - 2\,i\,\theta_\alpha \wedge \tau^\beta + 2\,i\,\tau_\alpha \wedge \theta^\beta\,,$$

where

$$\tau^\alpha = A^\alpha_{\overline{\beta}}\,\theta^{\overline{\beta}}\,, \quad \tau^{\overline{\alpha}} = \overline{\tau^\alpha}\,, \quad \tau_\alpha = g_{\alpha\overline{\beta}}\,\tau^{\overline{\beta}}\,, \quad \tau(T_{\overline{\beta}}) = A^\alpha_{\overline{\beta}}\, T_\alpha\,, \quad \theta_\alpha = g_{\alpha\overline{\beta}}\,\theta^{\overline{\beta}}\,.$$

By a result of S.M. Webster (cf. [**84**])

$$\Omega^\beta{}_\alpha = R_\alpha{}^\beta{}_{\lambda\overline{\mu}}\,\theta^\lambda \wedge \theta^{\overline{\mu}} + W^\beta{}_{\alpha\lambda}\,\theta^\lambda \wedge \eta - W^\beta{}_{\alpha\overline{\lambda}}\,\theta^{\overline{\lambda}} \wedge \eta \tag{12}$$

(compare to (1.90) in [**31**], p. 55) where

$$W^\beta{}_{\alpha\lambda} = g^{\beta\overline{\sigma}}\,\nabla_{\overline{\sigma}} A_{\alpha\lambda}\,, \quad W^\beta{}_{\alpha\overline{\lambda}} = g^{\beta\overline{\sigma}}\,\nabla_\alpha A_{\overline{\lambda}\overline{\sigma}}\,.$$

Covariant derivatives are meant with respect to the Tanaka-Webster connection ∇ of (M,η). A contact form η on M is *pseudo-Einstein* if the pseudohermitian Ricci tensor is proportional to the Levi form i.e.

$$R_{\mu\overline{\nu}} = \frac{R}{n}\, g_{\mu\overline{\nu}}\,. \tag{13}$$

Cf. J.M. Lee, [**54**]. Odd dimensional spheres $S^{2n+1} \subset \mathbb{C}^{n+1}$, (total spaces of) tangent sphere bundles $U(N^m(1))$ over Riemannian space forms $N^m(1)$, nondegenerate 3-dimensional CR manifolds [for which the pseudo-Einstein condition (13) is trivially satisfied] are examples of strictly pseudoconvex CR manifolds admitting (globally defined) pseudo-Einstein contact forms.

Lemma 2.1. *For any 3-dimensional orientable nondegenerate CR manifold M its CR structure $T_{1,0}(M)$ is a trivial complex line bundle.*

Proof. The Tanaka-Webster connection ∇ of (M,η) parallelizes $H(M)$ and J hence it descends to a connection in the vector bundle $T_{1,0}(M)$. Therefore one may use the curvature of ∇ in order to compute characteristic forms of $T_{1,0}(M)$ (cf. [**47**], Vol. II, pp. 307-308). Indeed the first Chern class of $T_{1,0}(M)$ is represented by

$$\gamma_1 = -\frac{1}{2\pi i}\left\{R_{\lambda\overline{\mu}}\,\theta^\lambda \wedge \theta^{\overline{\mu}} + W^\alpha{}_{\alpha\lambda}\,\theta^\lambda \wedge \eta - W^\alpha{}_{\alpha\overline{\mu}}\,\theta^{\overline{\mu}} \wedge \eta\right\}$$

i.e. $c_1\big(T_{1,0}(M)\big) = \big[\gamma_1\big] \in H^2(M,\mathbb{C})$. Compare to [**31**], p. 298. For any pseudo-Einstein contact form η

$$W^\alpha{}_{\alpha\lambda} = -\frac{i}{2n}\,R_\lambda\,, \quad W^\alpha{}_{\alpha\overline{\mu}} = \frac{i}{2n}\,R_{\overline{\mu}}\,,$$

where $R_\alpha = T_\alpha(R)$. Hence (for $n = 1$)

$$-2\,\pi\,i\,\gamma_1 = R\,g_{1\overline{1}}\,\theta^1 \wedge \theta^{\overline{1}} - \frac{i}{2}\,R_1\,\theta^1 \wedge \eta - \frac{i}{2}\,R_{\overline{1}}\,\theta^{\overline{1}} \wedge \eta$$

$$= R\,g_{1\overline{1}}\,\theta^1 \wedge \theta^{\overline{1}} - \frac{i}{2}\,(dR) \wedge \eta = \frac{1}{2i}\,R\,d\eta + \frac{1}{2i}\,(dR) \wedge \eta$$

so that $\gamma_1 = (1/4\pi)\,d\big(R\eta\big)$. This is exact, hence $c_1\big(T_{1,0}(M)\big) = 0$. Finally, a complex line bundle over an oriented manifold is trivial if and only if its first Chern class vanishes. $\square$

2.6 - *CR geometry* versus *Lorentzian geometry*

2.6.1 - `Radiative gravitational fields`

Let $(M,\, T_{1,0}(M))$ be an orientable nondegenerate 3-dimensional CR manifold (of CR dimension $n = 1$). Let η be a pseudohermitian structure on M. Let $\xi \in \mathfrak{X}(M)$ be the Reeb vector of (M,η). By Lemma 2.1 the complex line bundle $T_{1,0}(M)$ is trivial [i.e. $T_{1,0}(M) \approx M \times \mathbb{C}$, a complex vector bundle isomorphism]. Given a (globally defined) cross section $\sigma : M \to T_{1,0}(M)$ we set as customary $T_{1,x} = \sigma(x)$ for any $x \in M$. Let θ^1 be the complex 1-form on M determined by

$$\theta^1\big(T_1\big) = 1, \quad \theta^1\big(T_{\overline{1}}\big) = 0, \quad \theta^1\big(\xi\big) = 0,$$

where $T_{\overline{1}} = \overline{T}_1$ i.e. $\{\theta^1\}$ is an adapted coframe. Let $\mathfrak{M} = M \times \mathbb{R}$ and let $\pi : \mathfrak{M} \to M$ be the projection. Let us consider the tangent vector field

$$N = \rho\,\frac{\partial}{\partial t}\,, \quad \rho \in C^\infty\big(\mathfrak{M},\, \mathbb{R} \setminus \{0\}\big), \tag{14}$$

where $t : \mathfrak{M} \to \mathbb{R}$ is the projection. Given a pseudohermitian structure η on M, a C^∞ section $\sigma : M \to T_{1,0}(M)$, and C^∞ functions

$$P : \mathfrak{M} \to \mathbb{R} \setminus \{0\}, \quad H : \mathfrak{M} \to \mathbb{R}, \quad W : \mathfrak{M} \to \mathbb{C},$$

we consider the $(0,2)$-tensor field $g = g\big(\eta,\, \sigma,\, P,\, H,\, W\big)$ on $\mathfrak{M}$ defined by

$$g = 2P^2 \Big\{ \big(\pi^*\theta^1\big) \odot \big(\pi^*\theta^{\overline{1}}\big) + \big(\pi^*\eta\big) \odot \left(\frac{1}{\rho}\, dt + W\, \pi^*\theta^1 + \overline{W}\, \pi^*\theta^{\overline{1}} + H\, \pi^*\eta \right) \Big\} \tag{15}$$

where $\theta^{\overline{1}} = \overline{\theta^1}$. Compare to (1.5) in [**42**], p. 3135. As it turns out $g \in \mathrm{Lor}(\mathfrak{M})$ and solutions of the form (15) to (a class of) gravitational field equations are known to describe gravitational radiation, cf. I. Robinson and A. Trautman, [**71**]–[**72**], R. K. Sachs, [**73**]–[**74**], and A. Trautman, [**82**].

To consider the family of Lorentzian metrics (15) we followed the work by C.D. Hill and J. Lewandowski and P. Nurowski (cf. [**42**]). The basic properties of the metrics (15) are also described in [**42**], pp. 3135–3137, yet avoiding the language of flag and optical structures (cf. I. Robinson and A. Trautman, [**71**]). The construction in [**42**] is more general [it includes Levi flat CR manifolds $(M,\, T_{1,0}(M))$] yet only the case of a nondegenerate CR structure is of interest for the present paper. The notations and conventions in Theorem 2.2 below are made precise in Sections § 2-§ 3 where we also review the basic facts about flag and optical geometries, and their adapted Lorentzian metrics.

Given a metric $g \in \mathrm{Lor}(\mathfrak{M})$ and a tangent vector field $k \in \mathfrak{X}(\mathfrak{M})$ let $g(k) = k^\flat \in \Omega^1(\mathfrak{M})$ be the differential 1-form given by

$$k^\flat(X) = g(k, X), \quad X \in \mathfrak{X}(\mathfrak{M}).$$

Theorem 2.2. *Let $K \subset T(\mathfrak{M})$ and $L \subset T^*(\mathfrak{M})$ be the real line subbundles given by*

$$K_x = \mathbb{R}\, N_x\,, \quad L_x = \mathbb{R}\, g(N)_x\,, \quad x \in \mathfrak{M},$$

where $N \in \mathfrak{X}(\mathfrak{M})$ is given by (14) *and $g = g\big(\eta,\, \sigma,\, P,\, H,\, W\big) \in \mathrm{Lor}(\mathfrak{M})$ is the Lorentzian metric associated to the data $\big(\eta,\, \sigma,\, P,\, H,\, W\big)$ as in* (15). *Then*

i) *$(K,\, L)$ is a flag structure on M.*

ii) *$g \in \mathcal{A}$ i.e. g is adapted to $(K,\, L)$.*

Moreover let $B \subset \mathcal{A}$ be the class mod *$\mathcal{R}$ such that $g \in B$ and let us set $E = \mathrm{Ker}(L)/K$. Let $\phi : E \to E$ be the complex structure given by*

$$\phi_x\big(u + K_x\big) = J_{\pi(x)}(d_x\pi)u, \quad u \in \mathrm{Ker}(L)_x\,, \quad x \in \mathfrak{M}. \tag{16}$$

Let $\mathrm{GL}(1,\mathbb{C}) \to \mathcal{O} \to \mathfrak{M}$ *be the orientation of* E *determined by* ϕ*. Then*

iii) $\big((K,\, L),\ B,\ \mathcal{O}\big)$ *is an optical structure on* $\mathfrak{M}$.

iv) *If* $g' = g\big(\eta',\, \sigma',\, P',\, H',\, W'\big)$ *then* $g' \in B$.

Proof. i) Follows from $g(N,N) = 0$.

ii) For arbitrary sections $k \in C^\infty(K)$ and $\lambda \in C^\infty(L)$ there exist functions $\kappa,\, \gamma \in C^\infty(\mathfrak{M},\, \mathbb{R})$ such that $k = \kappa\, N$ and $\lambda = \gamma\, g(N)$. By the very definitions (14)–(15) one has $g(N) = P^2\, \pi^*\eta$. Hence $g(k) \wedge \lambda = 0$ i.e. $g \in \mathcal{A}$. Q.e.d.

iii) Let $\lambda \in C^\infty(L)$ so that [by Lemma 3.1 in § 2]

$$E_x = \mathrm{Ker}(\lambda_x)/K_x\,, \quad x \in \mathfrak{M}.$$

As $\lambda = \gamma\, P^2\, \pi^*\eta$, for each $u \in \mathrm{Ker}(\lambda_x)$ one has $(d_x\pi)u \in \mathrm{Ker}(\eta_x) = H(M)_x$. Hence there is a natural vector bundle epimorphism $E \to H(M)$ so that J [the complex structure on the Levi distribution $H(M)$] lifts to a complex structure ϕ on E [given by (16)].

iv) Let $g' = g\big(\eta',\, \sigma',\, P',\, H',\, W'\big)$ be the Lorentzian metric on $\mathfrak{M}$ determined by the data $\big(\eta',\, \sigma',\, P',\, H',\, W'\big)$. Let us set $T'_{1,x} = \sigma'(x)$ for any $x \in \mathfrak{M}$. As both η and η' are nowhere zero C^∞ sections of the same real line bundle [the conormal bundle $H(M)^\perp \to M$] there is $f \in C^\infty\big(M,\, \mathbb{R}\setminus\{0\}\big)$ such that $\eta' = f\,\eta$. Then the Reeb vector ξ' of $(M,\, \eta')$ is related to ξ by

$$\xi' = \frac{1}{f}\left(\xi + \frac{1}{2if}\, f^1\, T_1 - \frac{1}{2if}\, f^{\bar 1}\, T_{\bar 1}\right), \tag{17}$$

$$f^1 = g^{1\bar 1}\, f_{\bar 1}\,, \quad f_{\bar 1} = T_{\bar 1}(f), \quad g^{1\bar 1} = 1/g_{1\bar 1}\,, \quad g_{1\bar 1} = g_\eta\big(T_1\,,\, T_{\bar 1}\big).$$

Let $\{\theta'^1\}$ be the adapted frame of $T_{1,0}(M)^*$ determined by

$$\theta'^1\big(T'_1\big) = 1, \quad \theta'^1\big(T'_{\bar 1}\big) = 0, \quad \theta'^1\big(\xi'\big) = 0.$$

If $T'_1 = (1/h)\, T_1$ for some $h \in C^\infty\big(M,\, \mathbb{R}\setminus\{0\}\big)$ then [by (17)]

$$\theta'^1 = h\,\theta^1 + p\,\eta, \quad p \equiv \frac{ihf^1}{2f}\,. \tag{18}$$

Next, substitution from (18) into

$$g' = 2\,P'^2 \Big\{ \big(\pi^*\theta'^1\big) \odot \big(\pi^*\theta'^{\bar 1}\big)$$
$$+ \big(\pi^*\eta'\big) \odot \left(\frac{1}{\rho}\, dt + W'\,\pi^*\theta'^1 + \overline{W'}\,\pi^*\theta'^{\bar 1} + H'\,\pi^*\eta'\right)\Big\}$$

leads to

$$g' = \alpha^2 \, g + g(N) \odot \varphi \tag{19}$$

where $\alpha = |h| \, P'/P$ and

$$\begin{aligned}\varphi = {} & \frac{2P'^2}{\rho P^2} \left(f - |h|^2 \right) \, dt \\ & + \frac{2hP'^2}{P^2} \left(\overline{p} + f \, W' - \overline{h} \, W \right) \pi^* \theta^1 + \frac{2\overline{h}P'^2}{P^2} \left(p + f \, \overline{W'} - h \, \overline{W} \right) \pi^* \theta^{\overline{1}} \\ & + \frac{2P'^2}{P^2} \left(|p|^2 + f \, p \, W' + f \, \overline{p} \, \overline{W'} + f^2 \, H' - |h|^2 \, H \right) \pi^* \eta .\end{aligned}$$

As a consequence of (19) for every $\lambda = \gamma \, g(N) \in C^\infty(L)$

$$g' = \alpha^2 \, g + \mu \odot \lambda, \quad \mu \equiv \frac{1}{\gamma} \, \varphi,$$

so that [by Proposition 4.3 in § 2] g' is equivalent to g mod $\mathcal{R}$. Q.e.d. □

2.6.2 - Fefferman type metrics

Let $(M, \, T_{1,0}(M))$ be a strictly pseudoconvex $(2n+1)$-dimensional CR manifold, of CR dimension n. A complex valued differential p-form $\omega \in \Omega^p(M)$ is of *type* $(p, 0)$ if $T_{0,1}(M) \rfloor \omega = 0$. Let $\Lambda^{p,0}(M) \to M$ be the relevant bundle and let us set $K(M) = \Lambda^{n+1,0}(M)$ (the *canonical line bundle*). Sections in $K(M)$ are top degree forms of type $(p, 0)$ i.e. $p = n + 1$. There is a natural action of $\mathbb{R}_+ = \mathrm{GL}^+(1, \mathbb{R})$ (the positive reals) on $K(M) \setminus \{\text{zero section}\}$ so that the quotient space

$$C(M) = \big[K(M) \setminus \{\text{zero section}\} \big] / \mathbb{R}_+$$

is the total space of a principal circle bundle $S^1 \to C(M) \xrightarrow{\pi} M$ (the *canonical circle bundle*). Let η be a positively oriented contact form on M and G_η the corresponding Levi form. If ξ is the Reeb vector of (M, η) then we extend G_η to a (degenerate) bilinear form $\tilde{G}_\eta$ defined on the whole of $T(M)$ by requiring that

$$\tilde{G}_\eta(X, Y) = G_\eta(X, Y), \quad \tilde{G}_\eta(V, \xi) = 0,$$

for any $X, Y \in H(M)$ and $V \in T(M)$. Next let F_η be the $(0, 2)$-tensor field on $\mathfrak{M} = C(M)$ defined by

$$F_\eta = \pi^* \tilde{G}_\eta + 2 \left(\pi^* \eta \right) \odot \sigma, \tag{20}$$

$$\sigma = \frac{1}{n+2}\left\{ d\mathbf{s} + \pi^* \left(i\,\omega_\mu{}^\mu - \frac{i}{2}\, g^{\mu\overline{\nu}}\, dg_{\mu\overline{\nu}} - \frac{R}{4(n+1)}\,\eta \right)\right\}, \tag{21}$$

where $\mathbf{s}$ is a local fibre coordinate on $\mathfrak{M}$. Then $F_\eta \in \mathrm{Lor}(\mathfrak{M})$ [F_η is the *Fefferman metric* of (M,η)] (cf. [**31**], p. 128) whose *restricted conformal class* $\{e^{u\circ\pi}\, F_\eta : u \in C^\infty(M,\mathbb{R})\}$ is a CR invariant. By a result of C.R. Graham (cf. [**40**]) σ is a connection 1-form in the principal bundle $S^1 \to \mathfrak{M} \to M$. Let $X^\uparrow \in \mathfrak{X}(\mathfrak{M})$ be the horizontal lift of $X \in \mathfrak{X}(M)$ with respect to σ i.e.

$$X^\uparrow_p \in \mathrm{Ker}\big(\sigma_p\big), \quad (d_p\pi)X^\uparrow_p = X_{\pi(p)}\,, \quad p \in \mathfrak{M}.$$

Let $N \in \mathfrak{X}(\mathfrak{M})$ be the tangent to the S^1 action. The vector field $\xi^\uparrow - N$ is time-like, hence the Lorentzian manifold $(\mathfrak{M},\, F_\eta)$ is time-oriented. In particular, if M is 3-dimensional then $\big(\mathfrak{M},\, F_\eta\,,\, \xi^\uparrow - N\big)$ is a space-time. The Heisenberg group $M = \mathbb{H}_1 = \mathbb{C}\times\mathbb{R}$ is customarily endowed with the pseudohermitian structure

$$\eta_0 = dt + i\big(z\, d\overline{z} - \overline{z}\, dz\big)$$

whose corresponding Reeb vector is $\xi_0 = \partial/\partial t$. Let us set

$$\overline{L} = \frac{\partial}{\partial \overline{z}} - i\,z\,\frac{\partial}{\partial t}$$

[the (unsolvable) *Lewy operator*] so that $T_1 = L$ is a (globally defined) frame in $T_{1,0}(\mathbb{H}_1)$ and $\theta^1 = dz$ is an adapted coframe. If $g_{1\overline{1}} = G_{\eta_0}\big(T_1\,,\, T_{\overline{1}}\big)$ then $g_{1\overline{1}} = 1$ hence the 1-form $\sigma = \sigma_0$ given by (21) reads

$$\sigma_0 = \frac{1}{3}\, d\mathbf{s}.$$

Assume the local fibre coordinate $\mathbf{s}$ is chosen such that $N = (3/2)\,\partial/\partial\mathbf{s}$. Hence $\sigma_0(N) = 1/2$ and (20) for $\eta = \eta_0$ becomes

$$F_{\theta_0} = 2\left\{ \big(\pi^*\theta^1\big) \odot \big(\pi^*\theta^{\overline{1}}\big) + \frac{1}{3}\,\big(\pi^*\eta_0\big) \odot d\mathbf{s} \right\}. \tag{22}$$

As $\mathbb{H}_1 \approx \partial\mathcal{S}_2$ (a CR isomorphism, cf. Section § 2.2) the canonical circle bundle is trivial i.e. $C(\mathbb{H}_1) \approx \mathbb{H}_1 \times S^1$ (a principal bundle isomoprhism). Since

$$C(\mathbb{H}_1)_x = \big\{ \big[\lambda\big(\eta_0 \wedge \theta^1\big)_x\big] \,:\, \lambda \in \mathbb{C}\setminus\{0\}\big\}, \quad x \in \mathbb{H}_1\,,$$

an explicit principal bundle isomorphism $F : C(\mathbb{H}_1) \to \mathbb{H}_1 \times S^1$ is given by

$$F\left(\left[\lambda\left(\eta_0 \wedge \theta^1\right)_x\right]\right) = \left(x,\ \frac{\lambda}{|\lambda|}\right).$$

For every $\gamma_0 \in \mathbb{R}$ let us set $U(\gamma_0) = \{e^{i\gamma} : |\gamma - \gamma_0| < \pi\}$ and let us consider the parametrization $\psi_{\gamma_0} : (\gamma_0 - \pi,\ \gamma_0 + \pi) \to S^1$ given by $\psi(\gamma) = e^{i\gamma}$. Next let us cover S^1 by the local charts $\varphi_{\gamma_0} = \psi_{\gamma_0}^{-1} : U(\gamma_0) \to (\gamma_0 - \pi,\ \gamma_0 + \pi)$. The local fibre coordinate in (21) is

$$\mathbf{s} : F^{-1}\big(\mathbb{H}_1 \times U(\gamma_0)\big) \to \mathbb{R}, \quad \mathbf{s}\big([\lambda(\eta_0 \wedge \theta^1)_x]\big) = \varphi_{\gamma_0}\left(\frac{\lambda}{|\lambda|}\right).$$

Next let us consider the map

$$f = F^{-1} \circ \big(1_{\mathbb{H}_1} \times \psi_{\gamma_0}\big) : \mathbb{H}_1 \times \big(\gamma_0 - \pi,\ \gamma_0 + \pi\big) \to C(\mathbb{H}_1)$$

(a C^∞ diffeomorphism on the image). As a consequence of (22) the Lorentzian metric $g = f^* F_{\eta_0}$ belongs to the Robinson-Trautmann class (15).

By a result of C. R. Graham (cf. [**40**]) the Fefferman metric F_η of an arbitrary strictly pseudoconvex CR manifold M satisfies

i) N is null i.e. $F_\eta(N, N) = 0$ and $N_p \neq 0$ for any $p \in C(M)$.

ii) $\mathcal{L}_N F_\eta = 0$.

iii) $\mathrm{Ric}_{F_\eta}(N, N) = 2n$.

iv) $N \rfloor W = 0$ and $N \rfloor C = 0$.

Compare to Theorem 2.1 in [**40**], p. 856. Here W and C are respectively the Weyl and Cotton tensor fields of $(C(M),\ F_\eta)$ [cf. our § 5.1 below]. By a *Fefferman type metric* we mean a Lorentzian metric g on some $(2n+2)$-dimensional manifold $\mathfrak{M}$ which may be organized as the total space of a circle bundle $S^1 \to \mathfrak{M} \to M$ over a strictly pseudoconvex CR manifold such that $(\mathfrak{M},\ g)$ is conformally isometric to $\big(C(M),\ F_\eta\big)$. By an unpublished result of G. Sparling (cf. [**77**]) the properties (i)-(iv) provide a local characterization of Fefferman type metrics, and a globally defined conformal isometry may be produced provided a certain cohomology class in $H^1(M, S^1)$ (discovered by C. R. Graham, cf. [**40**], p. 872) vanishes.

3 - Flag structures and adapted Lorentzian metrics

Let $\mathfrak{M}$ be a 4-dimensional C^∞ manifold. A pair $(K,\ L)$ of real line subbundles $K \subset T(\mathfrak{M})$ and $L \subset T^*(\mathfrak{M})$ is a *flag structure* on $\mathfrak{M}$ if $\alpha(u) = 0$ for any $\alpha \in L_x$ and $u \in K_x$ and $x \in \mathfrak{M}$. Let $\mathrm{Lor}(\mathfrak{M})$ be the set of all Lorentzian metrics on $\mathfrak{M}$. A Lorentzian metric $g \in \mathrm{Lor}(\mathfrak{M})$ is *adapted* to the flag structure $(K,\ L)$ if $g(N) \wedge \lambda = 0$ for any $N \in C^\infty(K)$ and $\lambda \in C^\infty(L)$. We shall need

Lemma 3.1. *Let* (K, L) *be a flag structure and* $g \in \mathrm{Lor}(\mathfrak{M})$ *an adapted metric. Then for every* $x \in \mathfrak{M}$ *the line* K_x *is null i.e.* $g_x(u,u) = 0$ *for any* $u \in K_x$*. Moreover for any* $\lambda \in C^\infty(L)$

$$\mathrm{Ker}\big[\lambda(x)\big] = K_x^\perp \tag{23}$$

where $K_x^\perp = \big\{v \in T_x(\mathfrak{M}) : g_x(u,v) = 0, \quad u \in K_x\big\}$*. In particular the bundle* $\bigcup_{x\in\mathfrak{M}} \mathrm{Ker}\big[\lambda(x)\big]$ *doesn't depend upon the choice of* $\lambda \in C^\infty(L)$*.*

Cf. I. Robinson and A. Trautman, [**71**], p. 318. A differential p-form $F \in \Omega^p(\mathfrak{M})$ with $p \in \{1,2,3\}$ is *adapted* to the flag structure $(K,\ L)$ if

$$N \rfloor F = 0, \quad \lambda \wedge F = 0,$$

for any $N \in C^\infty(K)$ and $\lambda \in C^\infty(L)$. If g is an adapted metric then $g(N)$ is an adapted 1-form. If $(K,\ L)$ is a flag structure then L is *invariant* with respect to K if $\lambda \wedge \mathcal{L}_N \lambda = 0$ for any $N \in C^\infty(K)$ and $\lambda \in C^\infty(L)$, where $\mathcal{L}_N$ denotes the Lie derivative. The invariance property is described by

Theorem 3.1. *Let* $(K,\ L)$ *be a flag structure on* $\mathfrak{M}$*. The following statements are equivalent*

i) L *is invariant with respect to* K*.*

ii) $\lambda \wedge d\lambda$ *is adapted to* $(K,\ L)$ *for every* $\lambda \in C^\infty(L)$*.*

iii) *For every* $N \in C^\infty(K)$ *and any* $x \in \mathfrak{M}$ *the curves* $C_x(t) = \varphi_t^N(x)$ *are null geodesics with respect to any metric* $g \in \mathrm{Lor}(\mathfrak{M})$ *adapted to* $(K,\ L)$*.*

iv) *If* $F \in \Omega^2(\mathfrak{M})$ *is adapted then* $\lambda \wedge F = 0$ *for any* $\lambda \in C^\infty(L)$*.*

Cf. I. Robinson and A. Trautman, [**70**]. Here $\{\varphi_t^N\}_{|t|<\epsilon}$ is the local 1-parameter group of local transformations generated by $N \in \mathfrak{X}(\mathfrak{M})$. A flag structure $(K,\ L)$ possessing one of the equivalent properties (i)-(iv) is referred to as a *geodesic flag structure*. By Lemma 3.1 the notation $\mathrm{Ker}(L) = \bigcup_{x\in\mathfrak{M}} \mathrm{Ker}\big[\lambda(x)\big]$ [with $\lambda \in C^\infty(\mathfrak{M})$] is legitimate. The distribution $\mathrm{Ker}(L)$ is *integrable* if $\lambda \wedge d\lambda = 0$ for any $\lambda \in C^\infty(L)$.

Proposition 3.1. *If* $\mathrm{Ker}(L)$ *is integrable then* $(K,\ L)$ *is geodesic.*

Cf. I. Robinson and A. Trautman, [**71**], p. 319. According to the terminology adopted in theoretical physics, given a geodesic flag structure $(K,\ L)$, an adapted metric $g \in \mathrm{Lor}(\mathfrak{M})$, and a vector field $N \in C^\infty(K)$, the family of orbits of N [consisting of null geodesics of g, by (iii) in Theorem 3.1] is referred to as a *congruence of null geodesics* on $\mathfrak{M}$. The congruence in question is said to be *twisting* if $\mathrm{Ker}(L)$ is not integrable.

4 - Optical structures

Let $(K,\ L)$ be a flag structure and let $\mathcal{A} \subset \mathrm{Lor}(\mathfrak{M})$ be the set of all Lorentzian metrics adapted to $(K,\ L)$. Given $g \in \mathrm{Lor}(\mathfrak{M})$ let $*_g : \Omega^p(\mathfrak{M}) \to \Omega^{4-p}(\mathfrak{M})$ be the corresponding Hodge operator.

Proposition 4.1. *If $g \in \mathcal{A}$ and $F \in \Omega^p(\mathfrak{M})$ is an adapted p-form ($p \in \{1,2,3\}$) then $*_g F \in \Omega^{4-p}(\mathfrak{M})$ is an adapted $(4-p)$-form.*

Cf. I. Robinson and A. Trautman, [**71**], p. 319. Let $F \in \Omega^2(\mathfrak{M})$ be an adapted 2-form such that $F_x \neq 0$ for any $x \in \mathfrak{M}$. Let us consider the relation $\mathcal{R} \subset \mathcal{A} \times \mathcal{A}$ given by

$$\mathcal{R} = \big\{ \big(g,\ g'\big) \in \mathcal{A} \times \mathcal{A} \ :\ *_g F = *_{g'} F \big\}.$$

Then $\mathcal{R}$ is an equivalence relation on $\mathcal{A}$. The definition of $\mathcal{R}$ doesn't depend upon the choice of (nowhere vanishing, adapted) 2-from F, as a consequence of $\dim(\mathfrak{M}) = 4$. An *optical structure* is a synthetic object

$$\big\{(K,\ L),\ \ B,\ \ \mathcal{O}\big\}$$

consisting of a flag structure $(K,\ L)$, an equivalence class $B \in \mathcal{A}/\mathcal{R}$, and an orientation $\mathcal{O}$ of the real rank 2 vector bundle $E = \mathrm{Ker}(L)/K \to \mathfrak{M}$. The given orientation of $E = \mathrm{Ker}(L)/K \to \mathfrak{M}$ is a principal subbundle $G \to \mathcal{O} \to \mathfrak{M}$ of the principal bundle $\mathrm{GL}(2,\mathbb{R}) \to L\big(E\big) \to \mathfrak{M}$ of all frames ($\mathbb{R}$-linear isomorphisms) $\mathbf{u} : \mathbb{R}^2 \to E_x$ with $x \in \mathfrak{M}$, where $G \subset \mathrm{GL}^+(2,\mathbb{R})$ (a Lie subgroup).

Proposition 4.2. *The data $(B,\ \mathcal{O})$ is equivalent to the prescription of a complex structure on $\mathrm{Ker}(L)/K$ i.e. an endomorphism*

$$J : E \to E, \quad J^2 = -I.$$

We take into account

Lemma 4.1. *Let $g \in B$ and let $\hat{g}$ be given by*

$$\hat{g}_x\big(\hat{u}\,,\,\hat{v}\big) = g_x(u,v),$$

$$\hat{u} = u + K_x\,, \quad \hat{v} = v + K_x\,, \quad u,\ v \in \mathrm{Ker}(L)_x\,.$$

Then $\hat{g}$ is a Riemannian bundle metric on $E = \mathrm{Ker}(L)/K \to \mathfrak{M}$.

By Lemma 4.1 the metric $\hat{g}$ gives rise to a principal subbundle $\mathrm{O}(2) \to O(E, \hat{g}) \to \mathfrak{M}$ of $\mathrm{GL}(2, \mathbb{R}) \to L(E) \to \mathfrak{M}$ i.e. each $\mathbf{u} \in O(E, \hat{g})_x$ is a frame $\mathbf{u} \in L(E)_x$ such that

$$\hat{g}_x\big(\mathbf{u}(e_j)\,,\,\mathbf{u}(e_k)\big) = \delta_{jk}\,, \quad 1 \le j, k \le 2,$$

where $\{e_1\,,\,e_2\} \subset \mathbb{R}^2$ is the canonical linear basis. We may then outline the construction of J in Proposition 4.2 as follows. For every $x \in \mathfrak{M}$ and $\mathbf{u} \in \mathcal{O}_x \cap O\big(E, \hat{g}\big)_x$ we consider

$$J_x : E_x \to E_x\,, \tag{24}$$

$$J_x\big(\mathbf{u}(e_1)\big) = \mathbf{u}(e_2), \quad J_x\big(\mathbf{u}(e_2)\big) = -\mathbf{u}(e_1). \tag{25}$$

To build J_x one only uses frames at x adapted to $\mathrm{GL}^+(2, \mathbb{R}) \cap \mathrm{O}(2) \to \mathcal{O} \cap O(E, \hat{g}) \to \mathfrak{M}$ so that J_x is well defined. Indeed let $J_x^{\mathbf{u}}$ be a new temporary name for the map (24)–(25). If $\mathbf{v} \in \mathcal{O}_x \cap O\big(E, \hat{g}\big)_x$ is another frame then $\mathbf{v} = \mathbf{u}a$ for some $a \in \mathrm{GL}^+(2, \mathbb{R}) \cap \mathrm{O}(2)$ i.e.

$$\mathbf{v}(e_k) = a^j_k\,\mathbf{u}(e_j), \quad a = \big[a^j_k\big], \quad a^{-1} = a^{\mathrm{T}}\,, \quad \det(a) = 1.$$

Then

$$J_x^{\mathbf{u}}\big(\mathbf{v}(e_k)\big) = a^j_k\,J_x^{\mathbf{u}}\big(\mathbf{u}(e_j)\big) = a^1_k\,\mathbf{u}(e_2) - a^2_k\,\mathbf{u}(e_1)$$

$$= \big[a^1_k\,(a^{-1})^j_2 - a^2_k\,(a^{-1})^j_1\big]\,\mathbf{v}(e_j) = \sum_{j=1}^{2}\big[a^1_k\,a^2_j - a^2_k\,a^1_j\big]\,\mathbf{v}(e_j)$$

and

$$a^1_1\,a^2_j - a^2_1\,a^1_j = \begin{cases} 0 & \text{if } j = 1, \\ \det(a) & \text{if } j = 2, \end{cases} \quad a^1_2\,a^2_j - a^2_2\,a^1_j = \begin{cases} -\det(a) & \text{if } j = 1, \\ 0 & \text{if } j = 2, \end{cases}$$

hence $J_x^{\mathbf{u}} = J_x^{\mathbf{v}}$. Q.e.d.

Let $\big((K,\,L),\,B,\,\mathcal{O}\big)$ be an optical structure on $\mathfrak{M}$. By Proposition 4.2 the vector bundle $E = \mathrm{Ker}(L)/K$ may be organized as a complex line bundle over $\mathfrak{M}$, in a natural manner.

Proposition 4.3. *Let $\big((K,\,L),\,B,\,\mathcal{O}\big)$ be an optical structure on $\mathfrak{M}$ and let $g \in B$ and $g' \in \mathcal{A}$. The following statements are equivalent*

i) $g' \in B$.

ii) *For any $\lambda \in C^\infty(L)$ there exist a C^∞ function $\rho : \mathfrak{M} \to (0, +\infty)$ and a differential 1-form $\mu \in \Omega^1(\mathfrak{M})$ such that*

$$g' = \rho\,g + 2\,\mu \odot \lambda. \tag{26}$$

Cf. [**71**], pp. 319–320. Let $\mathfrak{M}$ and $\mathfrak{M}'$ be two 4-dimensional manifolds equipped with the optical structures $\big((K,\ L),\ B,\ \mathcal{O}\big)$ and $\big((K',\ L'),\ B',\ \mathcal{O}'\big)$. A C^∞ diffeomorphism $f : \mathfrak{M} \to \mathfrak{M}'$ is an *isomorphism* of optical structures if

$$f^* B' = B, \quad f^* L' = L, \quad (d_x f) K_x = K'_{f(x)}\,, \quad x \in \mathfrak{M},$$

and the vector bundle morphism $f_* : K \to f^{-1} K'$ descends to an orientation preserving vector bundle morphism

$$\hat{f}_* : E \to f^{-1} E'\,, \quad E = \mathrm{Ker}(L)/K, \quad E' = \mathrm{Ker}(L')/K'\,.$$

Orientations $\mathcal{O}$ and $\mathcal{O}'$ are principal subbundles

$$\begin{array}{ccccc}
\mathrm{GL}^+(2,\mathbb{R}) \to & \mathcal{O} & \hookrightarrow & L(E) & \leftarrow \mathrm{GL}(2,\mathbb{R}) \\
 & \downarrow & & \downarrow & \\
 & \mathfrak{M} & & \mathfrak{M} & \\
 & f\downarrow & & \downarrow f & \\
 & \mathfrak{M}' & & \mathfrak{M}' & \\
 & \uparrow & & \uparrow & \\
\mathrm{GL}^+(2,\mathbb{R}) \to & \mathcal{O}' & \hookrightarrow & L(E') & \leftarrow \mathrm{GL}(2,\mathbb{R})
\end{array}$$

The pair $\big(\hat{f}_*\,,\ f\big)$ induces a principal bundle morphism

$$L(f) : L(E) \to f^{-1} L(E'), \quad L(f)_x\, \mathbf{u} = \mathbf{u}' \in L(E')_{f(x)}\,, \quad x \in \mathfrak{M},$$

$$\mathbf{u} : \mathbb{R}^2 \to E_x\,, \quad \mathbf{u}' : \mathbb{R}^2 \to E'_{f(x)}\,, \quad \mathbf{u}' = \big(\hat{f}_*\big)_x \circ \mathbf{u}\,.$$

Then the "orientation preserving" condition above is $L(f)\big(\mathcal{O}\big) = f^{-1}\mathcal{O}'$.

Given a 4-dimensional manifold $\mathfrak{M}$ it is customary to define an optical structure on $\mathfrak{M}$ by fixing a Lorentzian metric $g \in \mathrm{Lor}(\mathfrak{M})$ and a null vector field $N \in \mathfrak{X}(\mathfrak{M})$ and setting

$$K_x = \mathbb{R} N_x\,, \quad L_x = \mathbb{R}\, N_x^\flat\,, \quad x \in \mathfrak{M},$$

where $\flat : \mathfrak{X}(\mathfrak{M}) \to \Omega^1(\mathfrak{M})$ is the musical isomorphism associated to g. Then specifying an orientation of $\mathrm{Ker}(L)/K$ gives an optical structure on $\mathfrak{M}$.

Let $\big((K,\ L),\ B,\ \mathcal{O}\big)$ be an optical structure on $\mathfrak{M}$ and let $g \in B$ be an adapted Lorentzian metric. Let us consider *Maxwell's equations*

$$dF = 0, \quad d *_g F = 0, \tag{27}$$

where solutions $F \in \Omega^2(\mathfrak{M})$ are to be looked for among 2-forms F adapted to $(K,\ L)$.

Proposition 4.4. *Let F be a solution to Maxwell's equations* (27). *Then*

$$\mathcal{L}_N F = 0, \quad \mathcal{L}_N *_g F = 0, \tag{28}$$

for any $N \in C^\infty(K)$.

Cf. [**71**], p. 320. In particular both F and $*_g F$ are invariant by the flow of N i.e.

$$\left(\varphi_t^N\right)^* F = F, \quad \left(\varphi_t^N\right)^* \left(*_g F\right) = *_g F.$$

By the second of these relations, together with the properties of the Hodge star

$$*_g F = \left(\varphi_t^N\right)^* \left(*_g F\right) = *_{\left(\varphi_t^N\right)^* g} F\,.$$

Provided that F is a nowhere vanishing solution, this means that the metrics g and $\left(\varphi_t^N\right)^* g$ are equivalent (mod $\mathcal{R}$). Hence each φ_t^N is an optical automorphism. In particular L is invariant with respect to K hence the flag structure $(K,\ L)$ is geodesic.

Proposition 4.5. *Let $\big((K,\ L),\ B,\ \mathcal{O}\big)$ be an optical structure and let $N \in C^\infty(K)$ and $g \in B$. The following statements are equivalent*

i) $\left(g\,,\ \left(\varphi_t^N\right)^* g\right) \in \mathcal{R}$ *for every t.*

ii) *For every $\lambda \in C^\infty(L)$ there exist a C^∞ function $\sigma : \mathfrak{M} \to \mathbb{R}$ and a* 1*-form $\nu \in \Omega^1(\mathfrak{M})$ such that*

$$\mathcal{L}_N g = \sigma\, g + 2\,\nu \odot \lambda\,. \tag{29}$$

Cf. [**71**], p. 320. An optical structure satisfying one of the equivalent requirements (i)–(ii) in Proposition 4.5 is said to be *shear-free*. We close the section by recalling a result by H. Bateman, [**15**] (emphasising the importance of optical structures)

Theorem 4.1. *An optical isomorphism maps any adapted solution F to Maxwell's equations into another adapted solution.*

The modern formulation of Bateman's theorem (within optical geometry) is due to A. Trautman, [**81**].

5 - CR structures associated to shear-free optical structures

5.1 - *Curvature properties*

Let $\mathfrak{M}$ be a real m-dimensional C^∞ manifold and let $g \in \mathrm{Lor}(\mathfrak{M})$ be a Lorentzian manifold. Let (U, x^i) be a local coordinate system on $\mathfrak{M}$ and let g_{ij} be the local components of g with respect to (U, x^i). Let ∇ be the Levi-Civita connection of $(\mathfrak{M}, g)$ and let R^∇ be the curvature tensor field of ∇ i.e.

$$R^\nabla(X,Y) = \big[\nabla_X\,,\,\nabla_Y\big] - \nabla_{[X,Y]}\,.$$

We adopt the following convention as to the local components of R^∇

$$R_k{}^\ell{}_{ij}\,\partial_\ell = R^\nabla\,(\partial_i\,,\,\partial_j)\,\partial_k \tag{30}$$

where ∂_i is short for $\partial/\partial x^i$. Given a tangent vector field $V \in \mathfrak{X}(\mathfrak{M})$ let $\nabla_i V^j$ be the local components of the covariant derivative ∇V i.e.

$$\nabla_{\partial_i} V = \big(\nabla_i V^j\big)\,\partial_j\,.$$

Given a $(1,1)$-tensor field J one denotes by J_i^j and $\nabla_i J_j{}^k$ respectively the local components of J and ∇J i.e.

$$J\partial_i = J_i^j\,\partial_j\,,\quad (\nabla_{\partial_i} J)\partial_j = \big(\nabla_i J_j{}^k\big)\,\partial_k\,.$$

The second order covariant derivative $\nabla\nabla V$ of a tangent vector field V is the (first order) covariant derivative of the $(1,1)$-tensor field $J = \nabla V$ [whose local components are $J_i^j = \nabla_i V^j$]. Its local components are denoted by $\nabla_i \nabla_j V^k$ i.e.

$$\big(\nabla_i\nabla_j V^k\big)\partial_k = \big(\nabla_{\partial_i}\nabla V\big)\partial_j\,.$$

The convention (30) was chosen such that

$$\nabla_i\nabla_j V^k - \nabla_j\nabla_i V^k = V^\ell\,R_\ell{}^k{}_{ij}\,.$$

The Ricci curvature of ∇ is

$$\mathrm{Ric}_\nabla\big(X\,,\,Y\big) = \mathrm{trace}\,\big\{V \mapsto R^\nabla\big(V,\,Y\big)X\big\}$$

and we set $R_{ij} = \mathrm{Ric}_\nabla(\partial_i\,,\,\partial_j)$ so that

$$R_{ij} = R_i{}^k{}_{kj} = g^{k\ell}\,R_{i\ell kj}\,.$$

The scalar curvature is $R = g^{ij}R_{ij}$. We shall need the *Weyl* and *Cotton tensor* fields

$$W_{ijk\ell} = R_{ijk\ell} - L_{jk}\,g_{i\ell} - L_{i\ell}\,g_{jk} + L_{j\ell}\,g_{ik} + L_{ik}\,g_{j\ell}\,,$$

$$C_{jk\ell} = \nabla_\ell L_{jk} - \nabla_k L_{j\ell}\,,$$

where

$$L_{jk} = \frac{1}{m-2}\left[R_{jk} - \frac{R}{2(m-1)}\, g_{jk}\right].$$

We shall need the following

Lemma 5.1 (C. R. Graham, [**40**]). *Let* $(\mathfrak{M}, g)$ *be a* $(2n+2)$*-dimensional Lorentzian manifold admiting a tangent vector field* $N \in \mathfrak{X}(\mathfrak{M})$ *which is both null (i.e.* $g(N,N) = 0$*) and Killing (i.e.* $\mathcal{L}_N g = 0$*). If* $N \rfloor W = 0$ *and* $N \rfloor C = 0$ *then* $\mathrm{Ric}_\nabla(N,N)$ *is a constant.*

Throughout it is tacitly assumed that $N_x \neq 0$ for any $x \in \mathfrak{M}$. Indeed, by a result of M. Sáncez (cf. [**75**]) if $K \in \mathfrak{X}(\mathfrak{M})$ is a non-spacelike [i.e. $g(K,K) \leq 0$ everywhere on $\mathfrak{M}$] Killing vector field with $K_{x_0} = 0$ for some $x_0 \in \mathfrak{M}$ then $K = 0$ everywhere. It should be also observed that the assumptions in Lemma 5.1 are conformally invariant.

We give a proof of Lemma 5.1 by following the calculations in [**40**], pp. 860–862. Let n^i be the local components of N with respect to (U, x^i) i.e. $N = n^i\,\partial_i$. Here $m = 2n+2$ hence the tensor field L_{jk} is given by

$$L_{jk} = \frac{1}{2n}\left[R_{jk} - \frac{R}{2(2n+1)}\, g_{jk}\right].$$

Let us contract with $n^j n^k$ and use $g_{jk} n^j n^k = 0$. We obtain

$$L_{jk} n^j n^k = \frac{1}{2n} R_{jk} n^j n^k$$

or

$$D = 2n\, L_{jk} n^j n^k \tag{31}$$

where we have set $D = \mathrm{Ric}_\nabla(N,N) \in C^\infty(\mathfrak{M})$. We also adopt the notation

$$\nu_i = L_{ij} n^j$$

so that equation (31) becomes

$$D = 2n\, \nu_k n^k\,. \tag{32}$$

We claim that

$$\nabla_j n_i = -\nabla_i n_j \tag{33}$$

where $\nabla_i n_j = g_{jk} \nabla_i n^k$. Equation (33) is the local expression of the Killing condition $\mathcal{L}_N g = 0$. Indeed (by $\nabla g = 0$)

$$\begin{aligned} 0 = \big(\mathcal{L}_N g\big)(X, Y) &= N(g(X, Y)) - g(\mathcal{L}_N X, Y) - g(X, \mathcal{L}_N Y) \\ &= g(\nabla_N X, Y) + g(X, \nabla_N Y) - g([N, X], Y) - g(X, [N, Y]) \end{aligned}$$

or (by $T_\nabla = 0$)

$$g(\nabla_X N, Y) + g(X, \nabla_Y N) = 0. \tag{34}$$

Next (34) for $X = \partial_i$ and $Y = \partial_j$ gives

$$0 = g(\nabla_{\partial_i} N, \partial_j) + g(\partial_i, \nabla_{\partial_j} N) = \big(\nabla_i n^k\big)\, g_{kj} + g_{ik}\, \big(\nabla_j n^k\big)$$

yielding (33). This proves the claim. Starting from $g(N, N) = 0$ one has (by $\nabla_X g = 0$)

$$0 = X(g(N, N)) = 2g(\nabla_X N, N)$$

yielding (for $X = \partial_i$)

$$0 = g\big(\nabla_{\partial_i} N, N\big) = \big(\nabla_i n^j\big)\, n^k\, g_{jk}$$

or

$$n^k\, \nabla_i n_k = 0. \tag{35}$$

Let us contract with n^j in $\nabla_i n_j + \nabla_j n_i = 0$ and use (35). We obtain

$$n^j\, \nabla_j n_i = 0. \tag{36}$$

The following property of Killing vector fields will be needed in the sequel

$$\nabla_k \nabla_j\, n_i = -R_{ijk\ell}\, n^\ell\, . \tag{37}$$

Cf. [**40**], p. 861. C. R. Graham attributes (37) to [**47**] (cf. Proposition 2.6, p. 235). However i) the quoted result in [**47**] is stated solely for Riemannian metrics and ii) (37) is but a corollary of Proposition 2.6 (in [**47**], p. 235) i.e. (37) is not explicitly reported there. Nevertheless the generalization (from Riemannian to Lorentzian geometry) of the result in [**47**] [and the proof of (37) as a corollary to that] is straightforward. Next let us contract with n^ℓ in

$$W_{ijk\ell} = R_{ijk\ell} - \big(L_{jk}\, g_{i\ell} + L_{i\ell}\, g_{jk} - L_{j\ell}\, g_{ik} - L_{ik}\, g_{j\ell}\big)$$

and take into account that $W_{ijk\ell}\, n^\ell = 0$ (by $N \rfloor W = 0$). We obtain

$$0 = R_{ijk\ell}\, n^\ell - L_{jk}\, n_i - L_{i\ell}\, n^\ell\, g_{jk} + L_{j\ell}\, n^\ell\, g_{ik} + L_{ik}\, n_j \tag{38}$$

where $n_i = g_{i\ell}\, n^\ell$. Let us substitute from (37) into (38) so that to obtain

$$\nabla_k \nabla_j n_i = -L_{jk}\, n_i - \nu_i\, g_{jk} + \nu_j\, g_{ik} + L_{ik}\, n_j\, . \tag{39}$$

Compare to (3.2) in [**40**], p. 861. Moreover we shall need

Lemma 5.2. (C. R. Graham, [**40**]) *The following identities hold*

$$(\nabla_j n_i)\,(\nabla^k n^j) = -n^j\,\nabla^k\nabla_j n_i = -\delta_i^k\,n^j\,\nu_j + \nu^k\,n_i + \nu_i\,n^k\,. \tag{40}$$

Compare to (3.3) in [**40**], p. 861. The first identity in (40) follows by taking the covariant derivative of the product $n^j\,\nabla_j n_i$ and making use of (36) i.e.

$$(\nabla_j n_i)\,(\nabla^k n^j) = g^{k\ell}\,(\nabla_j n_i)\,(\nabla_\ell n^j)$$

$$= g^{k\ell}\left[\nabla_\ell\left(n^j\,\nabla_j n_i\right) - n^j\,\nabla_\ell\nabla_j n_i\right] = -n^j\,\nabla^k\nabla_j n_i\,.$$

To prove the second identity in (40) one conducts the following calculation

$$n^j\,\nabla^k\nabla_j n_i = g^{k\ell}\,n^j\,\nabla_\ell\nabla_j n_i =$$

[by (39) with $k=\ell$]

$$= g^{k\ell}\,n^j\,\left[-L_{j\ell}\,n_i - \nu_i\,g_{j\ell} + \nu_j\,g_{i\ell} + L_{i\ell}\,n_j\right] =$$

[as $n^j n_j = 0$]

$$= g^{k\ell}\,\left[-\nu_\ell\,n_i - \nu_i\,n_\ell + n^j\,\nu_j\,g_{i\ell}\right]$$

or

$$n^j\,\nabla^k\nabla_j n_i = -\nu^k\,n_i - \nu_i\,n^k + \delta_i^k\,n^j\,\nu_j\,.$$

Q.e.d.

Let us contract i and k in $(\nabla_j n_i)\,(\nabla^k n^j)$ and use (40) in Lemma 5.2. We obtain

$$(\nabla_j n_i)\,(\nabla^i n^j) = -(2n+2)\,n^j\nu_j + \nu^i\,n_i + \nu_i\,n^i$$

or

$$(\nabla_j n_i)\,(\nabla^i n^j) = -2n\,n^j\,\nu_j\,. \tag{41}$$

Let us substitute from (32)–(33) into (41). We obtain

$$D = (\nabla_j n_i)(\nabla^j n^i). \tag{42}$$

Compare to (3.4) in [**40**], p. 861. Next let us differentiate in (42) with respect to x^ℓ. As D is a scalar field $\partial D/\partial x^\ell = \nabla_\ell D$ hence [by (42)]

$$\frac{\partial D}{\partial x^\ell} = \nabla_\ell\left[(\nabla_j n_i)(\nabla^j n^i)\right]$$

$$= (\nabla_\ell\nabla_j n_i)(\nabla^j n^i) + (\nabla_j n_i)(\nabla_\ell\nabla^j n^i)$$

and the last term is

$$\big(\nabla_j n_i\big)\big(\nabla_\ell \nabla^j n^i\big) = g_{ir}\, g_{js}\, \big(\nabla^s n^r\big)\big(\nabla_\ell \nabla^j n^i\big) = \big(\nabla^s n^r\big)\big(\nabla_\ell \nabla_s n_r\big)$$

hence

$$\frac{\partial D}{\partial x^\ell} = 2\, \big(\nabla_\ell \nabla_j n_i\big)\big(\nabla^j n^i\big). \tag{43}$$

Let us replace $\nabla_\ell \nabla_j n_i$ from (39) [with $k = \ell$] into (43)

$$\frac{\partial D}{\partial x^\ell} = 2\, \big(\nabla^j n^i\big)\, [-L_{j\ell}\, n_i - \nu_i\, g_{j\ell} + \nu_j\, g_{i\ell} + L_{i\ell}\, n_j] =$$

[by taking into account the identities (35) i.e. $\big(\nabla^j n^i\big)\, n_i = 0$ and (36) i.e. $\big(\nabla^j n^i\big)\, n_j = 0$]

$$= 2\, \big(\nabla^j n^i\big)\, [\nu_j\, g_{i\ell} - \nu_i\, g_{j\ell}]$$

or [as $\nabla^j n^i$ is skew symmetric]

$$\frac{\partial D}{\partial x^\ell} = -4\, \nu_i\, \nabla_\ell n^i\,. \tag{44}$$

At this point we also differentiate in (31) with respect to x^ℓ so that

$$\frac{\partial D}{\partial x^\ell} = \nabla_\ell D = 2n\, \nabla_\ell \Big(L_{jk}\, n^j\, n^k\Big) =$$

[as L_{jk} is symmetric]

$$= 2n\; n^j\, n^k\, \nabla_\ell\, L_{jk} + 4n\; L_{jk}\, n^k\, \nabla_\ell\, n^j$$

or

$$\frac{\partial D}{\partial x^\ell} = 2n\; n^j\, n^k\, \nabla_\ell\, L_{jk} + 4n\; \nu_j\, \nabla_\ell n^j\,. \tag{45}$$

Now let us replace $\nu_j\, \nabla_\ell n^j$ from (44) into (45) so that to obtain

$$\frac{\partial D}{\partial x^\ell} = 2n\; n^j\, n^k\, \nabla_\ell\, L_{jk} - n\, \frac{\partial D}{\partial x^\ell}$$

or

$$\frac{\partial D}{\partial x^\ell} = \frac{2n}{n+1}\, n^j\, n^k\, \nabla_\ell\, L_{jk}\,. \tag{46}$$

Next let us contract by n^j in $C_{jk\ell} = \nabla_\ell L_{jk} - \nabla_k L_{j\ell}$ and use $C_{jk\ell}\, n^j = 0$. We obtain

$$0 = n^j \nabla_\ell L_{jk} - n^j\, \nabla_k L_{j\ell}\,.$$

Let us contract again by n^k

$$0 = n^j\, n^k\, \nabla_\ell L_{jk} - n^j\, n^k\, \nabla_k L_{j\ell}\,.$$

Hence

$$n^j\, n^k\, \nabla_\ell L_{jk} = n^j\, n^k\, \nabla_k L_{j\ell} = n^k\, \nabla_k\left(n^j\, L_{j\ell}\right) - n^k\, L_{j\ell}\, \nabla_k n^j$$

or [by (36) i.e. $n^k\, \nabla_k n^j = 0$]

$$n^j\, n^k\, \nabla_\ell L_{jk} = n^k\, \nabla_k\, \nu_\ell\,. \tag{47}$$

Let us substitute $n^j\, n^k\, \nabla_\ell L_{jk}$ from (47) into (46). We obtain

$$\frac{\partial D}{\partial x^\ell} = \frac{2n}{n+1}\, n^k\, \nabla_k\, \nu_\ell\,. \tag{48}$$

Summing up [by (43) and (48), as $\nabla_\ell\, n_i$ is skew symmetric]

$$\frac{\partial D}{\partial x^\ell} = \frac{2n}{n+1}\, n^k\, \nabla_k \nu_\ell = 4\, \nu^i\, \nabla_i n_\ell\,. \tag{49}$$

Compare to (3.5) in [**40**], p. 861. Next

$$\nabla^j \nu_j = g^{jk}\, \nabla_k \nu_j =$$

[by taking the covariant derivative of $\nu_j = L_{j\ell}\, n^\ell$]

$$= g^{jk}\, \nabla_k\left(L_{j\ell}\, n^\ell\right) = g^{jk}\, \left[n^\ell\, \nabla_k L_{j\ell} + L_{j\ell}\, \nabla_k n^\ell\right]$$

$$= n^\ell\, \nabla^j\, L_{j\ell} + L_{j\ell}\, \nabla^j\, n^\ell$$

or [by $L_{j\ell}\, \nabla^j\, n^\ell = 0$, as a contraction of the symmetric tensor field $L_{j\ell}$ by the skew symmetric tensor field $\nabla^j\, n^\ell$]

$$\nabla^j \nu_j = n^\ell\, \nabla^j\, L_{j\ell}\,. \tag{50}$$

We claim that the right hand term in (50) vanishes i.e.

L e m m a 5.3 (C. R. Graham, [**40**]).

$$\nabla^j\, \nu_j = 0. \tag{51}$$

Proof. Relation (51) is a consequence of the curvature properties of $(\mathfrak{M}, g)$. We start from the second Bianchi identity

$$\sum_{XYZ} \big(\nabla_X R\big)(Y, Z) = 0 \tag{52}$$

where $\sum_{XYZ}$ is the cyclic sum over X, Y, Z. We adopt the following convention as to local calculations

$$\big(\nabla_{\partial_i} R\big)(\partial_j\,,\, \partial_k)\partial_\ell = \big(\nabla_i\, R_\ell{}^s{}_{jk}\big)\; \partial_s\,.$$

Then (52) for $X = \partial_i$, $Y = \partial_j$ and $Z = \partial_k$ reads

$$\nabla_i\, R_\ell{}^s{}_{jk} + \nabla_j\, R_\ell{}^s{}_{ki} + \nabla_k\, R_\ell{}^s{}_{ij} = 0$$

or [by contraction with g^{rs}]

$$\nabla_i\, R_{\ell rjk} + \nabla_j\, R_{\ell rki} + \nabla_k\, R_{\ell rij} = 0. \tag{53}$$

We shall also make use of the symmetries of the Riemann-Christoffel tensor

$$R_{\ell rjk} = -R_{\ell rkj}\,, \quad R_{\ell rjk} = -R_{r\ell jk}\,, \quad R_{\ell rjk} = R_{jk\ell r}\,. \tag{54}$$

Next let us contract with g^{rk} in (53) so that to obtain [by (54)]

$$-\nabla_i\, R_{\ell j} + \nabla_j\, R_{\ell i} + \nabla_k\, R_\ell{}^k{}_{ij} = 0.$$

Contracting once more with $g^{\ell j}$ yields

$$-\nabla_i R + \nabla_j\, R^j{}_i + \nabla_k\, R^{jk}{}_{ij} = 0 \tag{55}$$

and

$$R^{jk}{}_{ij} = g^{jr}\, R_r{}^k{}_{ij} = g^{jr}\, g^{ks}\, R_{rsij} = g^{jr}\, g^{ks}\, R_{srji} = g^{ks}\, R_{si} = R^k{}_i$$

so that (55) becomes

$$\frac{\partial R}{\partial x^i} = 2\, \nabla^j\, R_{ji}\,. \tag{56}$$

Taking the covariant derivative of L_{jk} and using $\nabla_\ell\, g_{jk} = 0$ we obtain

$$\nabla_\ell\, L_{jk} = \frac{1}{2n}\left[\nabla_\ell\, R_{jk} - \frac{1}{2(2n+1)}\, \frac{\partial R}{\partial x^\ell}\, g_{jk}\right]$$

hence [by contracting with $g^{j\ell}$]

$$\nabla^j L_{jk} = \frac{1}{4n}\left[\nabla^j R_{jk} - \frac{1}{2(2n+1)}\frac{\partial R}{\partial x^k}\right]. \tag{57}$$

Let us replace $\partial R/\partial x^k$ from (56) into (57). This yields

$$\nabla^j L_{jk} = \frac{1}{2(2n+1)}\frac{\partial R}{\partial x^k}. \tag{58}$$

Let us contract by n^k in (58). The resulting identity, together with (50), may be written

$$\nabla^j \nu_j = n^k \nabla^j L_{jk} = \frac{1}{2(2n+1)} n^k \frac{\partial R}{\partial x^k}. \tag{59}$$

On the other hand the fact that N is Killing ($\mathcal{L}_N g = 0$) yields $N(R) = 0$ i.e. locally $n^k\,(\partial R/\partial x^k) = 0$. Then Lemma 5.3 follows from (59). Q.e.d. □

The formula (40) in Lemma 5.3

$$\left(\nabla_j n_i\right)\left(\nabla^k n^j\right) = -\left(\nu_j n^j\right)\delta_i^k + \nu_i\, n^k + \nu^k\, n_i$$

may now be used together with $D = 2n\,\nu_j\, n^j$ to yield

$$\left(\nabla_j n_i\right)\left(\nabla^k n^j\right) = -\frac{1}{2n}\,\delta_i^k\, D + \nu_i\, n^k + \nu^k\, n_i\,.$$

Let us apply the covariant derivative ∇_ℓ so that to obtain

$$\left(\nabla_\ell \nabla_j n_i\right)\left(\nabla^k n^j\right) + \left(\nabla_j n_i\right)\left(\nabla_\ell \nabla^k n^j\right)$$

$$= -\frac{1}{2n}\,\delta_i^k\,\nabla_\ell D + \left(\nabla_\ell \nu_i\right) n^k + \nu_i\,\nabla_\ell\, n^k + \left(\nabla_\ell\, \nu^k\right) n_i + \nu^k\,\nabla_\ell\, n_i\,.$$

Next let us contract k and ℓ

$$\left(\nabla_k \nabla_j n_i\right)\left(\nabla^k n^j\right) + \left(\nabla_j\, n_i\right)\left(\nabla_k \nabla^k n^j\right) \tag{60}$$

$$= -\frac{1}{2n}\,\nabla_i D + \left(\nabla_k \nu_i\right) n^k + \left(\nabla_k \nu^k\right) n_i + \nu^k\,\nabla_k n_i$$

and use

$$\nabla_k \nu^k = 0, \quad \nu^k\,\nabla_k n_i = \frac{1}{4}\,\nabla_i D, \quad \left(\nabla_k \nu_i\right) n^k = \frac{n+1}{2n}\,\nabla_i D\,,$$

to substitute into (60). This yields

$$(61) \qquad \big(\nabla_k \nabla_j n_i\big)\big(\nabla^k n^j\big) + \big(\nabla_j\, n_i\big)\big(\nabla_k \nabla^k n^j\big) = \frac{3}{4}\, \nabla_i D.$$

On the other hand we may invoke (39)

$$\nabla_k \nabla_j n_i = -L_{jk}\, n_i - \nu_i\, g_{jk} + \nu_j\, g_{ik} + L_{ik}\, n_j$$

contract with g^{jk}

$$\nabla_k \nabla^k n_i = -L^k{}_k\, n_i - (2n+2)\, \nu_i + \nu_i + L_{ik}\, n^k$$

followed by contraction with g^{ij}

$$(62) \qquad \nabla_k \nabla^k n^j = -(2n+1)\nu^j + L^j{}_k\, n^k - L^k{}_k\, n^j\, .$$

Moreover, let us use (39) and (62) to conduct the following calculation

$$\begin{aligned}
&\big(\nabla_k \nabla_j n_i\big)\big(\nabla^k n^j\big) + \big(\nabla_j n_i\big)\big(\nabla_k \nabla^k n^j\big)\\
&= \big[\nu_j\, g_{ik} - \nu_i\, g_{jk} + L_{ik}\, n_j - L_{jk}\, n_i\big]\, \nabla^k n^j\\
&\quad + \big(\nabla_j n_i\big)\, \Big[-(2n+1)\, \nu^j + L^j{}_k\, n^k - L^k{}_k\, n^j\Big]\\
&= \nu_j\, \nabla_i n^j - \nu_i\, \nabla_j n^j + L_{ik}\, n_j\, \nabla^k n^j - n_i\, L_{jk}\, \nabla^k n^j\\
&\quad -(2n+1)\, \big(\nabla_j n_i\big)\, \nu^j + \big(\nabla_j n_i\big)\, L^j{}_k\, n^k - L^k{}_k\, n^j\, \nabla_j n_i\, .
\end{aligned}$$

The last expression greatly simplifies by observing that $\nabla_j n^j = 0$ (as $\nabla_i n_j$ is skew-symmetric) $n_j\, \nabla^k n^j = 0$ [by (35)] $L_{jk}\, \nabla^k n^j = 0$ (as a contraction of a symmetric tensor with an skew-symmetric tensor) $n^j\, \nabla_j n_i = 0$ [by (36)] and $L^j{}_k\, n^k = \nu^j$ (by the very definition of ν_i). In the end

$$\begin{aligned}
&\big(\nabla_k \nabla_j n_i\big)\big(\nabla^k n^j\big) + \big(\nabla_j n_i\big)\big(\nabla_k \nabla^k n^j\big)\\
&= \nu_j\, \nabla_i n^j - (2n+1)\, \big(\nabla_j n_i\big)\, \nu^j + \nu^j\, \nabla_j n_i =
\end{aligned}$$

(once again because $\nabla_i n_j$ is skew)

$$= -(2n+1)\, \big(\nabla_j n_i\big)\, \nu^j =$$

[as $\nu^j\, \nabla_j n_i = \frac{1}{4}\, \nabla_i D$ by (44)]

$$= -\frac{2n+1}{4}\, \nabla_i D.$$

Summing up

$$(63) \qquad \big(\nabla_k \nabla_j n_i\big)\big(\nabla^k n^j\big) + \big(\nabla_j n_i\big)\big(\nabla_k \nabla^k n^j\big) = -\frac{2n+1}{4}\, \frac{\partial D}{\partial x^i}\, .$$

Finally we may substitute from (63) into (61) to obtain $\partial D / \partial x^i = 0$ so that D is locally constant, and then constant ($\mathfrak{M}$ is tacitly assumed to be connected). Lemma 5.1 is proved.

5.2 - *f-Structures*

Let $\mathfrak{M}$ be a real $(2n+s)$-dimensional C^∞ manifold. An *f-structure* on $\mathfrak{M}$ is a $(1,1)$-tensor field f of rank $2n$ such that $f^3+f=0$. An *f-structure with s complemented frames* is a synthetic object $\big(f,\ \{\xi_a\,,\,\eta_a\ :\ 1\le a\le s\}\big)$ consisting of an f-structure and s vector fields $\xi_a\in\mathfrak{X}(\mathfrak{M})$ and s differential 1-forms $\eta^a\in\Omega^1(\mathfrak{M})$ such that

$$f\big(\xi_a\big)=0,\ \ \eta^a\big(\xi_b\big)=\delta^a_b\,,\ \ \eta^a\circ f=0,\ \ f^2=-I+\eta^a\otimes\xi_a\,. \tag{64}$$

An f-structure with s complemented frames is *normal* if

$$\big[f\,,\,f\big]+(d\eta^a)\otimes\xi_a=0. \tag{65}$$

As a consequence of (65) the structure vector fields ξ_a commute (i.e. $[\xi_a\,,\,\xi_b]=0$). A Riemannian metric g on $\mathfrak{M}$ is *compatible* to the f-structure $\big(f,\ \{\xi_a\,,\,\eta_a\ :\ 1\le a\le s\}\big)$ if

$$g(fX,fY)=g(X,Y)-\sum_{a=1}^{s}\eta_a(X)\,\eta_a(Y)$$

for any $X,Y\in\mathfrak{X}(\mathfrak{M})$. Given a compatible Riemannian metric g, let us consider the differential 2-form

$$\Omega(X,Y)=g(X,fY).$$

A *K-structure* is a normal f-structure with Ω closed (i.e. $d\Omega=0$). A theory of f-structures with complemented frames, in the presence of a compatible Riemannian metric g, was built by D. E. Blair (cf. [**18**] – [**19**]) and D. E. Blair and G. D. Ludden and K. Yano (cf. [**20**]) though confined to the case of a compact manifold $\mathfrak{M}$. A prototypical result is

Theorem 5.1 (D. E. Blair et al., [**20**]). *Let $\mathfrak{M}$ be a $(2n+s)$-dimensional compact, connected manifold with a regular normal f-structure. Then $\mathfrak{M}$ is the bundle space of a principal toroidal bundle over a complex n-dimensional manifold $\mathfrak{N}$. If in addition $\mathfrak{M}$ is a K-manifold then $\mathfrak{N}$ is a Kähler manifold.*

The theory of f-structures wasn't new[2] and the papers [**18**] – [**20**] certainly build on previous work by S. I. Goldberg (cf. [**37**]), S. I. Goldberg and K. Yano (cf. [**39**]), and A. Morimoto (cf. [**60**]).

Recovering the results by D. E. Blair and collaborators (cf. *op. cit.*) to f-structures in the presence of a Lorentzian metric g, compatible to the given

[2]Perhaps the first to explicitly introduce the notion of a f-structure in differential geometry practice was K. Yano, [**85**]. Cf. also K. Yano and S. Ishihara, [**86**]. Previous to that, credit ought to be given to S. S. Chern, [**24**].

f-structure in an appropriate manner, is an open problem. To motivate the necessity of an attempt one has the discovery by G. Sparling (cf. [**77**]) and C. R. Graham (cf. [**40**]) that a $(2n+2)$-dimensional Lorentzian manifold $(\mathfrak{M}, g)$ admitting a null Killing vector field N such that $N \rfloor W = 0$ and $N \rfloor C = 0$ carries a natural f-structure $(J,\ N,\ V,\ \theta,\ \sigma)$ with two complemented frames to which g is *compatible* in the sense that

$$g(JX, JY) = g(X, Y) - 2\big(\theta \odot \sigma\big)(X, Y)$$

for any $X, Y \in \mathfrak{X}(\mathfrak{M})$. As (by Lemma 5.1) $\mathrm{Ric}(N, N)$ is constant we may normalise N such that $\mathrm{Ric}(N, N) = 2n$ and then

$$\nu_j \, n^j = 1. \tag{66}$$

Let $V \in \mathfrak{X}(\mathfrak{M})$ and $\theta,\ \sigma \in \Omega^1(\mathfrak{M})$ be the tangent vector field and differential 1-forms on $\mathfrak{M}$ locally given by

$$V = \nu^i \, \partial_i \,, \quad \theta = n_i \, dx^i \,, \quad \sigma = \nu_i \, dx^i \,.$$

Next let $J : T(\mathfrak{M}) \to T(\mathfrak{M})$ be the $(1,1)$-tensor field defined by

$$JX = \nabla_X N, \quad X \in \mathfrak{X}(\mathfrak{M}).$$

If $J\partial_j = J^i{}_j \, \partial_i$ then $J^i{}_j = \nabla_j n^i$. For further use let us set

$$H = \mathrm{Ker}(\theta) \cap \mathrm{Ker}(\sigma).$$

We collect a few properties of J in the following

Proposition 5.1 (C.R. Graham, [**40**]).
i) $JV = 0$ *and* $JN = 0$.
ii) $\theta \circ J = 0$ *and* $\sigma \circ J = 0$ *i.e.* $\mathrm{Range}(J) \subset H$.
iii) $J^2 = -I + \theta \otimes V + \sigma \otimes N$.
iv) $J^2 = -I$ *on* H.

Proof. i) By (44) and Lemma 5.1

$$JV = \nu^j \left(\nabla_j n^i\right) \partial_i = \frac{1}{4} \frac{\partial D}{\partial x^i} = 0.$$

Similarly [by (36)]

$$JN = n^j \left(\nabla_j n^i\right) \partial_i = 0.$$

ii) By (35) and (44)

$$\theta\big(J\partial_j\big) = \left(\nabla_j n^i\right) n_i = 0, \quad \sigma\big(J\,\partial_j\big) = \left(\nabla_j n^i\right) \nu_i = 0.$$

iii) One has

$$J^2\partial_j = \left(\nabla_j n^k\right)\left(\nabla_k n^i\right)\partial_i$$

and [by (40)]

$$\left(\nabla_j n^k\right)\left(\nabla_k n^i\right) = g^{ir}\, g_{js}\left(\nabla_k n_r\right)\left(\nabla^s n^k\right)$$

$$= g^{ir}\, g_{js}\left[-\nu_\ell\, n^\ell\, \delta^s_r + \nu_r\, n^s + \nu^s\, n_r\right] = -\nu_\ell\, n^\ell\, \delta^i_j + \nu^i\, n_j + \nu_j\, n^i$$

hence (by $\nu_\ell\, n^\ell = 1$)

$$J^2\partial_j = \left(-\delta^i_j + \nu^i\, n_j + \nu_j\, n^i\right)\partial_i\,.$$

Q.e.d.

iv) Follows from (ii)–(iii). □

As a consequence of Proposition 5.1

$$T(\mathfrak{M}) = H \oplus \mathbb{R}N \oplus \mathbb{R}V \tag{67}$$

and the f-structure J respects the decomposition (67) i.e. J annihilates N and V and determines a complex structure on H.

Lemma 5.4. *V is null i.e.* $g(V,V) = 0$.

Proof. As $JV = 0$ and $\theta(V) = 1$ one has [by (iii) in Proposition 5.1]

$$0 = J^2 V = -V + \theta(V)V + \sigma(V)N = \sigma(V)N$$

hence (as $N_x \neq 0$ for any $x \in \mathfrak{M}$)

$$0 = \sigma(V) = \nu_i \nu^i = g(V,V).$$

Q.e.d. □

Let us set for further use

$$\Omega(X,Y) = g(X,JY), \quad X,Y \in \mathfrak{X}(\mathfrak{M}).$$

Lemma 5.5. $\Omega = -d\theta$.

Proof. One has

$$(d\theta)(\partial_i\,,\,\partial_j) = \frac{1}{2}\left\{\partial_i\left(\theta\,\partial_j\right) - \partial_j\left(\theta\,\partial_i\right)\right\} = \frac{1}{2}\left(\nabla_i n_j - \nabla_j n_i\right) = \nabla_i n_j$$

as $\nabla_i n_j$ is skew. On the other hand

$$g\left(J\partial_i\,,\,\partial_j\right) = \left(\nabla_i n^k\right) g_{kj} = \nabla_i n_j$$

so that

$$g(JX,Y) = (d\theta)(X,Y). \tag{68}$$

Q.e.d. □

For further use let us define G_θ by setting

$$G_\theta(X,Y) = (d\theta)(X,JY)$$

for any $X, Y \in \mathfrak{X}(\mathfrak{M})$.

Lemma 5.6. *For any $X, Y \in \mathfrak{X}(\mathfrak{M})$*

$$g(JX,JY) = g(X,Y) - \theta(X)\,\sigma(Y) - \theta(Y)\,\sigma(X), \tag{69}$$

$$g = G_\theta + 2\,\theta \odot \sigma\,.$$

In particular the bilinear form G_θ is positive definite on H.

Proof. First of all note that

$$\theta(X) = g(X,N), \quad \sigma(X) = g(X,V). \tag{70}$$

Moreover

$$g(JX,JY) = \qquad \text{[by replacing } Y \mapsto JY \text{ in (68)]}$$

$$= (d\theta)(X,JY) = -(d\theta)(JY,X) =$$

[by replacing $X \mapsto JY$ and $Y \mapsto X$ in (68)]

$$= g(Y,X) - \theta(Y)\,g(X,V) - \sigma(Y)\,g(N,X)$$

followed by use of (70). Q.e.d.

Next [by (69)]

$$g(X,Y) = g(JX,JY) + 2(\theta \odot \sigma)(X,Y) =$$

[by (68)]

$$= (d\theta)(X,JY) + 2(\theta \odot \sigma)(X,Y)$$

yielding the second statement in Lemma 5.6. Finally, as the decomposition $T(\mathfrak{M}) = H \oplus \mathbb{R}N \oplus \mathbb{R}V$ is orthogonal with respect to g and g has signature $(2n+1,1)$ it must be that G_θ has signature $(2n,0)$ so it is positive definite on H. Q.e.d. □

As N is everywhere nonzero it determines a codimension 3 foliation $\mathcal{F}$ of $\mathfrak{M}$ whose leaves are the maximal integral curves of N. Let $\nu(\mathcal{F}) = T(\mathfrak{M})/T(\mathcal{F})$ be the transverse bundle and $\pi : T(\mathfrak{M}) \to \nu(\mathcal{F})$ the canonical projection. Let

$$\Omega^0_B(\mathcal{F}) \xrightarrow{d_B} \Omega^1_B(\mathcal{F}) \xrightarrow{d_B} \Omega^2_B(\mathcal{F}) \xrightarrow{d_B} \Omega^3_B(\mathcal{F})$$

be the *basic complex* of $(\mathfrak{M}, \mathcal{F})$. Cf. e.g. Ph. Tondeur, [**80**], p. 119. Let us set $\mathfrak{X}_{\mathcal{F}} = C^\infty\big(T(\mathcal{F})\big)$. Elements $f \in \Omega^0_B(\mathcal{F})$ are *basic functions* i.e. C^∞ functions $f : \mathfrak{M} \to \mathbb{R}$ such that $X(f) = 0$ for any $X \in \mathfrak{X}_{\mathcal{F}}$. An element $\omega \in \Omega^k_B(\mathcal{F})$ is a **basic** *k-form* $(1 \leq k \leq 3)$ i.e. a differential k-form $\omega \in \Omega^k(\mathfrak{M})$ such that $X \rfloor \omega = 0$ and $X \rfloor d\omega = 0$ (equivalently[3] $X \rfloor \omega = 0$ and $\mathcal{L}_X \omega = 0$) for any $X \in \mathfrak{X}_{\mathcal{F}}$. The following result holds (compare to [**40**], p. 864)

Lemma 5.7. i) *$\theta(N) = 0$ and $N \rfloor d\theta = 0$ hence θ is a basic 1-form i.e. $\theta \in \Omega^1_B(\mathcal{F})$.*

ii) *One has*

$$\mathcal{L}_N g = 0, \quad \mathcal{L}_N J = 0, \quad \mathcal{L}_N \Omega = 0, \quad \mathcal{L}_N \sigma = 0,$$

hence g, J, Ω and σ are invariant under sliding along the leaves of $\mathcal{F}$.

Proof. i) Locally $\theta(N) = n_i n^i = g(N, N) = 0$ and [by (68) and $JN = 0$]

$$(d\theta)(N, X) = g(JN, X) = 0.$$

ii) $\mathcal{L}_N g = 0$ by our assumption that N is Killing. Moreover (by the very definition of the Lie derivative)

$$g\big((\mathcal{L}_N J)X, Y\big) = \big(\mathcal{L}_N d\theta\big)(X, Y) - \big(\mathcal{L}_N g\big)(JX, Y) = 0$$

for any $X, Y \in \mathfrak{X}(\mathfrak{M})$. Yet g is nondegenerate hence $\big(\mathcal{L}_N J\big)X = 0$. Similarly

$$\big(\mathcal{L}_N \Omega\big)(X, Y) = \big(\mathcal{L}_N g\big)(X, JY) + g\big(X, (\mathcal{L}_N J)Y\big) = 0.$$

To prove the last statement in (ii) of Lemma 5.7 we first observe that

$$[N, V] = 0. \tag{71}$$

Indeed

$$[N, V] = \left(n^i \frac{\partial \nu^j}{\partial x^i} - \nu^i \frac{\partial n^j}{\partial x^i}\right) \partial_j =$$

(as ∇ is torsion-free)

$$= \big(n^i\, \nabla_i \nu^j - \nu^i\, \nabla_i n^j\big)\, \partial_j = 0$$

[3] By Cartan's formula $\mathcal{L}_X = i_X \circ d + d \circ i_X$.

because of $n^i \nabla_i \nu^j = 0$ [a consequence of (49) and $D =$ constant] and $\nu^i \nabla_i n^j = 0$ [following from (44) and the fact that $\nabla_i n_j$ is skew]. Finally

$$\begin{aligned}(\mathcal{L}_N \sigma)X = N\big(\sigma X\big) - \sigma\big(\mathcal{L}_N X\big) = \quad &[\text{by } (70)]\\ = N\big(g(X, V)\big) - g\big(\mathcal{L}_N X, V\big) = \quad &[\text{by } \mathcal{L}_N g = 0 \text{ and } (71)]\\ = g\big(X\,,\, \mathcal{L}_N V\big) = 0.\end{aligned}$$

Q.e.d. □

Recall that $N \rfloor \sigma = 1$ so that (despite $\mathcal{L}_N \sigma = 0$) σ is not a basic 1-form on $(\mathfrak{M},\, \mathcal{F})$.

By Proposition 5.1 the restriction of $J : T(\mathfrak{M}) \to T(\mathfrak{M})$ to H is H-valued and $J : H \to H$ is a complex structure along H. Let $J^{\mathbb{C}}$ be the $\mathbb{C}$-linear extension of J to $H \otimes \mathbb{C}$. Then $J^2 X = -X$ for any $X \in H$ yields $\big(J^{\mathbb{C}}\big)^2 Z = -Z$ for any $Z \in H \otimes \mathbb{C}$ so that $\mathrm{Spec}\big(J^{\mathbb{C}}\big) = \{\pm i\}$ (with $i = \sqrt{-1}$). Let

$$H^{1,0} = \mathrm{Eigen}\big(J^{\mathbb{C}};\ i\big), \quad H^{0,1} = \mathrm{Eigen}\big(J^{\mathbb{C}};\ -i\big),$$

be the corresponding eigenbundles. It may be shown that

Theorem 5.2. *Let $(\mathfrak{M}, g)$ be a $(2n+2)$-dimensional Lorentzian manifold admitting a nowhere zero null Killing vector field N such that $N \rfloor W = 0$ and $N \rfloor C = 0$. Let $\mathcal{F}$ be the foliation tangent to N and $\pi : T(\mathfrak{M}) \to \nu(\mathcal{F})$ the projection. Then*

i) *$\mathcal{H} = \pi\, H^{1,0} \subset \nu(\mathcal{F}) \otimes \mathbb{C}$ is a transverse CR structure on $(\mathfrak{M},\, \mathcal{F})$.*

ii) *If the leaf space $O = \mathfrak{M}/\mathcal{F}$ is an orbifold then $H^{1,0}$ projects on a CR structure $T_{1,0}(O) \subset T(O) \otimes \mathbb{C}$.*

For a discussion of foliations with a transverse CR structure one may see the authors [**10**]. CR structures on orbifolds (or V-manifolds, cf. I. Satake, [**76**]) were studied by I. Masamune et al., [**30**]. We do not prove Theorem 5.2 here and only report on the particular case considered by C. R. Graham (cf. [**40**]) where N is regular (in the sense of R. S. Palais, [**64**]) so that the leaf space $M = \mathfrak{M}/\mathcal{F}$ is a C^∞ manifold. To start with we recall a few elements of R. S. Palais theory (cf. *op. cit.*). Let $\mathfrak{M}$ be a m-dimensional C^∞ manifold. A local coordinate system $\chi = \big(x^1\,, \cdots, x^m\big) : U \to \mathbb{R}^m$ is *cubical*, of *breath* $2a$, *centered at* $x_0 \in U$, if $\chi(x_0) = 0$ and

$$\chi(U) = \big\{\big(t^1\,, \cdots, t^m\big) \in \mathbb{R}^m \ : \ \big|t^j\big| < a,\ \ 1 \le j \le m\big\}.$$

Let $1 \le p \le m-1$ and $t = \big(t^{p+1}\,, \cdots, t^m\big) \in \mathbb{R}^{m-p}$ such that $\big|t^j\big| < a$ for any $1 \le j \le m-p$. The set

$$\Sigma_t = \{x \in U \ : \ x^{p+j}(x) = t^{p+j}\,,\ \ 1 \le j \le m-p\}$$

is a *p-dimensional slice* of (U, χ). A tangent vector field $N \in \mathfrak{X}(\mathfrak{M})$ is *regular* if there is a C^∞ atlas $\mathcal{A}$ such that every local chart $(U, x^i) \in \mathcal{A}$ is cubical and the intersection of U with each maximal integral curve of N is a 1-dimensional slice of (U, x^i). We recall the following

Theorem 5.3 (R. S. Palais, [**64**]). *If N is regular then the leaf space $\mathfrak{M}/N$ admits a C^∞ manifold structure such that the projection $\Pi : \mathfrak{M} \to \mathfrak{M}/N$ is a C^∞ map.*

Next we recall a few elements of Boothby-Wang theory (cf. [**23**]). By Proposition 1.5 in [**47**], p. 13, for each point $x_0 \in \mathfrak{M}$ there is an open neighborhood $U \subset \mathfrak{M}$ of x_0, a positive number $\epsilon > 0$, and a local 1-parameter group of local transformations $\varphi_t : U \to \mathfrak{M}$, $|t| < \epsilon$, inducing N i.e.

$$\frac{dC_x}{dt}(0) = N_x\,, \quad x \in U, \quad C_x(t) = \varphi_t(x), \quad |t| < \epsilon.$$

The local 1-parameter group $\{\varphi_t\}_{|t|<\epsilon}$ is *generated* by N and N is *complete* if it generates a global 1-parameter group of transformations $\varphi_t : \mathfrak{M} \to \mathfrak{M}$, $t \in \mathbb{R}$. Compactness of $\mathfrak{M}$ implies completeness of N. The tangent vector field N is *closed* if C_x is a closed curve for any $x \in \mathfrak{M}$. The *period function* $\lambda_N : \mathfrak{M} \to \mathbb{R}$ of a closed regular vector field N is defined by

$$\lambda_N(x) = \inf\big\{t > 0 \;:\; \varphi_t(x) = x\big\}, \quad x \in \mathfrak{M}.$$

As a consequence of regularity $\lambda_N(x) > 0$ for every $x \in \mathfrak{M}$. By a result in [**23**] the period function is smooth i.e. $\lambda_N \in C^\infty(\mathfrak{M}, \mathbb{R})$. The relevance of the period function is emphasised by

Theorem 5.4 (A. Morimoto, [**60**]). *Let $(\phi,\ \xi,\ \eta)$ be a normal almost contact structure on $\mathfrak{M}$ whose structure vector ξ is closed and regular, and has a constant period function λ_ξ. Then there exist a complex manifold $\mathfrak{N}$ and a C^∞ map $\Pi : \mathfrak{M} \to \mathfrak{N}$ such that*

i) *$\mathfrak{M}$ is the total space of a principal circle bundle $S^1 \to \mathfrak{M} \overset{\Pi}{\to} \mathfrak{N}$.*

ii) *η is a connection 1-form on $\mathfrak{M}$.*

iii) *ξ is a vertical vector field i.e. $\xi \in \mathrm{Ker}(d\Pi)$.*

5.3 - *CR structures*

5.3.1 - `Graham-Sparling construction`

W. M. Boothby's results (cf. *op. cit.*) do not apply to the situation at hand, as θ is not a contact form and the compactness assumption is dropped, in general. Neither does Theorem 5.4 apply, as $(J,\ N,\ \theta)$ is not an almost contact structure, to start with. To avoid the abrupt generalization of the results in [**23**], [**60**] and [**20**] to the realm of Lorentzian geometry, or confine the discussion to purely local considerations, the approach in [**40**] only deals with the case where $(\mathfrak{M},\ g)$ is a $(2n+2)$-dimensional Lorentzian manifold, admitting a null Killing vector field N such that $N \rfloor W = 0$, $N \rfloor C = 0$, the leaf space $M = \mathfrak{M}/N$ is a $(2n+1)$-dimensional C^∞ manifold, the canonical projection $\pi : \mathfrak{M} \to M$ is C^∞ and readily organizes $\mathfrak{M}$ as the total space of a principal circle bundle $S^1 \to \mathfrak{M} \xrightarrow{\pi} M$ whose vertical bundle $\mathrm{Ker}(d\pi) \to \mathfrak{M}$ is the span of N i.e. $\mathrm{Ker}(d\pi) = \mathbb{R}N$. If this is the case then, by the main construction in [**77**] and [**40**], the data $(J,\ N,\ V,\ \theta,\ \sigma)$ induces a nondegenerate CR structure on M. It is our purpose in the present section to recall Sparling's construction of an almost CR structure $T_{1,0}(M) \subset T(M) \otimes \mathbb{C}$ [got by projecting (H, J) on M] and prove its integrability.

Let $p \in M$ and $u \in T_p(M)$. Let $x \in \pi^{-1}(p)$. As $\mathfrak{M}$ is a principal S^1-bundle over M, its projection π is a C^∞ submersion, so there is $v \in T_x(\mathfrak{M})$ such that $(d_x\pi)v = u$. Next we define $\eta \in \Omega^1(M)$ by setting

$$\eta_p(u) = \theta_x(v).$$

The definition of $\eta_p(u)$ doesn't depend upon the choice of x and v. Indeed if $y \in \pi^{-1}(p)$ and $w \in T_y(\mathfrak{M})$ with $(d_y\pi)w = u$ then there is a unique $a \in S^1$ such that $y = x \cdot a$ and $\pi \circ R_a = \pi$ thus yielding

$$(d_y\pi)w = u = (d_x\pi)v = (d_y\pi) \circ (d_x R_a)v$$

hence $w = v + \lambda N_y$ for some $\lambda \in \mathbb{R}$. Finally $N \rfloor \theta = 0$ implies $\theta_x(v) = \theta_y(w)$. Q.e.d.

Let us set $H(M) = \mathrm{Ker}(\eta)$ and define $J_M : H(M) \to H(M)$ by

$$J_{M,\,p}u = (d_x\pi)J_x v\,,$$

$$p \in M, \quad x \in \pi^{-1}(p), \quad u \in T_p(M), \quad v \in (d_x\pi)^{-1}(u).$$

Similar to the above $J_{M,p}u$ is well defined because of $JN = 0$. Next $J^2 = -I$ on H yields $J_M^2 = -I_M$ on $H(M)$, where I_M is the identical transformation of $H(M)$. We set as customary

$$T_{1,0}(M) = \mathrm{Eigen}\big(J_M^{\mathbb{C}},\ i\big) \subset H(M) \otimes \mathbb{C}.$$

Proposition 5.2 (C. R. Graham, [**40**]). *$T_{1,0}(M)$ is a strictly pseudoconvex CR structure on M, of CR dimension n.*

Proof. If we set $T_{0,1}(M) = \mathrm{Eigen}\big(J_M^{\mathbb{C}},\ -i\big)$ then $T_{0,1}(M) = \overline{T_{1,0}(M)}$ where an overbar denotes complex conjugation. Hence

$$T_{1,0}(M) \cap T_{0,1}(M) = (0)$$

i.e. $T_{1,0}(M)$ is an almost CR structure on M. To prove integrability

$$Z, W \in C^\infty(T_{1,0}(M)) \Longrightarrow [Z, W] \in C^\infty(T_{1,0}(M))$$

we ought to compute the Nijenhuis tensor field of J

$$N_J(X,Y) = [JX, JY] + J^2[X,Y] - J\big\{[JX,Y] + [X,JY]\big\},$$

$$X, Y \in \mathfrak{X}(\mathfrak{M}).$$

Locally

$$\begin{aligned} N_{jk}{}^i &= J^\ell{}_j \, \nabla_\ell J^i{}_k - J^\ell{}_k \, \nabla_\ell J^i{}_j - J^i{}_\ell \, \nabla_j J^\ell{}_k + J^i{}_\ell \, \nabla_k J^\ell{}_j \\ &= \big(\nabla_j n^\ell\big) \, \nabla_\ell \nabla_k n^i - \big(\nabla_k n^\ell\big) \, \nabla_\ell \nabla_j n^i \\ &\quad -\big(\nabla_\ell n^i\big) \, \nabla_j \nabla_j n^\ell + \big(\nabla_\ell n^i\big) \, \nabla_k \nabla_j n^\ell \, . \end{aligned}$$

Let us substitute the covariant derivatives of $N = n^i \, \partial_i$ from (39) and observe the cancellation of terms. We obtain

$$\begin{aligned} N_{jk}{}^i = {} & 2\nu^i \, \nabla_k n_j + \big(L_{jk} \, \nabla_k n^\ell - L_{k\ell} \, \nabla_j n^\ell\big) \, n^i \\ & + \big(L^\ell{}_k \, \nabla_\ell n^i - L^i{}_\ell \, \nabla_k n^\ell\big) \, n_j + \big(L^i{}_\ell \, \nabla_j n^\ell - L^\ell{}_j \, \nabla_\ell n^i\big) \, n_k \, . \end{aligned} \tag{72}$$

Lemma 5.8. *If* $\theta(X) = \theta(Y) = 0$ *then* $N_J(X,Y) \in \mathrm{Span}\big\{N,\ V\big\}$.

Proof. Let us contract with $X^j Y^k$ in (72) and use $n_j X^j = n_k Y^k = 0$. We obtain

$$N_J(X,Y) = 2\,\{(\nabla_Y \theta) X\} \ V + \big\{L\big(X,\ \nabla_Y N\big) - L\big(\nabla_X N,\ Y\big)\big\} \ N.$$

Q.e.d. □

Lemma 5.9.

$$\big[C^\infty\big(T_{1,0}(M)\big)\,,\ C^\infty\big(T_{1,0}(M)\big)\big] \subset C^\infty\big(H(M) \otimes \mathbb{C}\big).$$

Proof. Let $X, Y \in C^\infty(H(M))$ and let $\tilde{X}, \tilde{Y} \in C^\infty(H)$ be respectively lifts of X and Y i.e.

$$(d_x\pi)\tilde{X}_x = X_{\pi(x)}\,, \quad (d_x\pi)\tilde{Y}_x = Y_{\pi(x)}\,, \quad x \in \mathfrak{M}.$$

The identities

$$(d\tilde{\theta})(J\tilde{X},\, J\tilde{Y}) = (d\theta)(\tilde{X},\, \tilde{Y}), \quad (d\theta)(J\tilde{X},\, \tilde{Y}) = -(d\theta)(\tilde{X},\, J\tilde{Y}),$$

yield

$$(d\eta)(J_M X,\, J_M Y) = (d\eta)(X, Y), \quad (d\eta)(J_M X, Y) = -(d\eta)(X, J_M Y),$$

so that

$$(d\eta)(X - iJ_M X,\; Y - iJ_M Y) = 0.$$

Hence [as $\eta(X - iJ_M X) = \eta(Y - iJ_M Y) = 0$]

$$0 = (d\eta)(X - iJ_M X,\; Y - iJ_M Y) = -\frac{1}{2}\,\eta\big([X - iJX,\; Y - iJ_M Y]\big)$$

i.e. $[X - iJX,\; Y - iJ_M Y] \in \mathrm{Ker}(\eta) \otimes \mathbb{C} = H(M) \otimes \mathbb{C}$. Q.e.d. $\square$

Let us set

$$\tilde{Z} = \Big[\tilde{X} - iJ\tilde{X}\,,\, \tilde{Y} - iJ\tilde{Y}\Big]$$

so that $\tilde{Z}$ is a lift of $Z = \big[X - iJ_M X\,,\, Y - iJ_M Y\big]$. Then

$$\tilde{Z} + i\,J\,\tilde{Z} =$$

$$= \big[\tilde{X}\,,\, \tilde{Y}\big] - \big[J\tilde{X}\,,\, J\tilde{Y}\big] + J\big[J\tilde{X}\,,\, \tilde{Y}\big] + J\big[\tilde{X}\,,\, J\tilde{Y}\big]$$
$$+\, i\Big\{J\big[\tilde{X}\,,\, \tilde{Y}\big] - J\big[J\tilde{X}\,,\, J\tilde{Y}\big] - \big[J\tilde{X}\,,\, \tilde{Y}\big] - \big[\tilde{X}\,,\, J\tilde{Y}\big]\Big\} =$$

[by $J^2 = -I + \theta \otimes V + \sigma \otimes N$ and $J^3 + J = 0$]

$$= \big(-J^2 + \theta \otimes V + \sigma \otimes N\big)\big[\tilde{X}\,,\, \tilde{Y}\big]$$
$$-\big[J\tilde{X}\,,\, J\tilde{Y}\big] + J\big[J\tilde{X}\,,\, \tilde{Y}\big] + J\big[\tilde{X}\,,\, J\tilde{Y}\big]$$
$$+\, i\Big\{-J^3\big[\tilde{X}\,,\, \tilde{Y}\big] - J\big[J\tilde{X}\,,\, J\tilde{Y}\big]$$
$$+\big(J^2 - \theta \otimes V - \sigma \otimes N\big)\Big(\big[J\tilde{X}\,,\, \tilde{Y}\big] + \big[\tilde{X}\,,\, J\tilde{Y}\big]\Big)\Big\}$$

or [by the very definition of N_J]

$$\tilde{Z} + i\,J\tilde{Z} = -N_J\big(\tilde{X}\,,\, \tilde{Y}\big) - i\,J\,N_J\big(\tilde{X}\,,\, \tilde{Y}\big) \tag{73}$$

$$+\theta\left([\tilde{X}\,,\tilde{Y}]-i[J\tilde{X}\,,\tilde{Y}]-i[\tilde{X}\,,J\tilde{Y}]\right)V$$
$$+\sigma\left([\tilde{X}\,,\tilde{Y}]-i[J\tilde{X}\,,\tilde{Y}]-i[\tilde{X}\,,J\tilde{Y}]\right)N.$$

Yet $\tilde{X},\ \tilde{Y}\in H$ hence $\theta(\tilde{X})=\theta(\tilde{Y})=0$ yielding (by Lemma 5.8) $N_J(\tilde{X}\,,\tilde{Y})\in\mathbb{R}N\oplus\mathbb{R}V$. Together with (73) this yields

$$\tilde{Z}+iJ\tilde{Z}\in\mathbb{C}N\oplus\mathbb{C}V. \tag{74}$$

Previously one has shown that $\mathcal{L}_N V=[N,V]=0$ so that V projects on M i.e. there is a tangent vector field $T\in\mathfrak{X}(M)$ such that

$$(d_x\pi)V=T_{\pi(x)}\,,\quad x\in\mathfrak{M}.$$

Then (74) yields

$$Z+iJ_M Z\in\mathbb{C}T \tag{75}$$

i.e. $Z+iJ_M Z=fT$ for some C^∞ function $f:M\to\mathbb{C}$. On the other hand (by Lemma 5.9) $Z\in H(M)\otimes\mathbb{C}$ [and $H(M)=\mathrm{Ker}(\eta)$] hence

$$0=\eta\big(Z+iJ_M Z\big)=f\,\eta(T)=f$$

(as $\eta(T)_{\pi(x)}=\theta(V)_x=1$). It follows that $Z+iJ_M Z=0$ hence $Z\in T_{1,0}(M)$ or

$$\big[X-iJ_M X\,,\,Y-iJ_M Y\big]\in T_{1,0}(M).$$

Q.e.d. Finally the Levi form

$$G_\eta(X,Y)=(d\eta)(X,J_M Y),\quad X,Y\in H(M),$$

is positive definite because G_θ is positive definite on H (cf. Lemma 5.6 above). Q.e.d. □

5.3.2 - Robinson-Trautman construction

Let $\mathfrak{M}$ be a 4-dimensional (i.e. $n=1$) manifold and (K,L) a flag structure on $\mathfrak{M}$. Let us assume that (K,L) is geodesic, so that [by Theorem 3.1] every section $N\in C^\infty\big(K^0\big)$ is a null Killing vector field with respect to any adapted metric $g\in\mathcal{A}\subset\mathrm{Lor}(\mathfrak{M})$. Here $K^0=K\setminus\{\text{zero section}\}$. Moreover, we assume there is a section $N\in C^\infty\big(K^0\big)$ which is a regular vector field, in the sense of R. Palais (cf. [**64**] and our Section § 5.2). Then [by Theorem 5.3] the leaf space $M=\mathfrak{M}/N$ may be organized as a C^∞ manifold such that the canonical projection $\pi : \mathfrak{M} \to M$ is a C^∞ submersion. As (K, L) is geodesic, L is invariant with respect to K so that $\lambda \wedge \mathcal{L}_N \lambda = 0$ for any $\lambda \in C^\infty(L)$ and then L is invariant with respect to the flow $\{\varphi_t^N\}_{|t|<\epsilon}$ generated by N. Consequently $L \subset T^*(\mathfrak{M})$ projects on a line bundle $L_N \subset T^*(M)$ such that for any section $\lambda \in C^\infty(L_N)$ the pullback of λ by π is a section in L i.e. $\pi^*\lambda \in C^\infty(L)$. Given $\lambda \in C^\infty(L_N)$ is a nowhere zero section in L_N we set as customary

$$\mathrm{Ker}(L_N)_p = \mathrm{Ker}\big(\lambda_p\big), \quad p \in M,$$

so that $\mathrm{Ker}(L_N) \to M$ is a rank 2 subbundle of $T(M) \to M$ [which doesn't depend upon the choice of λ].

Let now $\big((K, L),\ J\big)$ be a shear free optical structure, based on the flag structure (K, L). Here $J : E \to E$ is a complex structure on $E = \mathrm{Ker}(L)/K$. As J is invariant with respect to the flow $\{\varphi_t^N\}_{|t|<\epsilon}$ it projects onto a complex structure $J_M : \mathrm{Ker}(L_N) \to \mathrm{Ker}(L_N)$ hence $T_{1,0}(M) = \{X - iJ_M X : X \in T(M)\}$ is a CR structure on M.

5.3.3 - `Graham-Sparling and Robinson-Trautman constructions compared`

We may parallel the Graham-Sparling and Robinson-Trautman constructions, as follows. Let $\mathfrak{M}$ be a 4-dimensional manifold. Let $g \in \mathrm{Lor}(\mathfrak{M})$ be a Lorentzian metric on $\mathfrak{M}$ and let $N \in \mathfrak{X}(\mathfrak{M})$ be a nowhere zero Killing vector field on $(\mathfrak{M},\, g)$ i.e. $\mathcal{L}_N g = 0$. Next, let us assume that N is null [i.e. $g(N, N) = 0$] and that $N \rfloor W = 0$ and $N \rfloor C = 0$ where W are respectively the Weyl and Cotton tensor fields of $(\mathfrak{M},\, g)$. This is of course the starting data in the Graham-Sparling construction (with $n = 1$). Let $\theta \in \Omega^1(\mathfrak{M})$ be defined by

$$\theta(X) = g(X, N), \quad X \in \mathfrak{X}(\mathfrak{M}), \tag{76}$$

and let us consider the line subbundles

$$K \subset T(\mathfrak{M}), \quad L \subset T^*(\mathfrak{M}),$$

$$K_x = \big\{a\, N_x \ :\ a \in \mathbb{R}\big\}, \quad L_x = \big\{a\, \theta_x \ :\ a \in \mathbb{R}\big\}, \quad x \in \mathfrak{M}.$$

As $\theta(N) = 0$ the pair (K, L) is a flag structure on $\mathfrak{M}$. By the very definition (76) one has $N^\flat = \theta$ hence trivially $N^\flat \wedge \theta = 0$ i.e. g is adapted to (K, L). If $k \in C^\infty(K)$ and $\lambda \in C^\infty(L)$ then $k = \rho N$ and $\lambda = f\theta$ for some $\rho,\ f \in C^\infty(\mathfrak{M}, \mathbb{R})$ so that

$$\mathcal{L}_k \lambda = \mathcal{L}_{\rho N} \lambda = \rho\, \mathcal{L}_N \lambda,$$

$$\mathcal{L}_N \lambda = \mathcal{L}_N\big(f\theta\big) = N(f)\,\theta + f\,\mathcal{L}_N \theta,$$

$$\lambda \wedge \mathcal{L}_k \lambda = f^2\, \theta \wedge \mathcal{L}_N \theta,$$

and $\mathcal{L}_N \theta = 0$ [as $\theta \in \Omega^1_B(\mathcal{F})$, cf. Section § 5.2]. Then $\lambda \wedge \mathcal{L}_k \lambda = 0$, or L is invariant with respect to K i.e. the flag structure (K, L) is geodesic. Also

$$\lambda \wedge d\lambda = f^2\, \theta \wedge d\theta$$

hence $\mathrm{Ker}(L)$ isn't integrable. That is, the congruence by null curves defined by the flow is twisting. Let $J : T(\mathfrak{M}) \to T(\mathfrak{M})$ be the $(1,1)$-tensor field defined by

$$JX = \nabla_X N, \quad X \in \mathfrak{X}(\mathfrak{M}). \tag{77}$$

Then [by Proposition 5.1] J is an f-structure on $\mathfrak{M}$ and $(J,\, N,\, V,\, \theta,\, \sigma)$ is an f-structure with two complemented frames. Next let us consider the real rank 2 bundle $E = \mathrm{Ker}(L)/K$. As $\theta \circ J = 0$ the bundle morphism J is $\mathrm{Ker}(L)$-valued. Also, as $JN = 0$ the endomorphism $J : \mathrm{Ker}(L) \to \mathrm{Ker}(L)$ descends to a vector bundle morphism $J : E \to E$ with $J^2 = -I$ i.e. a complex structure on E. Therefore $\big\{(K,\, L),\ J\big\}$ is an optical structure on $\mathfrak{M}$. Let $B \subset \mathcal{A}$ be the class $(\bmod\ \mathcal{R})$ of g (which makes sense because g is already adapted). Also let $\mathcal{O}$ be the natural orientation of E determined by the complex structure $J : E \to E$ so that the optical structure $\big\{(K, L),\ J\big\}$ may also be thought of as the synthetic object $\big\{(K, L),\ B,\ \mathcal{O}\big\}$. Let $\lambda = f\,\theta \in C^\infty(L)$ be an arbitrary nowhere zero section [hence $f(x) \neq 0$ for any $x \in \mathfrak{M}$]. Then for any $k = \rho\, N \in C^\infty(K)$

$$\mathcal{L}_k g = 2\, \nu \odot \lambda$$

which is (29) with $\sigma = 0$ (and $\nu = f^{-1}\, d\rho$). Thus the optical structure $\big\{(K, L),\ J\big\}$ is shear-free.

Finally, let us assume now that the given null Killing vector is also regular (in the sense of R. Palais) so that $M = \mathfrak{M}/N$ is a manifold and $\pi : \mathfrak{M} \to M$ is C^∞. Then L projects (by the invariance of L with respect to K) on a line bundle $L_N \subset T^*(M)$ and (by $JN = 0$, $\mathcal{L}_N J = 0$) $J : E \to E$ projects on a complex structure $J : \mathrm{Ker}(L_N) \to \mathrm{Ker}(L_N)$ leading to the CR structure on M arising from the Robinson-Trautman construction. Therefore, aside from being confined to the case of 4-dimensional manifolds $\mathfrak{M}$, the Robinson-Trautman is more general in the sense that it associates a CR structure on the orbit space $M = \mathfrak{M}/N$ to any complex structure $J : \mathrm{Ker}(L)/K \to \mathrm{Ker}(L)/L$ (provided that J is invariant by the flow of N) rather than to the particular complex structure (77) alone.

6 - Petrov classification

6.1 - *Petrov classification*

Petrov's classification (also known as the *Petrov-Pirani-Penrose classification*) as discovered by A. Z. Petrov (cf. [**66**]) and F. Pirani (cf. [**67**]) introduces six types

$$\text{I},\ \text{II},\ \text{D},\ \text{III},\ \text{N},\ \text{O}, \tag{78}$$

for the Weyl tensor $W_{\mu\nu\sigma\rho}(x)$ at a point $x \in \mathfrak{M}$ of the given space-time. The Weyl tensor is *algebraically general* at x if it is of Petrov type I at x, while in the remaining cases the Weyl tensor is *algebraically special*. Rather than providing the definition of the Petrov classes (78) we recall their description due to L. Bel and R. Debever (cf. e.g. M. Ortaggio, [**63**]). Let $W^*_{\mu\nu\sigma\rho}$ be the dual of the Weyl tensor (cf. e.g. M.N. Novello and J. Duarte De Oliveira, [**62**]). Let T be one of the types (78). If $W_{\mu\nu\sigma\rho}$ is of type T at x we write $W \in \mathrm{T}_x$. Then (cf. [**63**])

$$W \in \mathrm{N}_x \Longleftrightarrow \begin{cases} \exists\, n^\mu\,(\partial_\mu)_x \in T_x(\mathfrak{M}) \setminus \{0\} \\ g_{\mu\nu}(x) n^\mu n^\nu = 0, \\ W_{\mu\nu\sigma\rho}(x) n^\rho = 0. \end{cases}$$

If $W \notin \mathrm{N}_x$ then

$$W \in \mathrm{III}_x \Longleftrightarrow \begin{cases} \exists\, n^\mu\,(\partial_\mu)_x \in T_x(\mathfrak{M}) \setminus \{0\} \\ g_{\mu\nu}(x) n^\mu n^\nu = 0, \\ W_{\mu\nu\sigma\rho}(x) n^\nu n^\rho = 0, \\ W^*_{\mu\nu\sigma\rho}(x) n^\nu n^\rho = 0. \end{cases}$$

Moreover

$$W \in \mathrm{II}_x \Longleftrightarrow \begin{cases} \exists\, n^\mu\,(\partial_\mu)_x \in T_x(\mathfrak{M}) \setminus \{0\} \\ \exists\, \alpha,\, \beta \in \mathbb{R} \setminus \{0\} \\ g_{\mu\nu}(x) n^\mu n^\nu = 0, \\ W_{\mu\nu\sigma\rho}(x) n^\nu n^\rho = \alpha\, n_\mu n_\sigma\,, \\ W^*_{\mu\nu\sigma\rho}(x) n^\nu n^\rho = \beta\, \nu_\mu n_\sigma\,. \end{cases}$$

$$W \in \mathrm{D}_x \Longleftrightarrow \begin{cases} \exists\, n^\mu\,(\partial_\mu)_x\,,\ n'^\mu(\partial_\mu)_x \in T_x(\mathfrak{M}) \setminus \{0\} \\ \exists\, \alpha,\, \beta,\, \gamma,\, \delta \in \mathbb{R} \setminus \{0\} \\ n^\mu\,(\partial_\mu)_x\,,\ n'^\mu(\partial_\mu)_x \text{ linearly independent} \\ g_{\mu\nu}(x) n^\mu n^\nu = 0, \quad g_{\mu\nu}(x) n'^\mu n'^\nu = 0, \\ W_{\mu\nu\sigma\rho}(x) n^\nu n^\rho = \alpha\, n_\mu n_\sigma\,, \\ W^*_{\mu\nu\sigma\rho}(x) n^\nu n^\rho = \beta\, n_\mu n_\sigma\,, \\ W_{\mu\nu\sigma\rho}(x) n'^\nu n'^\rho = \gamma\, n'_\mu n'_\sigma\,, \\ W^*_{\mu\nu\sigma\rho}(x) n'^\nu n'^\rho = \delta\, n'_\mu n'_\sigma\,. \end{cases}$$

$$W \in \mathrm{O}_x \Longleftrightarrow W_{\mu\nu\sigma\rho}(x) = 0.$$

The result (obtained independently) in [**66**] and [**67**] is that the Weyl tensor of a given Lorentzian metric falls (at a point x) in one of the six classes (78). The proof is a linear algebra inspection of the spectrum of the mapping $v'^{\mu\nu} = W_{\mu\nu\sigma\rho}(x) v^{\sigma\rho}$ on 2-vectors [i.e. on $(2,0)$-tensor fields at a point x] determined by the Weyl tensor at x.

6.2 - *Goldberg-Sachs theorem*

For our needs in Section § 6 we recall the *Goldberg-Sachs theorem* according to which a vacuum solution of the Einstein field equations admits a shearfree null geodesic congruence if and only if the Weyl tensor is algebraically special (cf. J. N. Goldberg and R. K. Sachs, [**38**]). It should be mentioned that Goldberg-Sachs theorem doesn't include linearized gravity: by a result of S. Dain and O. M. Moreschi (cf. [**25**]) there exist solutions to linearized Einstein field equations admitting a shearfree null geodesic congruence which aren't algebraically special.

7 - Dirac equation and shearfree geodesic null congruences

7.1 - *Dirac equation on a curved space-time*

The classical[4] *Dirac equation* is

$$\gamma^\mu\, \partial_\mu \Psi = -\frac{imc}{\hbar}\, \Psi. \tag{79}$$

[4]That is the four component wave equation as discovered by P. Dirac himself, cf. [**27**].

The *conserved probability current* is

$$J^\mu = c\,\Psi^+ A\,\gamma^\mu\,\Psi. \tag{80}$$

An early study of (79)–(80) in a Minkowski space-time was performed by W. Pauli (cf. [**65**]) and W. Kofink (cf. [**50**]). In this context, if $\mathfrak{M} = \left(\mathbb{R}^4\,,\, g_0\right)$ is the 4-dimensional Minkowski space-time[5]

$$g_0 = \sum_{\mu=0}^{3} \epsilon_\mu\, dx^\mu \otimes dx^\mu\,, \quad \epsilon_0 = 1 = -\epsilon_j\,, \quad j \in \{1,2,3\},$$

then the *Dirac field* Ψ, governed by equation (79), is a function $\Psi : \mathfrak{M} \to \mathbb{C}^4$ and we set $\partial_\mu \Psi = \partial\Psi/\partial x^\mu$. The *Dirac gamma matrices* γ^μ act[6] on Ψ and satisfy the anticommutation formula

$$\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2\,\eta^{\mu\nu}\,\mathbf{I}_4 \tag{81}$$

where

$$\left[\eta^{\mu\nu}\right] = \left[\eta_{\mu\nu}\right]^{-1}, \quad \eta_{\mu\nu} = \epsilon_\mu\,\delta_{\mu\nu}\,, \quad \mathbf{I}_4 = \left[\delta_{\mu\nu}\right].$$

Also m, c and $\hbar$ denote respectively *mass*, *speed of light* and *Planck's constant*, while Ψ^+ is the complex conjugate transpose of Ψ and A is the *hermitizing matrix* (for the Dirac gamma matrices γ^μ) i.e.

$$A = A^+\,, \quad \left(\gamma^\mu\right)^+ = A\,\gamma^\mu\,A^{-1}\,. \tag{82}$$

The *similarity transformation* (or *spin-base transformation*) is

$$\tilde{\Psi} = S^{-1}\Psi, \quad \tilde{\gamma}^\mu = S^{-1}\,\gamma^\mu\,S, \quad \tilde{A} = S^+\,A\,S, \tag{83}$$

where S is a constant 4×4 matrix with complex entries. Equations (79)–(82) are invariant under any similarity transformation (83).

To generalize Dirac's equation to an arbitrary space-time, in a manner consistent with its Minkowski space-time version, one replaces the ordinary partials ∂_μ by covariant derivatives $D_\mu = \partial_\mu + \Gamma_\mu$ where Γ_μ are 4×4 complex matrices (the *spin connection matrices*) acting on the Dirac field Ψ. Also Minkowski 4-space is replaced by an arbitrary 4-dimensional Lorentzian manifold $(\mathfrak{M}, g)$ so that the Minkowski metric $g_0 = \eta_{\mu\nu}\,dx^\mu \otimes dx^\nu$ is replaced by an arbitrary

[5] The reader should be warned that through section § 7 we switch from signature convention $(- + ++)$ [for a given Lorentzian metric] to convention $(+ - --)$. A unified presentation is of course desirable yet, since we give no proofs, we wish to respect the conventions in the sources (such as [**46**]) quoted through § 7.

[6] That is γ^μ are 4×4 matrices with complex entries.

Lorentzian metric $g = g_{\mu\nu}\, dx^\mu \otimes x^\nu$ on $\mathfrak{M}$ i.e. one replaces $\eta^{\mu\nu}$ by $g^{\mu\nu}$ in the anticommutation formula (81) for the Dirac gamma matrices

$$\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2\, g^{\mu\nu}\, \mathbf{I}_4 \, . \tag{84}$$

It is customary (cf. e.g. M. Arminjon and F. Reifler, [**7**]) to additionally request that $g_{00} > 0$ and that the 3×3 matrix $\left[g_{jk} \right]$ be negative definite. While these are considered to be mild restrictions imposed on g, satisfied by most space-times appearing in mathematical practice, they notably do not hold for Gödel's (cf. [**36**]) space-time.

One further generalizes Dirac's equation by allowing the coefficient matrices $(\gamma^\mu \, , \, A)$ to be functions of the space-time point $x \in \mathfrak{M}$. Moreover, to get the right invariance properties of equation (79) one adds to (83) the transformation law of the spin connection matrices Γ_μ under similarity transformations (associated to matrices S which are also allowed to be functions of the space-time point)

$$\tilde{\Gamma}_\mu = S^{-1} \left(\partial_\mu + \Gamma_\mu \right) S \tag{85}$$

so that covariant derivatives transform as $\tilde{D}_\mu = S^{-1}\, D_\mu\, S$. According to the terminology adopted in [**7**] the transformations given by (83) and (85) are referred to as local similarity transformations of the *first kind*. Local similarity transformations of the *second kind* are defined by adding to (83) the transformation law

$$\tilde{\Gamma}_\mu = \Gamma_\mu \tag{86}$$

instead of (85). For transformations of the form (83) and (86) one has $\tilde{D}_\mu = D_\mu$. Two Dirac equations are (*classically*) *equivalent* if there is a local similarity transformation of any of the two kinds above which maps solutions to one equation into solutions of the other.

The *Dirac Lagrangian* is

$$\begin{aligned} & L\big(x^\mu, \ \Psi, \ \partial_\mu \Psi\big) \qquad (87) \\ & \equiv \sqrt{-G}\, \frac{i\hbar c}{2} \left[\Psi^+ A \gamma^\mu (D_\mu \Psi) - (D_\mu \Psi)^+ A \gamma^\mu \Psi + \frac{2mc}{\hbar}\, i\, \Psi^+ A \Psi \right] \end{aligned}$$

with $G = \det[\, g_{\mu\nu}]$. The Euler-Lagrange equations of the variational principle

$$\delta \int L\big(x^\mu, \ \Psi, \ \partial_\mu \Psi\big)\, d^4 x = 0$$

are

$$\gamma^\mu\, D_\mu \Psi + \frac{1}{2}\, A^{-1} D_\mu \big(A \gamma^\mu\big) \Psi = - \frac{imc}{\hbar}\, \Psi \tag{88}$$

where

$$D_\mu \Psi \equiv \partial_\mu \Psi + \Gamma_\mu \Psi,$$
$$D_\mu \gamma^\nu \equiv \nabla_\mu \gamma^\nu + \Gamma_\mu \gamma^\nu - \gamma^\nu \Gamma_\mu \, ,$$
$$D_\mu A \equiv \partial_\mu A - \Gamma_\mu^+ A - A\Gamma_\mu \, ,$$

and ∇_μ is the covariant derivative with respect to the Levi-Civita connection of $g_{\mu\nu}$ i.e.

$$\nabla_\mu \Psi \equiv \partial_\mu \Psi,$$
$$\nabla_\mu \gamma^\nu \equiv \partial_\mu \gamma^\nu + \left\{ \begin{array}{c} \nu \\ \rho\mu \end{array} \right\} \gamma^\rho \, ,$$
$$\nabla_\mu A \equiv \partial_\mu A,$$

where $\left\{ \begin{array}{c} \nu \\ \rho\mu \end{array} \right\}$ are the Christoffel symbols of $g_{\mu\nu}$. Equation (88) is the (*generalized*) *Dirac equation.* According to the terminology adopted by M. Arminjon and F. Reifler (cf. [**5**]) the *class QRD-0* consists of the generalized Dirac equations (88) with $\Gamma = 0$ where $\Gamma \equiv \gamma^\mu \Gamma_\mu$ is the *contracted spin connection matrix.* Dirac equations in the class QRD-0 read

$$\gamma^\mu \, \partial_\mu \Psi + \frac{1}{2} \, A^{-1} \nabla_\mu \big(A \gamma^\mu \big) \Psi = - \frac{imc}{\hbar} \, \Psi. \tag{89}$$

If $\nabla_\mu \big(A\gamma^\mu \big) = 0$ then (89) takes the *normal form*

$$\gamma^\mu \, \partial_\mu \Psi = - \frac{imc}{\hbar} \, \Psi. \tag{90}$$

We recall

Theorem 7.1 (M. Arminjon and F. Reifler, [**7**]). *Let (U, x^μ) be a local coordinate system on a space-time $(\mathfrak{M}, g)$ such that $g_{00}(x) > 0$ and the matrix $[\, g_{jk}(x)]_{1 \le jk \le 3}$ is positive definite, for any $x \in U$. Then for any choice of C^∞ coefficient fields $(\gamma^\mu \, , \, A)$ and any choice of covariant derivatives $D_\mu = \partial_\mu + \Gamma_\mu$ acting on C^∞ Dirac fields Ψ defined on U, there exist a C^∞ local similarity transformation S of the first kind and a C^∞ local similarity transformation T of the second kind such that $\Psi \longmapsto (TS)^{-1} \Psi$ transforms the generalized Dirac equation* (88) *with coefficient fields $(\gamma^\mu \, , \, A, \, \Gamma_\mu)$ into an equivalent normal Dirac equation* (90) *of class* QRD-0*, in some open neighborhood of each point $x_0 \in U$.*

By Theorem 7.1 one may think of normal QRD-0 equations (90) as *canonical forms* for the (generalized) Dirac equations (88), in an open neighborhood of each point of the given space-time. The proof of Theorem 7.1 relies on the theory of linear hyperbolic PDEs (cf. e.g. R. Geroch, [**34**]).

7.2 - *Separation of variables for Dirac equation*

The successful treatment of the relativistic hydrogen atom problem (cf. P. A. M. Dirac, [**27**]) relies on the fact that Dirac's equation on Minkowski space can be solved by separation of variables (in spherical coordinates). The purpose of this section is to review the basic geometric structure needed to study the separability properties of the massive charged Dirac equation on a curved space-time

$$\mathbf{H}_D \Psi = 0,$$

$$\mathbf{H}_D \Psi \equiv \left[i\, \gamma^\mu \, (D_\mu - i\, e\, A_\mu) - \sqrt{2}\, m_e\, I \right] \Psi, \tag{91}$$

where $\{\gamma^\mu\}$ is a set of Dirac matrices associated to a Lorentzian metric $g_{\mu\nu}$, and D_μ denotes covariant differentiation with respect to a connection (cf. e.g. A. Lichnerowicz, [**56**]) on four-spinors corresponding to the choice of γ^μ and to the Levi-Civita connection of $g_{\mu\nu}$, and where $A = A_\mu \, dx^\mu$ is a differential 1-form. We follow the work by N. Kamran and R. G. McLenaghan (cf. [**46**]) and argue that the needed geometric background is a space-time admitting a 2-parameter abelian orthogonally transitive isometry group and a pair of shearfree geodesic null congruences. Precisely we consider the class of Lorentzian metrics g of the form

$$g = 2\left(\Theta^1 \odot \Theta^2 - \Theta^3 \odot \Theta^4\right) \tag{92}$$

where

$$\Theta^1 = \frac{\left|Z(w,x)\right|^{1/2}}{\sqrt{2}\, T(w,x)} \left\{ \frac{f\, W(w)}{Z(w,x)} \left[\epsilon_1\, du + m(x)\, dv\right] + \frac{1}{g^2\, W(w)}\, dw \right\}, \tag{93}$$

$$\Theta^2 = \frac{\left|Z(w,x)\right|^{1/2}}{\sqrt{2}\, T(w,x)} \left\{ \frac{W(w)}{Z(w,x)} \left[\epsilon_1\, du + m(x)\, dv\right] - \frac{f}{g^2\, W(w)}\, dw \right\}, \tag{94}$$

$$\Theta^3 = \frac{\left|Z(w,x)\right|^{1/2}}{\sqrt{2}\, T(w,x)} \left\{ \frac{X(x)}{Z(w,x)} \left[\epsilon_2\, du + p(w)\, dv\right] - \frac{i}{X(x)}\, dx \right\}, \tag{95}$$

$$\Theta^4 = \overline{\Theta^3}\,, \tag{96}$$

and

$$A = \frac{T(w,x)}{\sqrt{2}\left|Z(w,x)\right|^{1/2}} \left\{ \frac{H(w)}{g^2\, W(w)} \left(f\, \theta^1 + \theta^2\right) + \frac{G(x)}{X(x)} \left(\theta^3 + \theta^4\right) \right\}, \tag{97}$$

(98) $$Z(w,x) = \epsilon_1\, p(w) - \epsilon_2\, m(x), \quad g = \frac{1}{\sqrt{2}}\left(1+f^2\right)^{1/2}.$$

Here $\epsilon_1, \epsilon_2, f \in \mathbb{R}$ are constants with $\left(\epsilon_1,\, \epsilon_2\right) \neq (0,0)$. Also $\left(x^\mu\right) \equiv \left(u,\, v,\, w,\, x\right)$. All functions involved are real valued.

By a result of R. Debever (cf. [**26**]) the class of Lorentzian metrics (92) consists precisely of all Lorentzian metrics that admit

i) a 2-dimensional abelian group of local isometries acting orthogonally and transitively,

ii) two shear-free congruences of null geodesics.

Let $\mathfrak{D}$ be the class of all solutions to Einstein's vacuum and electrovac field equations with cosmological constant

(99) $$R_{\mu\nu} - \frac{1}{2}\, R\, g_{\mu\nu} + \Lambda\, g_{\mu\nu} = F_{\mu\rho}\, F_\nu{}^\rho - \frac{1}{4}\, g_{\mu\nu}\, F_{\sigma\rho}\, F^{\sigma\rho},$$

(100) $$\nabla^\nu F_{\mu\nu} = 0, \quad \nabla_{[\sigma} F_{\mu\nu]} = 0, \quad F_{\mu\nu} = 2\, \nabla_{[\mu} A_{\nu]},$$

[where we have set as customary

$$\nabla_{[\sigma} F_{\mu\nu]} \equiv \nabla_\sigma F_{\mu\nu} + \nabla_\nu F_{\sigma\mu} + \nabla_\mu F_{\nu\sigma},$$
$$\nabla_{[\mu} A_{\nu]} \equiv \nabla_\mu A_\nu - \nabla_\nu A_\mu,$$

and we allow for the cosmological constant and the electromagnetic tensor field to vanish i.e. we allow for $\Lambda = 0$ and $F_{\mu\nu} = 0$] that satisfy the following requirements

(H1) The Weyl tensor is everywhere of Petrov type D,

(H2) If the Maxwell tensor $F_{\mu\nu}$ is nonzero then it is nonsingular and its principal null directions are aligned with the repeated principal null directions of the Weyl tensor,

(H3) The assumptions in the generalized Goldberg-Sachs theorem are satisfied, insuring that the null congruences associated to the principal directions of the Weyl tensor are geodesic and shearfree.

Theorem 7.2. *For every $(g,\, F) \in \mathcal{D}$ and every point $x_0 \in \mathfrak{M}$ there is a local coordinate system $(U,\, x^\mu)$, $\left(x^\mu\right) \equiv \left(u,\, v,\, w,\, x\right)$, and a local frame $\{\Theta^a : 1 \leq a \leq 4\} \subset C^\infty(U,\, T^*(\mathfrak{M}) \otimes \mathbb{C})$ of complex valued null 1-forms such that $x_0 \in U$ and*

$$g = 2\left(\Theta^1 \odot \Theta^2 - \Theta^3 \odot \Theta^4\right),$$
$$F = B(w,x)\left(\Theta^1 \wedge \Theta^2 - \Theta^3 \wedge \Theta^4\right),$$

for some complex valued C^∞ function $B(w,x)$, where the 1-forms Θ^a and the real vector potential A are respectively given by (93)–(96) *and* (97)–(98).

The result is due to R. Debever and N. Kamran and R. G. MacLenaghan (cf. Theorem 1 in [**46**], p. 1020). Theorem 7.2 is the reason for which separability properties of Dirac's equation $\mathbf{H}_D \Psi = 0$ are studied for solutions $(g,\, F) \in \mathfrak{D}$ to equations (99)–(100). Equations (99)–(100) were solved by the same authors, producing the general solution expressed in terms of the tetrad $\{\Theta^a\}$ and the local coordinates (u, v, w, x) in Theorem 7.2. We recall

Theorem 7.3 (N. Kamran and R. G. MacLenaghan, [**46**]). *Let $(g, F) \in \mathcal{D}$. The following statements are equivalent*

1) *The Dirac equation* (91) *admits a solution of the form*

$$\Psi = e^{i(\alpha\, u + \beta\, v)}\, T^{3/2}\, Z^{-1/4} \begin{pmatrix} e^{i\,\mathcal{B}}\, R_1(x)\, S_2(w) \\ e^{i\,\mathcal{B}}\, R_2(x)\, S_1(w) \\ e^{-i\,\mathcal{B}}\, R_1(x)\, S_1(w) \\ e^{-i\,\mathcal{B}}\, R_2(x)\, S_2(w) \end{pmatrix} \tag{101}$$

where $\alpha, \beta \in \mathbb{R}$ are arbitrary constants and where

$$d\mathcal{B} = \frac{1}{4Z}\, \left[\epsilon_1\, m'(x)\, dw + \epsilon_2\, p'(w)\, dx\right]. \tag{102}$$

2) i) *The Petrov type* D *condition* (H1) *is satisfied.*

ii) *There exist real valued functions $h(w)$ and $g(x)$ such that*

$$Z^{1/2}\, T^{-1}\, e^{2\,i\,\mathcal{B}} = h(w) + i\, g(x). \tag{103}$$

It should be emphasized that under the requirement (H1) the equation (102) possesses solutions $\mathcal{B}$. The relevance of the CR manifolds M_a $(a \in \{1, 2\})$ associated to the two shearfree null geodesic congruences [such as springing from assumptions (H1)–(H3) above] is as yet unclear. The detailed analysis of the CR and pseudohermitian geometry of M_a is missing in the present mathematical physics literature.

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Elisabetta Barletta
Università degli Studi della Basilicata
Dipartimento di Matematica, Informatica ed Economia
Via dell'Ateneo Lucano 10
Potenza, 85100, Italy
e-mail: elisabetta.barletta@unibas.it

Sorin Dragomir
Università degli Studi della Basilicata
Dipartimento di Matematica, Informatica ed Economia
Via dell'Ateneo Lucano 10
Potenza, 85100, Italy
e-mail: sorin.dragomir@unibas.it