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From complex to real analytic geometry

Abstract. In this survey paper we look at the emergence of a good notion of real analytic space. Then we consider the global Nullstellensätze that have been proved both in the complex and in the real case. Finally we look at relations between a real global Nullstellensatz and Hilbert's 17^{th} problem for global analytic functions and we give some conclusions.

Keywords. Real analytic space, Global Nullstellensatz, 17^{th} Hilbert problem.

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This text resumes the talk given by the second author in the Conference in honour of Giuseppe Tomassini, who was our advisor of master theses in 1971.

Giuseppe Tomassini is a specialist in Complex Analysis, but we deal with real analytic geometry because at the beginning of his career he worked also in real analytic geometry, in particular with our common friend, Alberto Tognoli.

The paper is divided in three parts.

Part 1: Emergence of the notion of real analytic space

The theory of real analytic spaces was born in the fifties of the last century, when the complex theory was fully developed by people as Oka, Cartan, Whitney, Bruhat, Grothendieck, Rückert, Remmert, etc.

Complex analysis is the study of holomorphic functions and their zero sets.

A central notion is the one of analytic space, in particular of a Stein space. Among the most important results in this setting are Cartan's theorems A and B for coherent sheafs on Stein spaces.

The passage from $\mathbb{C}$ to $\mathbb{R}$ in the theory of analytic spaces quickly presented some criticism. Cartan remarked that taking the same definition for real analytic spaces as in the complex case, the resulting cathegory had a lot of pathological examples: for instance the structural sheaf is not coherent in general, so that theorems A and B cannot hold true. Also there is not a clear notion of irreducible components and irreducible objects may have not constant dimension, as already appears in real algebraic geometry.

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In this situation, for instance Grothendieck said: *L'intérêt des espaces analytiques, lorsque k n'est pas algébriquement clos, est d'ailleurs douteux.* (See [**Gr**])

Cartan did not agree. Even if he found the worst examples, see [**BC1**], [**BC2**], he tried to find what were the obstructions to get a good cathegory.

He remarked that a lot of bad examples were constructed using the local nature of the definition.

> *"...la seule notion de sous-ensemble analytique réel (d'une variété analytique-réelle V) qui ne conduise pas à des propriétés pathologiques doit se référer à l'espace complexe ambiant: il faut considérer les sous-ensembles fermés E de V tels qu'il existe une complexification W de V et un sous-ensemble analytique-complexe E' de W, de manière que $E = W \cap E'$. On démontre que ce sont aussi les sous-ensembles de V qui peuvent être définis globalement par un nombre fini d'équations analytiques. La notion de sous-ensemble analytique-réel a ainsi un caractère essentiellement global, contrairement à ce qui avait lieu pour les sous-ensembles analytiques-complexes."* (see [**C3**]).

Cartan uses complex notions to describe real pathologies: for instance he defines the *complexification* of a germ of real analytic space V_x at a point $x \in \mathbb{R}^n$ and proves that V_x is coherent if and only if the complexification of V_x induces the complexification of V_y on points y close to x.

Also Cartan proved that Theorems A and B are preserved by taking direct limits in the sense that if a closed set $X \subset \mathbb{R}^n$ has a fundamental system of Stein neighbourhoods in $\mathbb{C}^n$ then Theorems A and B hold true for X.

All these considerations led Cartan to the following characterization of what a good cathegory of real analytic sets in $\mathbb{R}^n$ should be.

For a real analytic subset $X \subset \mathbb{R}^n$ the following sentences are equivalent.

1. The set X is the zero set of finitely many real analytic functions.
2. There is a coherent ideal subsheaf of $\mathcal{O}_{\mathbb{R}^n}$ whose zero set is X.
3. There is an open neighbourhood Ω of $\mathbb{R}^n$ in $\mathbb{C}^n$ and a closed subspace $Y \subset \Omega$ such that $Y \cap \mathbb{R}^n = X$.

Since $\mathbb{R}^n$ has a fundamental system of Stein neighbourhoods in $\mathbb{C}^n$ and complex analytic sets in a Stein open set do the same, for real analytic subsets of $\mathbb{R}^n$ having the properties above Theorems A and B hold true. See [**C2**].

Bruhat and Whitney extend the notion of complexification to analytic manifolds and introduce the name *C-analytic* for analytic subsets of a real analytic manifold M induced by intersection with M of an analytic subset of the complexification of M. Finally Tognoli extended the notion of complexification of a real analytic space admitting a coherent structure, in analogy with condition 2 of Cartan. See [**WB**], [**T**].

Tognoli distinguished 3 types of real analytic spaces: the coherent ones, whose reduced structure is coherent, the ones carrying one (or several) coherent structure (for instance Whitney and Cartan umbrellas) and the ones not admitting any coherent structure (wild examples of Cartan, Bruhat-Cartan, etc, see [**BC1**], [**BC2**]).

This way emerges the first idea: globality. A *good* real analytic space is the zero set of finitely many global equations in a suitable ambient space.

Remark 1. 1) An important difference between the complex and the real case is that while the algebra of holomorphic functions on an open set of $\mathbb{C}^n$ is closed in the algebra of continuous complex valued functions with respect to the compact open topology, the algebra of real analytic functions is dense in the algebra of continuous functions both in the compact open and in the stronger Whitney topology (Whitney theorem). This implies on the one side that approximation is a standard trick in real analytic geometry, on the other side in a limit process to get an analytic result one has to make the limit process in $\mathbb{C}$, and then go back to $\mathbb{R}$.

2) Note that in 1958 Grauert (see [**Gra**]) proved that a Stein manifold is analytically isomorphic to a closed complex submanifold of $\mathbb{C}^{2n+1}$, where n is the complex dimension of the manifold. This implies that a real analytic manifold M admits a proper analytic injective map $f : M \to \mathbb{R}^{2n+1}$ which has maximum rank n at all points of M.

Part 2. Global Nullstellensätze

Looking at global rings of analytic functions, some difficulties appear even in the complex setting. Take as an example a theorem as the Nullstellensatz.

The algebra $A = H^0(X, \mathcal{O}_X)$ of holomorphic functions on a Stein space does not have good algebraic properties: it is not noetherian, nor factorial. Moreover there are in A proper prime ideals with empty zero set. For such ideals cannot be true a Nullstellensatz similar to the one by Hilbert in the polynomial case.

Forster in [**Fo**], 1964, proved a Nullstellensatz for A using the fact that A becomes a Frechet space when endowed with the compact open topology, so that one can define *closed ideals*. A characterization by Cartan says that an ideal $\mathfrak{a} \subset A$ is closed if and only if $\mathfrak{a} = H^0(X, \mathfrak{a}\mathcal{O}_X)$.

First of all Forster proves that for a closed primary ideal Hilbert's Nullstellensatz holds true.

Theorem 1. *Let* $\mathfrak{q}$ *be a closed primary ideal in* A. *Then* $V(\mathfrak{q}) \neq \emptyset$ *and* $\mathcal{I}(V(\mathfrak{q})) = \{f \in A : f \text{ vanishes on the zero set of } \mathfrak{q}\} = \sqrt{\mathfrak{q}}$. *Moreover there is a positive integer* n *such that* $\sqrt{\mathfrak{q}}^n \subset \mathfrak{q}$.

Also he proves that a closed ideal admits a primary decomposition, similar to the one in noetherian rings, with the same good properties, even if in general this decomposition is not finite, but only locally finite, that is, the zero sets of these closed primary ideals are a locally finite family.

The second result of Forster can be resumed in this way

Theorem 2. *Let* $\mathfrak{a}$ *be a closed ideal in* A. *Then* $\mathfrak{a} = \bigcap_i \mathfrak{q}_i$ *where all* $\mathfrak{q}_i$ *are closed primary ideals, the decomposition is irredundant and the prime ideals*

$\mathfrak{p}_i = \sqrt{\mathfrak{q}_i}$ *are pairwise distinct. Let* n_i *be the smallest positive integer such that* $\mathfrak{p}_i^{n_i} \subset \mathfrak{q}_i$. *Then* $\mathcal{I}(V(\mathfrak{a})) = \sqrt{\mathfrak{a}}$ *if and only if the sequence* $\{n_i\}$ *is bounded.*

Remark 2. Note that in Forster's result there is a numerical function that controls whether Nullstellensatz holds true or not, namely the exponent n_i.

A similar primary decomposition was found by P. de Bartolomeis for saturated ideals in the algebra of global analytic functions on a C-analytic space. Here we call *saturated* an ideal $\mathfrak{a}$ such that $\mathfrak{a} = H^0(X, \mathfrak{a}\mathcal{O})$ (see [**dB1**]).

To find results in the real case one has to wait till 1970 when Risler proved a Nullstellensatz for real polynomials, namely.

Theorem 3. *Consider the algebra* $A = \mathbb{R}[x_1, \ldots, x_n]$ *and let* $\mathfrak{a} \subset A$ *be an ideal. Then* $\mathcal{I}(V(\mathfrak{a})) = \sqrt[r]{\mathfrak{a}} = \{f \in \mathfrak{a} : f^{2m} + \sum_i a_i^2 \in \mathfrak{a}\}$ [1].

Risler proved a completely analogous result for the algebra of real convergent power series. For both results see [**R2**].

Now we come to a global real Nullstellensatz (see [**ABF2**]) where, besides globality, positivity comes out.

First we tried to find the same result as Risler without success. We had to find another way.

In the real setting the classical Łojasievicz inequality compares the local behaviour of two analytic functions f and g such that $V(f) \subset V(g)$. More precisely.

Theorem 4. *Let* K *be a compact set in* $\mathbb{R}^n$ *and* f, g *be analytic functions such that* $V(f) \cap K \subset V(g) \cap K$. *Then there are an integer* k *and a constant* c *such that* $|g^k| \leq c|f|$ *on* K.

The first step is to extend this result to a global situation. The main idea is that the integer k should work in a neighbourhood of the zero set of f minus a closed analytic subset where the multiplicity of f increases. In fact we get.

Theorem 5 (see [**ABS**]). *Let* f, g *be analytic on* $\mathbb{R}^n$ *and such that* $V(f) \subset V(g)$. *Then for each compact set* $K \subset \mathbb{R}^n$ *there are a positive integer* k *and an analytic function* h *not vanishing on* K *such that* $|g^k \cdot h| \leq |f|$ *on* $\mathbb{R}^n$.

This result allowed us to have a Nullstellensatz considering another radical.

Namely, take a saturated ideal $\mathfrak{a}$, consider its convex hull

$$\mathfrak{b} = \{f : \exists h \in \mathfrak{a} \,\text{such that}\, |f| \leq |h|\},$$

then take its radical $\sqrt{\mathfrak{b}}$. This is a real radical ideal and it is easy to prove that it contains the real radical $\sqrt[r]{\mathfrak{a}}$. This ideal is called *Łojasiewicz radical of* $\mathfrak{a}$ and denoted as $\sqrt[\text{Ł}]{\mathfrak{a}}$. It is a natural candidate to be $\mathcal{I}(V(\mathfrak{a}))$ because its elements vanish on the zero set of $\mathfrak{a}$. In fact we prove

[1] An ideal $\mathfrak{a} \subset A$ is a *real* ideal if $a_1^2 + \cdots + a_k^2 \in \mathfrak{a}$ implies $a_i \in \mathfrak{a}$ for each $i = 1, \ldots, k$. If $\mathfrak{a}$ is a real ideal then it is radical and $\sqrt[r]{\mathfrak{a}} = \mathfrak{a}$.

Theorem 6. *Let* $\mathfrak{a}$ *be a saturated ideal in* $\mathcal{O}(\mathbb{R}^n)$*. Then* $\mathcal{I}(V(\mathfrak{a}))$ *is the saturation of* $\sqrt[Ł]{\mathfrak{a}}$.

The idea of the proof is the following. Let g be a function vanishing on the zero set of the ideal $\mathfrak{a}$. If we are able to find a function $f \in \mathfrak{a}$ whose zero set is the zero set of $\mathfrak{a}$ (for instance if $\mathfrak{a}$ is a principal ideal), we are reduced to compare the zero set of g with the one of f and we can apply the global Łojasiewicz inequality. So, we get that for all $x \in \mathbb{R}^n$ the germ g_x belongs to the stalk of the ideal sheaf generated by the Łojasiewicz radical of the ideal $\mathfrak{a}$, hence g belongs to its saturation and we are done.

If $\mathfrak{a}$ is finitely generated we can take as f the sum of squares of the generators of $\mathfrak{a}$. If $\mathfrak{a}$ has countably many generators, say $f_1, f_2, \dots$, we can find a suitable sequence of analytic functions a_i in such a way that the series $\sum_i a_i^2 f_i^2$ converges to an analytic function f. Of course if $\mathfrak{a}$ is not saturated f does not belong to $\mathfrak{a}$ but it belongs to its saturation. This is the reason why we consider only saturated ideals. And that's all.

Part 3. Relations between real Nullstellensatz and Hilbert's 17^{th} problem

At this point a natural question is whether the real radical and the Łojasiewicz radical of an ideal $\mathfrak{a}$ coincide or not.

Here it is where Hilbert's 17^{th} problem enters in our scene. In other words, as positivity is expressed as *convexity* in Łojasiewicz inequality, here we want to express positivity in terms of *sums of squares*.

Let us recall what Hilbert's 17^{th} problem asks for.

Problem. Let $p \in \mathbb{R}[x_1, \dots, x_n]$ be a polynomial which is ≥ 0 at any point $x \in \mathbb{R}^n$. Is p a sum of squares of polynomials or at least a sum of squares in the field of rational functions $\mathbb{R}(x_1, \dots, x_n)$?

Of course the same question can be asked in any ring of real functions on $\mathbb{R}^n$.

In 1927 Artin gave a positive answer in the field of rational functions. Later the same result was proved for the ring of germs of meromorphic functions in n variables. ([**Ar**], [**Las**]).

For global analytic functions the answer is known only in some particular cases, depending on the zero set of the function. See for instance [**BKS**] for a discrete zero set, [**Jw**] and [**Rz**] for a compact zero set, [**ABFR**] when the zero set is a countable union of disjoint compact sets, [**ABF1**] for further reductions.

In any case the better result one can expect is with denominators, that is with meromorphic functions.

What we are able to prove is the following.

Theorem 7. *Let* $\mathfrak{a} \subset \mathcal{O}(\mathbb{R}^n)$ *be a saturated ideal and let* X *be its zero set. Assume that any positive semidefinite analytic function having* X *as its zero set is a sum of squares of meromorphic functions. Then* $\sqrt[Ł]{\mathfrak{a}} = \sqrt[r]{\mathfrak{a}}$

We give a sketch of the proof in the case $\mathfrak{a}$ is saturated, prime and real. Indeed, the case $\mathfrak{a}$ saturated and real reduces easily to the previous case and the general case of a saturated ideal follows considering its real radical.

Consider the sheaf of ideals $\mathfrak{a}\mathcal{O}_{\mathbb{R}^n}$ generated by $\mathfrak{a}$. There is an invariant open Stein neighbourhood Ω of $\mathbb{R}^n$ in $\mathbb{C}^n$ to which this sheaf extends to a coherent sheaf of ideals $\mathcal{I}$. For all $x \in \mathbb{R}^n$ one has by construction $\mathcal{I}_x = \mathfrak{a}\mathcal{O}_{\mathbb{R}^n,x} \otimes \mathbb{C}$. Since $\mathfrak{a}$ is prime and real, one can prove that the ideal $\mathfrak{p} = H^0(\Omega, \mathcal{I})$ is a closed prime ideal.

Assume by contradiction that there is $g \in \sqrt[\text{Ł}]{\mathfrak{a}} \setminus \mathfrak{a}$. Then, after shrinking Ω if necessary, g extends to a holomorphic function G on Ω and $G \notin \mathfrak{p}$ because $g \notin \mathfrak{a}$.

Hence if we denote by Z the zero set of $\mathfrak{p}$, by Forster's Nullstellensatz, G does not vanish on Z and this implies there is a real point $x_0 \in \mathbb{R}^n$ such that G does not vanish on Z_{x_0}, that is Z_{x_0} is not included in $V(G)_{x_0}$.

Next, since $\mathfrak{a}$ is saturated, there is $f \in \mathfrak{a}$, which we may assume to be positive semidefinite, such that $V(f) = V(\mathfrak{a})$. Since g vanishes on $V(f)$ and $\{x_0\}$ is compact, there is a positive integer m and a function h such that $h(x_0) \neq 0$ and $f_0 = f - h^2g^{2m} \geq 0$. Taking $f_1 = f - h_1^2g^{2m}$ instead of f_0, where $h_1 = \dfrac{h}{\sqrt{1+h^2g^{2m}}}$, it is easy to check that $V(f_1) = V(f) = V(\mathfrak{a}) = X$. So we can assume that f_1 is a sum of squares of meromorphic functions, that is $b^2(f - h_1^2g^{2m}) = \sum a_i^2$. Since $f \in \mathfrak{a}$ this implies $b^2h_1^2g^{2m} + \sum a_i^2 \in \mathfrak{a}$. We could conclude $g \in \sqrt[r]{\mathfrak{a}}$ if we had $b \notin \mathfrak{a}$ because h_1, which does not vanish on x_0, does not belong to $\mathfrak{a}$. Indeed, assume $b \notin \mathfrak{a}$. Then, $bh_1g^m \in \mathfrak{a}$ implies $g^m \in \mathfrak{a}$, and finally $g \in \mathfrak{a}$ because $\mathfrak{a}$ is prime.

So, we need to modify b in order to ensure it does not belong to $\mathfrak{a}$. To do this we use that Z_{x_0} is not included in $V(G)_{x_0}$. Note that if F_1 is a holomorphic extension of f_1 again Z_{x_0} is not included in $V(F_1)_{x_0}$. Indeed, assume F_1 vanishes on Z_{x_0}, since $F \in \mathfrak{p}$ and $H_1(x_0) \neq 0$, we deduce

$$Z_{x_0} \subset V(F)_{x_0} \cap V(F_1))_{x_0} = V(F - F_1)_{x_0} = V(H_1^2G^{2m})_{x_0} = V(G)_{x_0},$$

which is a contradiction.

Now, we have two holomorphic functions B and F_1 and a complex analytic set Z whose germ at x_0 is not included in $V(F_1)_{x_0}$.

In this situation we prove that there exists an analytic diffeomorphism $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ such that

1. $f_1 \circ \varphi = uf_1$ for some positive unit $u \in \mathcal{O}(\mathbb{R}^n)$.
2. If B_1 is a holomorphic extension of $b_1 = b \circ \varphi$ then Z_{x_0} is not included in $V(B_1)_{x_0}$.

Next, take a positive unit $v \in \mathcal{O}(\mathbb{R}^n)$ such that $v^2 = u^{-1}$. Finally

$$b_1^2 f = b_1^2h_1^2g^{2m} + b_1^2f_1 = b_1^2h_1^2g^{2m} + \sum_{i=1}^{k}(a_i \circ \varphi)^2.$$

Now, $b_1^2 f \in \mathfrak{a}$ while $b_1^2 h_1^2 \notin \mathfrak{a}$ because Z_{x_0} is not included in their complex zero set, hence g belongs to the real radical of $\mathfrak{a}$ which is $\mathfrak{a}$, as required.

We think that, as the representation of positive semidefinite polynomials as sums of squares is a central point in real algebraic geometry, a similar representation for positive semidefinite analytic functions would be a central point in real analytic geometry: but this is still an open problem.

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