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On a nonhomogeneous sublinear-superlinear fractional equation in $\mathbb{R}^N$

Abstract. The existence of a positive solution to a nonhomogeneous fractional sublinear-superlinear problem in the whole space is proved by combining a minimization method, Nehari manifold and the fibering map methods, and the concentration-compactness lemma. We also study the continuity of solutions in the perturbation parameter f at 0.

Keywords. Fractional Laplacian, Sublinear growth, Nehari manifold, Concentration-compactness.

Mathematics Subject Classification (2010): 35A15, 35J60, 35R11, 45G05.

1 - Introduction

In this paper we focus our attention on the existence of positive solutions to the following nonhomogeneous sublinear-superlinear fractional Laplacian problem

$$\begin{cases} (-\Delta)^s u + |u|^{r-2}u = |u|^{q-2}u + f & \text{in } \mathbb{R}^N, \\ u \in H^s(\mathbb{R}^N) \cap L^q(\mathbb{R}^N), \end{cases} \tag{1.1}$$

with $s \in (0,1)$, $N > 2s$, $1 < q < 2 \leq r < 2^*_s$, where $2^*_s = \dfrac{2N}{N-2s}$ is the fractional critical exponent, and f is a perturbative term that satisfies

$$f \in L^{\frac{q}{q-1}}(\mathbb{R}^N) \cap L^{\frac{2^*_s}{2^*_s-1}}(\mathbb{R}^N) \text{ and } f > 0 \text{ a.e. in } \mathbb{R}^N. \tag{f}$$

The nonlocal operator $(-\Delta)^s$ appearing in (1.1) is the well-known fractional Laplacian operator which is defined for any $u : \mathbb{R}^N \to \mathbb{R}$ smooth enough by

$$(-\Delta)^s u(x) = c_{N,s} P.V. \int_{\mathbb{R}^N} \frac{u(x)-u(y)}{|x-y|^{N+2s}}\, dy,$$

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where

$$c_{N,s} = \left(\int_{\mathbb{R}^N} \frac{1 - \cos(x_1)}{|x|^{N+2s}} \, dx \right)^{-1}.$$

We can also give the definition of fractional Laplacian by means of Fourier transform

$$\mathcal{F}((-\Delta)^s u)(k) = |k|^{2s} \mathcal{F}(u)(k), \quad k \in \mathbb{R}^N,$$

for any u belonging to the Schwarz class $\mathcal{S}(\mathbb{R}^N)$. For more details on the fractional Laplacian we refer the interested reader to [**21**, **34**].

One of the main reasons to face with problem (1.1) comes from the study of standing wave solutions, that is solutions of the form $\psi(t,x) = e^{-\imath ct} u(x)$ where c is a constant, to the nonlinear time-dependent fractional Schrödinger equation

$$\imath \frac{\partial \psi}{\partial t} = (-\Delta)^s \psi + V(x)\psi - g(x, |\psi|), \quad (t,x) \in \mathbb{R} \times \mathbb{R}^N, \tag{1.2}$$

where V is an external potential and g is a suitable nonlinearity. Equation (1.2) has been introduced by Laskin in [**31**, **32**] due to its relevance in fractional quantum mechanics in the study of particles on stochastic fields modeled by Lévy processes; see [**19**] for more details. When $s = 1$, eq. (1.1) reduces to a classical nonlinear Schrödinger equation of the form

$$-\Delta u + V(x) u = g(x,u) \quad \text{in } \mathbb{R}^N, \tag{1.3}$$

which has been extensively studied in the last twenty years by many authors; see [**3**, **11**, **13**, **26**, **37**] and the references therein for some existence and multiplicity results under different assumptions on the potential V and the nonlinearity g. In particular, a great interest [**1**, **2**, **10**, **12**, **15**, **17**, **30**, **41**, **42**] has been devoted to the study of (1.3) when $g(x,u) = f(x,u) + h(x)$ and $h \not\equiv 0$ is a suitable integrable function. In this case, (1.3) does not admit the trivial solution and its study is rather tricky. Indeed, while there are some general methods to study the analogue of (1.3) in bounded domains (see [**9**, **36**, **39**, **40**]), these arguments break down in the whole space due to the lack of the compactness of Sobolev embedding, and then a more delicate investigation is needed to obtain existence and multiplicity results.

Coming back to the fractional setting, we would like to point out that recently a remarkable attention has been devoted to the study of fractional Schrödinger equations. Indeed, several existence and multiplicity results have been established. For instance, Felmer, Quaas and Tan [**24**] (see also [**22**]) studied the existence and symmetry of positive solutions to $(-\Delta)^s u + u = f(u)$, when

f has subcritical growth and satisfies the Ambrosetti-Rabinowitz condition. Dávila, del Pino and Wei [**20**] proved the existence of standing-wave solutions to a fractional nonlinear Schrödinger equation by using the Lyapunov-Schmidt reduction method. Figueiredo and Siciliano [**25**] obtained a multiplicity result for a fractional Schrödinger equation via Ljusternick-Schnirelmann and Morse theory. Ambrosio [**7**] considered the existence of positive solutions for a fractional Schrödinger equation under the assumption that the nonlinearity is either asymptotically linear or superlinear at infinity.

Differently from the local case, only few papers considered fractional Schrödinger equations in presence of a perturbative term. For instance, Pucci, Xiang and Zhang [**35**] investigated the existence of multiple solutions for a nonhomogeneous fractional p-Laplacian equation of Schrödinger-Kirchhoff type involving a nonlinearity satisfying the Ambrosetti-Rabinowitz condition, a positive potential V satisfying suitable assumptions, and in presence of a $L^{\frac{p}{p-1}}$-perturbation term; see also [**6**, **27**] and the references therein for some interesting results involving $(-\Delta)^s_p$. Ambrosio and Hajaiej [**8**] proved the existence of at least two positive solutions to the following fractional Schrödinger equation $(-\Delta)^s u + u = k(x)f(u) + h(x)$, provided that $|h|_2$ is sufficiently small. In [**28**] the author studied the existence and uniqueness of a positive solution for a sublinear fractional equation with a L^2-perturbation term; see also [**29**] for a multiplicity result for this kind of problems. We also mention [**4**, **18**, **38**] for some multiplicity results for nonhomogeneous fractional problems in bounded domains, whose techniques are not adaptable in our situation due to the unboundedness of the domain.

Motivated by the above results, in this paper we aim to continue the study started in [**28**,**29**] related to nonhomogeneous fractional problems. In this paper, we deal with sublinear-superlinear nonlinearities. Our main result can be stated as follows:

Theorem 1.1. *Assume that* (f) *holds true. Then, problem* (1.1) *has a positive weak solution* $u \in H^s(\mathbb{R}^N) \cap L^q(\mathbb{R}^N)$ *that converges to zero in* $H^s(\mathbb{R}^N) \cap L^q(\mathbb{R}^N)$ *as* $|f|_{L^{\frac{q}{q-1}}(\mathbb{R}^N)}$ *tends to zero.*

Problem (1.1) involves the fractional Laplacian $(-\Delta)^s$, with $0 < s < 1$, which is a nonlocal operator. To study these kind of problems many authors used the Caffarelli-Silvestre [**14**] extension method which permits to transform (1.1) into a local degenerate elliptic equation in one more dimension with a nonlinear Neumann boundary condition. Anyway, in this paper we prefer to investigate (1.1) directly in $H^s(\mathbb{R}^N)$ via suitable variational methods, in order to resemble some arguments developed to study the case $s = 1$. Clearly, a more accurate inspection will be needed with respect to the classical case and some ideas contained in [**28**, **29**] will play a fundamental role to achieve the desired result. In order to investigate the existence of positive solutions for (1.1) we will combine a minimization argument, Nehari manifold and the fibering map methods [**23**], and the concentration-compactness by Lions [**33**].

The paper is organized as follows. In Section 2 we give some preliminary results and we develop our variational arguments. Section 3 is devoted to the proof of the main result of this work.

2 - Preliminary

2.1 - *Fractional Sobolev spaces*

In this section, we give some basic properties of the fractional Sobolev spaces that will be used in this paper.

Let $1 \leq p \leq \infty$ and $A \subset \mathbb{R}^N$. We denote by $|u|_{L^p(A)}$ the $L^p(A)$-norm of a function $u : \mathbb{R}^N \to \mathbb{R}$ belonging to $L^p(A)$. When $A = \mathbb{R}^N$, we simply write $|u|_p$.

For any $s \in (0,1)$ we define $\mathcal{D}^{s,2}(\mathbb{R}^N)$ as the completion of the $\mathcal{C}_c^\infty(\mathbb{R}^N)$-functions under the norm

$$[u]_s^2 = \iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\, dxdy,$$

and we denote by $H^s(\mathbb{R}^N)$ the Sobolev space defined by

$$H^s(\mathbb{R}^N) = \left\{ u \in L^2(\mathbb{R}^N) : \frac{|u(x)-u(y)|}{|x-y|^{\frac{N+2s}{2}}} \in L^2(\mathbb{R}^{2N}) \right\}$$

endowed with the natural norm

$$\|u\|_{H^s(\mathbb{R}^N)}^2 = [u]_s^2 + |u|_2^2.$$

We recall the following embeddings of the fractional Sobolev spaces into Lebesgue spaces.

Theorem 2.1 ([**21**]). *Let $s \in (0,1)$ and $N > 2s$. Then there exists a sharp constant $S_* > 0$ such that for any $u \in \mathcal{D}^{s,2}(\mathbb{R}^N)$*

$$|u|_{2_s^*}^2 \leq S_*^{-1} [u]_s^2.$$

Moreover $H^s(\mathbb{R}^N)$ is continuously embedded in $L^t(\mathbb{R}^N)$ for any $t \in [2, 2_s^*]$ and compactly in $L^t_{loc}(\mathbb{R}^N)$ for any $t \in [1, 2_s^*)$.

Lemma 2.1 ([**24**]). *Let $N > 2s$ and $r \in [2, 2^*_s)$. If $\{u_n\}_{n\in\mathbb{N}}$ is a bounded sequence in $H^s(\mathbb{R}^N)$ and if*

$$\lim_{n\to\infty} \sup_{y\in\mathbb{R}^N} \int_{B_R(y)} |u_n|^r \, dx = 0$$

where $R > 0$, then $u_n \to 0$ in $L^t(\mathbb{R}^N)$ for all $t \in (2, 2^*_s)$.

Now we introduce the space $\mathbb{E} := H^s(\mathbb{R}^N) \cap L^q(\mathbb{R}^N)$ endowed with the norm

$$\|u\| := \|u\|_{\mathbb{E}} = [u]_s + |u|_q.$$

The functional associated with (1.1) is given by

$$\mathcal{I}(u) = \frac{1}{2}[u]_s^2 + \frac{1}{r}|u|_r^r - \frac{1}{q}|u|_q^q - \int_{\mathbb{R}^N} f u \, dx.$$

It is easy to see that $\mathcal{I} \in \mathcal{C}^1(\mathbb{E}, \mathbb{R})$ and its differential $\mathcal{I}'$ is given by

$$\langle \mathcal{I}'(u), \varphi \rangle = \iint_{\mathbb{R}^{2N}} \frac{(u(x) - u(y))(\varphi(x) - \varphi(y))}{|x - y|^{N+2s}} \, dx dy + \int_{\mathbb{R}^N} |u|^{r-2} u \varphi \, dx - \int_{\mathbb{R}^N} |u|^{q-2} u \varphi \, dx - \int_{\mathbb{R}^N} f \varphi \, dx,$$

for any $u, \varphi \in \mathbb{E}$. Clearly, the solutions to (1.1) correspond to critical points of $\mathcal{I}$ in $\mathbb{E}$.

2.2 - *Minimization argument*

Before considering (1.1), we investigate the following homogeneous problem

$$\begin{cases} (-\Delta)^s u + |u|^{r-2} u = |u|^{q-2} u \text{ in } \mathbb{R}^N \\ u \in H^s(\mathbb{R}^N) \cap L^q(\mathbb{R}^N). \end{cases} \tag{2.1}$$

Then, we can see that

Lemma 2.2. *Problem* (2.1) *only possesses the trivial solution in* $\mathbb{E}$.

Proof. Let u be a weak solution to (2.1). Using the Pohozaev identity for the fractional Laplacian [**5**, **16**] and recalling that $1 < q < 2$, we can see that

$$\left(\frac{N - 2s}{2N} - \frac{1}{q} \right) [u]_s^2 + \left(\frac{1}{r} - \frac{1}{q} \right) |u|_r^r = 0,$$

which implies that $u = 0$. $\square$

From now on, we will consider (1.1) looking for critical points of $\mathcal{I}$. Since $\mathcal{I}$ is not bounded below on $\mathbb{E}$, we introduce a suitable open set of $\mathbb{E}$. Take $\alpha > 1$ and define

$$\mathcal{N}_\alpha := \left\{ u \in \mathbb{E} : [u]_s^2 + |u|_r^r - \alpha |u|_q^q > 0 \quad \text{and} \quad \langle \mathcal{I}'(u), u \rangle = 0 \right\}.$$

For any $u \in \mathbb{E} \setminus \{0\}$, we introduce the fibering map $h_u : [0, +\infty) \to \mathbb{R}$ as follows:

$$h_u(t) := \mathcal{I}(tu).$$

Then we can prove that

Lemma 2.3. $\mathcal{N}_\alpha \neq \emptyset$.

Proof. Let T be the unique root of the equation

$$[Tu]_s^2 + |Tu|_r^r - \alpha |Tu|_q^q = 0. \tag{2.2}$$

Let $\psi \in \mathcal{C}_c^\infty(\mathbb{R}^N)$ such that $\psi \geq 0$ and $\psi \not\equiv 0$. Let $\sigma > 0$ be a constant and consider the function $u(x) = \psi(\sigma(x - x_0))$. Then we have

$$[u]_s^2 = \frac{1}{\sigma^{N-2s}} [\psi]_s^2 \quad \text{and} \quad |u|_t^t = \frac{1}{\sigma^N} |\psi|_t^t \quad \text{with } t \in \{q, r\}. \tag{2.3}$$

Taking into account that $f > 0$ a.e. in $\mathbb{R}^N$, there exists $x_0 \in \mathbb{R}^N$ such that $f(x_0) > 0$, therefore

$$\int_{\mathbb{R}^N} f u \, dx = \frac{1}{\sigma^N} \int_{\mathbb{R}^N} f\left(x_0 + \frac{x}{\sigma}\right) \psi(x) \, dx \geq 0.$$

Moreover, there exists a positive R independent of σ such that

$$\int_{\mathbb{R}^N} f\left(x_0 + \frac{x}{\sigma}\right) \psi(x) \, dx \geq R \quad \text{and} \quad \int_{\mathbb{R}^N} f u \, dx > \frac{R}{\sigma^N} \quad \text{for } \sigma \text{ large enough.} \tag{2.4}$$

Putting together (2.2) and (2.3), and taking into account that $1 < q < 2 \leq r < 2_s^*$, we deduce that $T \in (0, 1]$ and

$$T < \left(\frac{\alpha |\psi|_q^q}{\sigma^{2s} [\psi]_s^2 + |\psi|_r^r} \right)^{\frac{1}{r-q}}. \tag{2.5}$$

In the light of the definition of h_u and gathering (2.4) and (2.5) we can infer

$$h'_u(T) = \langle \mathcal{I}'(Tu), u\rangle = T[u]_s^2 + T^{r-1}|u|_r^r - T^{q-1}|u|_q^q - \int_{\mathbb{R}^N} fu\,dx$$
$$= (\alpha - 1)T^{q-1}|u|_q^q - \int_{\mathbb{R}^N} fu\,dx < 0.$$

Now, since $h'_u(0) < 0$ and $h'_u(t) \to +\infty$ as $t \to \infty$, we deduce that there exists a minimum $t_u > T$ of $h_u(t)$, that is $\langle \mathcal{I}'(t_u u), u\rangle = 0$. Combining the fact that $t_u > T$ and the definition of T we get

$$[t_u u]_s^2 + |t_u u|_r^r > \alpha |t_u u|_q^q.$$

This implies that $v := t_u u \in \mathcal{N}_\alpha$. □

Since we are looking for critical points of $\mathcal{I}$, we need the following result.

Lemma 2.4. *Let $\{u_n\}_{n\in\mathbb{N}} \subset \mathbb{E}$ be such that $\langle \mathcal{I}'(u_n), u_n\rangle = 0$ for any $n \in \mathbb{N}$ and $\{\mathcal{I}(u_n)\}_{n\in\mathbb{N}}$ is bounded. Then $\{u_n\}_{n\in\mathbb{N}}$ is bounded in $\mathbb{E}$.*

Proof. Using Hölder and Young inequality we can infer

$$\tilde{C} \geq \frac{1}{q}\langle \mathcal{I}'(u_n), u_n\rangle - \mathcal{I}(u_n)$$
$$= \left(\frac{1}{q} - \frac{1}{2}\right)[u_n]_s^2 + \left(\frac{1}{q} - \frac{1}{r}\right)|u_n|_r^r - \left(\frac{1}{q} - 1\right)\int_{\mathbb{R}^N} fu_n\,dx$$
$$\geq \left(\frac{1}{q} - \frac{1}{2}\right)[u_n]_s^2 + \left(\frac{1}{q} - \frac{1}{r}\right)|u_n|_r^r - \left(\frac{1}{q} - 1\right)\left(1 - \frac{1}{r}\right)|f|_{\frac{r}{r-1}}^{\frac{r}{r-1}} - \left(\frac{1}{q} - 1\right)\frac{1}{r}|u_n|_r^r$$
$$= \left(\frac{1}{q} - \frac{1}{2}\right)[u_n]_s^2 + \frac{1}{q}\left(1 - \frac{1}{r}\right)|u_n|_r^r - \left(\frac{1}{q} - 1\right)\left(1 - \frac{1}{r}\right)|f|_{\frac{r}{r-1}}^{\frac{r}{r-1}}$$

which gives the boundedness of $\{u_n\}_{n\in\mathbb{N}}$ in $H^s(\mathbb{R}^N)$ and also in $L^r(\mathbb{R}^N)$.

Now, combining $\langle \mathcal{I}'(u_n), u_n\rangle = 0$ and the definition of $\mathcal{I}(u_n)$ we get

$$\frac{1}{2}[u_n]_s^2 + \frac{r-1}{r}|u_n|_r^r + \mathcal{I}(u_n) = \left(1 - \frac{1}{q}\right)|u_n|_q^q,$$

and taking into account that $\{u_n\}_{n\in\mathbb{N}}$ is bounded in $H^s(\mathbb{R}^N)$ and in $L^r(\mathbb{R}^N)$, and that $\{\mathcal{I}(u_n)\}_{n\in\mathbb{N}}$ is bounded, we deduce the thesis. □

Now, let $\overline{\mathcal{N}}_\alpha$ be the closure of $\mathcal{N}_\alpha$, and set

$$c := \inf_{u \in \overline{\mathcal{N}}_\alpha} \mathcal{I}(u). \tag{2.6}$$

Lemma 2.5. $-\infty < c < 0$.

Proof. Taking $v = t_u u \in \mathcal{N}_\alpha$ (as in Lemma 2.3) and using the fact that t_u is a minimum for h_u, we can infer

$$c \leq \mathcal{I}(v) = h_u(t_u) < h_u(0) = 0,$$

that is $c < 0$. Now we prove that $c > -\infty$.

Assume by contradiction that there exists a sequence $\{u_n\}_{n\in\mathbb{N}} \subset \overline{\mathcal{N}}_\alpha$ such that $\mathcal{I}(u_n) \to -\infty$. Taking into account that $\langle \mathcal{I}'(u_n), u_n\rangle = 0$ and $1 < q < 2 \leq r$, we get

$$\begin{aligned} \mathcal{I}(u_n) &= \mathcal{I}(u_n) - \langle \mathcal{I}'(u_n), u_n\rangle \\ &= -\frac{1}{2}[u_n]_s^2 + \left(\frac{1}{r} - 1\right)|u_n|_r^r + \left(1 - \frac{1}{q}\right)|u_n|_q^q \\ &\geq -\frac{1}{2}[u_n]_s^2 + \left(\frac{1}{r} - 1\right)|u_n|_r^r \end{aligned}$$

thus $[u_n]_s^2 + |u_n|_r^r \to +\infty$. Exploiting again $\langle \mathcal{I}'(u_n), u_n\rangle = 0$ and $u_n \in \overline{\mathcal{N}}_\alpha$ we have

$$1 = \frac{[u_n]_s^2 + |u_n|_r^r}{[u_n]_s^2 + |u_n|_r^r} = \frac{|u_n|_q^q + \displaystyle\int_{\mathbb{R}^N} f u_n \, dx}{[u_n]_s^2 + |u_n|_r^r} \leq \frac{1}{\alpha} + \frac{\displaystyle\int_{\mathbb{R}^N} f u_n \, dx}{[u_n]_s^2 + |u_n|_r^r}. \tag{2.7}$$

Now, we distinguish two cases. If $[u_n]_s \to \infty$, then using the Hölder inequality and Theorem (2.1) we have

$$\frac{\displaystyle\int_{\mathbb{R}^N} f u_n \, dx}{[u_n]_s^2 + |u_n|_r^r} \leq \frac{|f|_{\frac{2_s^*}{2_s^*-1}} |u_n|_{2_s^*}}{[u_n]_s^2} \leq \frac{|f|_{\frac{2_s^*}{2_s^*-1}}}{S_*^{\frac{1}{2}}[u_n]_s} \to 0 \text{ as } n \to \infty.$$

Let us assume $|u_n|_r \to \infty$. Observing that from $f \in L^{\frac{q}{q-1}}(\mathbb{R}^N) \cap L^{\frac{2_s^*}{2_s^*-1}}(\mathbb{R}^N)$ it follows that $f \in L^{\frac{r}{r-1}}(\mathbb{R}^N)$, we can apply the Hölder inequality to infer

$$\frac{\displaystyle\int_{\mathbb{R}^N} f u_n \, dx}{[u_n]_s^2 + |u_n|_r^r} \leq |f|_{\frac{r}{r-1}} |u_n|_r^{1-r} \to 0 \text{ as } n \to \infty.$$

Therefore, taking the limit as $n \to \infty$ in (2.7) we deduce that $\alpha \leq 1$, and this leads to a contradiction. $\square$

In the next lemma we show that $\inf_{u \in \overline{\mathcal{N}}_\alpha} \mathcal{I}(u) = \inf_{u \in \mathcal{N}_\alpha} \mathcal{I}(u)$ for some $\alpha > 1$.

Lemma 2.6. *Let $\alpha = 1 + \varepsilon$, with $\varepsilon > 0$ sufficiently small. Then we have*

$$c = \inf_{u \in \mathcal{N}_\alpha} \mathcal{I}(u).$$

Proof. Let $\{u_n\}_{n \in \mathbb{N}} \subset \overline{\mathcal{N}}_\alpha$ be a sequence such that $\mathcal{I}(u_n) \to c$, $\langle \mathcal{I}'(u_n), u_n \rangle = 0$ and

$$[u_n]_s^2 + |u_n|_r^r - \alpha |u_n|_q^q = 0.$$

Clearly, since $\{u_n\}_{n \in \mathbb{N}}$ is bounded in $\mathbb{E}$ we can find $M > 0$ such that $|u_n|_q^q < M$ for any $n \in \mathbb{N}$. Let us note that

$$\begin{aligned}
\mathcal{I}(u_n) = \mathcal{I}(u_n) - \langle \mathcal{I}'(u_n), u_n \rangle &= -\frac{1}{2}[u_n]_s^2 + \left(\frac{1}{r} - 1\right)|u_n|_r^r + \left(1 - \frac{1}{q}\right)|u_n|_q^q \\
&= \left(\frac{1}{r} - \frac{1}{2}\right)|u_n|_r^r + \left(1 - \frac{1}{q} - \frac{\alpha}{2}\right)|u_n|_q^q \\
&\geq -\left(\frac{1}{q} - \frac{\alpha}{r} + \alpha - 1\right)|u_n|_q^q \\
&\geq -\left(\frac{1}{q} - \frac{\alpha}{r} + \alpha - 1\right)M,
\end{aligned}$$

where $\dfrac{1}{q} - \dfrac{\alpha}{r} + \alpha - 1 > 0$.

Now, let $\eta \in C_c^\infty(\mathbb{R}^N)$ be such that

$$M < [\eta]_s^2 + |\eta|_r^r < |\eta|_q^q \quad \text{and} \quad \int_{\mathbb{R}^N} f\eta\, dx > 0. \tag{2.8}$$

We can see that $h'_\eta(0) < 0$, $h'_\eta(1) < 0$ and $h'_\eta(t) \to +\infty$ as $t \to \infty$, thus there exists $T_1 > 1$ such that $h'_\eta(T_1) = 0$ and $v := T_1\eta \in \mathcal{N}_\alpha$. In particular, it follows from (2.8) that $|v|_q^q > M$. Now,

$$\begin{aligned}\mathcal{I}(v) &= -\frac{1}{2}[v]_s^2 + \left(\frac{1}{r} - 1\right)|v|_r^r + \left(1 - \frac{1}{q}\right)|v|_q^q \\ &< \left(\frac{1}{r} - \frac{1}{2}\right)|v|_r^r + \left(1 - \frac{1}{q} - \frac{\alpha}{2}\right)|v|_q^q \\ &< -\left(\frac{1}{q} + \frac{\alpha}{2} - 1\right)M \\ &\leq -\left(\frac{1}{q} - \frac{\alpha}{r} + \alpha - 1\right)M \leq \mathcal{I}(u_n),\end{aligned}$$

and taking into account that $\mathcal{N}_\alpha \subset \overline{\mathcal{N}}_\alpha$, we have

$$c = \inf_{u \in \overline{\mathcal{N}}_\alpha} \mathcal{I}(u) \leq \inf_{u \in \mathcal{N}_\alpha} \mathcal{I}(u) \leq \mathcal{I}(v) < c,$$

which gives a contradiction. Therefore $u_n \in \mathcal{N}_\alpha$ for any $n \in \mathbb{N}$. □

3 - Proof of Theorem 1.1

Applying Ekeland's variational principle to (2.6), we deduce the existence of two sequences $\{u_n\}_{n\in\mathbb{N}} \subset \overline{\mathcal{N}}_\alpha$ and $\{\lambda_n\}_{n\in\mathbb{N}} \subset \mathbb{R}$ such that

$$\mathcal{I}(u_n) \to c \quad \text{and} \quad \mathcal{I}'(u_n) - \lambda_n \mathcal{J}'(u_n) \to 0 \text{ in } \mathbb{E}'$$

where

$$\mathcal{J}(u_n) = [u_n]_s^2 + |u_n|_r^r - |u_n|_q^q - \int_{\mathbb{R}^N} f u_n \, dx.$$

Similarly to Lemma 3.5 in **[28]** it is possible to prove the following.

L e m m a 3.1. *Let α be fixed as in Lemma* 2.6. *Then, any (PS)-sequence $\{u_n\}_{n\in\mathbb{N}}$ at level c for $\mathcal{I}$ restricted to $\mathcal{N}_\alpha$ is a (PS)-sequence for $\mathcal{I}$ on $\mathbb{E}$.*

Now we are ready to prove the following result.

T h e o r e m 3.1. *Under the same assumptions of Theorem* 1.1, *problem* (1.1) *admits a positive solution $u \in \overline{\mathcal{N}}_\alpha$ with $\mathcal{I}(u) = c$ and*

$$[u]_s^2 + |u|_r^r - \alpha|u|_q^q \geq 0. \tag{3.1}$$

Proof. Using Lemma 3.1 we can assume that there exists $\{u_n\}_{n\in\mathbb{N}} \subset \mathcal{N}_\alpha$ with $\mathcal{I}(u_n) \to c$ and $\langle \mathcal{I}'(u_n), u_n\rangle = 0$. Taking into account Lemma 2.4 we have that

$$\begin{aligned} &u_n \rightharpoonup u \quad \text{in } \mathbb{E},\\ &u_n \to u \quad \text{in } L^t_{loc}(\mathbb{R}^N), \ \text{ for any } t \in [1, 2^*_s),\\ &u_n \to u \quad \text{a.e. in } \mathbb{R}^N. \end{aligned} \tag{3.2}$$

First, we prove that u is a critical point for $\mathcal{I}$ in $\mathbb{E}$. From Lemma 3.1 we know that $\{u_n\}_{n\in\mathbb{N}}$ is a (PS)-sequence for $\mathcal{I}$ on $\mathbb{E}$. Now, let $\eta \in C^\infty_c(\mathbb{R}^N)$, then $\langle \mathcal{I}'(u_n), \eta\rangle \to 0$. From (3.2) we can infer that

$$\iint\limits_{\mathbb{R}^{2N}} \frac{(u_n(x) - u_n(y))(\eta(x) - \eta(y))}{|x-y|^{N+2s}}\, dxdy + \int\limits_{\mathbb{R}^N} |u_n|^{r-2} u_n \eta\, dx$$
$$\to \iint\limits_{\mathbb{R}^{2N}} \frac{(u(x) - u(y))(\eta(x) - \eta(y))}{|x-y|^{N+2s}}\, dxdy + \int\limits_{\mathbb{R}^N} |u|^{r-2} u \eta\, dx.$$

Taking into account that $u_n \to u$ in $L^q(\operatorname{supp}\eta)$, there exist a subsequence, still denoted by $\{u_n\}_{n\in\mathbb{N}}$, and a function $h \in L^q(\mathbb{R}^N)$ such that $|u_n| \le |h|$ and

$$\begin{aligned} &|u_n|^{q-2} u_n \eta \to |u|^{q-2} u\eta \quad \text{a.e. in } \mathbb{R}^N,\\ &|u_n|^{q-1}|\eta| \le |h|^{q-1}|\eta| \in L^1(\mathbb{R}^N), \end{aligned}$$

and applying the Dominated Convergence Theorem we can infer that

$$\int\limits_{\mathbb{R}^N} |u_n|^{q-2} u_n \eta\, dx \to \int\limits_{\mathbb{R}^N} |u|^{q-2} u\eta\, dx.$$

Thus we can conclude that $\langle \mathcal{I}'(u_n), \eta\rangle \to \langle \mathcal{I}'(u), \eta\rangle = 0$.

Now, we aim to prove that $\mathcal{I}(u) = c$. We know that

$$\|u\| \le \liminf_{n\to\infty} \|u_n\|. \tag{3.3}$$

In order to show that the equality holds in (3.3), let us suppose by contradiction that

$$\|u\| < \liminf_{n\to\infty} \|u_n\|.$$

Set $v_n(x) := u_n(x) - u(x)$ such that $v_n \rightharpoonup 0$ in $\mathbb{E}$. First, we prove that there exists an unbounded sequence $\{y_n\}_{n\in\mathbb{N}} \subset \mathbb{R}^N$ such that

$$v_n(x + y_n) \rightharpoonup v \neq 0 \quad \text{in } \mathbb{E}. \tag{3.4}$$

Assume by contradiction that for any $R > 0$

$$\sup_{y\in\mathbb{R}^N} \int\limits_{B_R(y)} |v_n|^q \, dx \to 0 \quad \text{as } n\to\infty.$$

Then, by Lemma 2.1 we have that $v_n \to 0$ in $L^t(\mathbb{R}^N)$ for all $t \in [q, 2^*_s)$. Taking into account that $\{u_n\}_{n\in\mathbb{N}}$ is a $(PS)_c$-sequence for $\mathcal{I}$, we have that $\langle \mathcal{I}'(u_n), v_n\rangle \to 0$, that is

$$\iint\limits_{\mathbb{R}^{2N}} \frac{(u_n(x)-u_n(y))(v_n(x)-v_n(y))}{|x-y|^{N+2s}} \, dxdy + \int\limits_{\mathbb{R}^N} |u_n|^{r-2} u_n v_n \, dx$$

$$- \int\limits_{\mathbb{R}^N} |u_n|^{q-2} u_n v_n \, dx - \int\limits_{\mathbb{R}^N} f v_n \, dx \to 0. \tag{3.5}$$

Recalling that $v_n \rightharpoonup 0$ in $\mathbb{E}$, we have

$$\int\limits_{\mathbb{R}^N} |u_n|^{q-1} |v_n| \, dx \le |u_n|_q^{q-1} |v_n|_q \le C |v_n|_q \to 0,$$

and

$$\int\limits_{\mathbb{R}^N} |f||v_n| \, dx \le |f|_{\frac{q}{q-1}} |v_n|_q \to 0.$$

Thus from (3.5) we deduce that

$$\iint\limits_{\mathbb{R}^{2N}} \frac{(u_n(x)-u_n(y))(v_n(x)-v_n(y))}{|x-y|^{N+2s}} \, dxdy + \int\limits_{\mathbb{R}^N} |u_n|^{r-2} u_n v_n \, dx \to 0. \tag{3.6}$$

Moreover, let us also observe that

$$\iint\limits_{\mathbb{R}^{2N}} \frac{(u(x)-u(y))(v_n(x)-v_n(y))}{|x-y|^{N+2s}} \, dxdy + \int\limits_{\mathbb{R}^N} |u|^{r-2} u v_n \, dx \to 0. \tag{3.7}$$

Putting together (3.6) and (3.7) we get

$$\iint\limits_{\mathbb{R}^{2N}} \frac{[(u_n(x)-u_n(y)) - (u(x)-u(y))](v_n(x)-v_n(y))}{|x-y|^{N+2s}} \, dxdy$$

$$+ \int\limits_{\mathbb{R}^N} (|u_n|^{r-2} u_n - |u|^{r-2} u) v_n \, dx \to 0.$$

Being $r \geq 2$ we can use the following inequality

$$C_r|x-y|^r \leq (|x|^{r-2}r - |y|^{r-2}y, x-y) \quad \forall x, y \in \mathbb{R}^N,$$

to deduce that

$$\int_{\mathbb{R}^N} (|u_n|^{r-2}u_n - |u|^{r-2}u)v_n \, dx \geq C_r|u_n - u|_r^r.$$

Then, recalling that $v_n = u_n - u$, we can see that

$$[u_n - u]_s^2 + |u_n - u|_r^r \to 0$$

that is $\|u_n\| \to \|u\|$, and this gives a contradiction. Thus, there exist $R > 0$, $\beta > 0$, and a sequence $\{y_n\}_{n\in\mathbb{N}} \subset \mathbb{R}^N$ such that

$$\int_{\mathbb{R}^N} |v_n|^q \, dx \geq \beta > 0,$$

therefore we have

$$\int_{B_R(0)} |v_n(x+y_n)|^q \, dx \to \int_{\mathbb{R}^N} |v|^q \, dx \geq \beta > 0,$$

which gives $v \neq 0$ in $\mathbb{E}$.

Now we prove that $\{y_n\}_{n\in\mathbb{N}}$ is not bounded. Assume by contradiction that $\{y_n\}_{n\in\mathbb{N}}$ is bounded. Then, up to a subsequence $y_n \to y$. Let $\phi \in \mathcal{C}_c^\infty(\mathbb{R}^N)$. Since $y_n \to y$ and $v_n(x) \rightharpoonup 0$ in $\mathbb{E}$, it follows that

$$\int_{\mathbb{R}^N} \phi(x-y_n)v_n(x) \, dx \to 0 \tag{3.8}$$

and using (3.4) we can infer

$$\int_{\mathbb{R}^N} \phi(x-y_n)v_n(x) \, dx = \int_{\mathbb{R}^N} \phi(x)v_n(x+y_n) \, dx \to \int_{\mathbb{R}^N} \phi(x)v(x) \, dx \quad \forall \phi \in \mathcal{C}_c^\infty(\mathbb{R}^N). \tag{3.9}$$

Combining (3.8) and (3.9) we deduce that

$$\int_{\mathbb{R}^N} \phi(x)v(x) \, dx = 0 \quad \forall \phi \in \mathcal{C}_c^\infty(\mathbb{R}^N).$$

This gives a contradiction in view of (3.4).

Now, we aim to prove that

$$u_n(\cdot + y_n) \rightharpoonup v \quad \text{in } \mathbb{E}\,. \tag{3.10}$$

From the boundedness of $\{u_n\}_{n\in\mathbb{N}}$ it follows that $u(x+y_n)$ is bounded in $\mathbb{E}$, so there exists $w \in \mathbb{E}$ such that $u(x+y_n) \rightharpoonup w(x)$ in $\mathbb{E}$ and for all $\phi \in \mathcal{C}_c^\infty(\mathbb{R}^N)$ we have

$$\int_{\mathbb{R}^N} u(x+y_n)\phi(x)\,dx \to \int_{\mathbb{R}^N} w(x)\phi(x)\,dx.$$

Recalling that $|y_n| \to \infty$ we also have

$$\int_{\mathbb{R}^N} u(x+y_n)\phi(x)\,dx \to 0,$$

and combining the previous relations we deduce that

$$\int_{\mathbb{R}^N} w(x)\phi(x)\,dx = 0,$$

that implies that $w = 0$ a.e. in $\mathbb{R}^N$. Thus, (3.10) holds true.

Since $\{u_n\}_{n\in\mathbb{N}}$ is a $(PS)_c$-sequence for $\mathcal{I}$, we have that $\langle \mathcal{I}'(u_n), \phi(\cdot - y_n)\rangle \to 0$, or equivalently

$$\begin{aligned}
&\iint_{\mathbb{R}^{2N}} \frac{(u_n(x+y_n)-u_n(y+y_n))(\phi(x)-\phi(y))}{|x-y|^{N+2s}}\,dxdy \\
&\quad + \int_{\mathbb{R}^N} |u_n(x+y_n)|^{r-2}u_n(x+y_n)\phi(x)\,dx \\
&\quad - \int_{\mathbb{R}^N} |u_n(x+y_n)|^{q-2}u_n(x+y_n)\phi(x)\,dx \;-\int_{\mathbb{R}^N} f(x)\phi(x-y_n)\,dx \to 0.
\end{aligned}$$

From (3.10) we can infer

$$\begin{aligned}
&\iint_{\mathbb{R}^{2N}} \frac{(u_n(x+y_n)-u_n(y+y_n))(\phi(x)-\phi(y))}{|x-y|^{N+2s}}\,dxdy \\
&\quad + \int_{\mathbb{R}^N} |u_n(x+y_n)|^{r-2}u_n(x+y_n)\phi(x)\,dx \\
&\to \iint_{\mathbb{R}^{2N}} \frac{(v(x)-v(y))(\phi(x)-\phi(y))}{|x-y|^{N+2s}}\,dxdy \;+\int_{\mathbb{R}^N} |v(x)|^{r-2}v(x)\phi(x)\,dx.
\end{aligned}$$

Furthermore,

$$\int_{\mathbb{R}^N} |u_n(x+y_n)|^{q-2}u_n(x+y_n)\phi(x)\,dx \to \int_{\mathbb{R}^N} |v(x)|^{q-2}v(x)\phi(x)\,dx.$$

Indeed, since $u_n(x+y_n)\to v(x)$ in $L^q(\operatorname{supp}\phi)$, there exist a subsequence, still denoted by u_n, and a function $h \in L^q(\mathbb{R}^N)$ such that $|u_n(x+y_n)|^{q-2}\,u_n\,(x+y_n)\,\phi(x) \to |v(x)|^{q-2}\,v(x)\,\phi(x)$ a.e. in $\mathbb{R}^N$ and $|u_n\,(x+y_n)|^{q-1}\,|\phi(x)| \le |h(x)|^{q-1}|\phi(x)| \in L^1(\mathbb{R}^N)$. At this point, applying the Dominated Convergence Theorem we get the thesis.

Finally, we note that,

$$\int_{\mathbb{R}^N} f(x)\phi(x-y_n)\,dx \to 0.$$

Gathering the above relations of limit we have

$$\iint_{\mathbb{R}^{2N}} \frac{(v(x)-v(y))(\phi(x)-\phi(y))}{|x-y|^{N+2s}}\,dxdy + \int_{\mathbb{R}^N} |v(x)|^{r-2}v(x)\phi(x)\,dx$$
$$- \int_{\mathbb{R}^N} |v(x)|^{q-2}v(x)\phi(x)\,dx = 0,$$

that is v is a weak solution to (2.1), and from Lemma 2.2 we deduce that $v=0$ a.e. in $\mathbb{R}^N$. This gives a contradiction. Hence we have that the equality holds in (3.3), and, up to a subsequence we can infer that $\|u\| = \lim_{n\to\infty}\|u_n\|$. Now,

$$\begin{aligned}|u|_q &\le \liminf_{n\to\infty}|u_n|_q \le \limsup_{n\to\infty}|u_n|_q\\ &\le \limsup_{n\to\infty}([u_n]_s + |u_n|_q) - \liminf_{n\to\infty}[u_n]_s\\ &\le ([u]_s + |u|_q) - \liminf_{n\to\infty}[u_n]_s^2 \le |u|_q,\end{aligned}$$

therefore $u_n \to u$ in $L^q(\mathbb{R}^N)$.

On the other hand $\|u\| = \lim_{n\to\infty}\|u_n\|$, so

$$\iint_{\mathbb{R}^{2N}} \frac{|u_n(x)-u_n(y)|^2}{|x-y|^{N+2s}}\,dxdy \to \iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\,dxdy.$$

Applying the Brezis-Lieb Lemma we can infer that

$$[u_n-u]_s^2 = [u_n]_s^2 - [u]_s^2 + o_n(1),$$

thus $[u_n - u]_s \to 0$ and we can conclude that $u_n \to u$ in $\mathbb{E}$. Let us also note that by interpolation we can also see that $|u_n - u|_r^r \le |u_n - u|_q^{r\alpha} |u_n - u|_{2_s^*}^{r(1-\alpha)} \to 0$ in view of (3.2). Finally,

$$\int_{\mathbb{R}^N} f u_n \, dx \to \int_{\mathbb{R}^N} f u \, dx,$$

therefore $\mathcal{I}(u_n) \to \mathcal{I}(u) = c$ as $n \to \infty$.

Our aim is to prove that u is a nonnegative weak solution to (1.1). Indeed, let us consider the function $h_{|u|}(t)$. We can see that $h'_{|u|}(0) < 0$ and $h'_{|u|}(t) \to \infty$ as $t \to \infty$, thus there exists $t_{|u|} > 0$ such that $h'_{|u|}(t_{|u|}) = 0$ and $h_{|u|}(t_{|u|}) = \inf_{t \ge 0} h_{|u|}(t)$. Now,

$$h'_{|u|}(1) = \int_{\mathbb{R}^N} f(u - |u|) \, dx \le 0.$$

If by contradiction $h'_{|u|}(1) < 0$, then $t_{|u|} > 1$ which together with (3.1) implies that $t_{|u|}|u| \in \mathcal{N}_\alpha$. Hence we have

$$c \le \mathcal{I}(t_{|u|}|u|) = h_{|u|}(t_{|u|}) < h_{|u|}(1) = \mathcal{I}(|u|) \le \mathcal{I}(u) = c,$$

which is a contradiction. Therefore $h'_{|u|}(1) = 0$ and $\displaystyle\int_{\mathbb{R}^N} f(u - |u|) \, dx = 0$. Using the fact that $f > 0$ a.e. in $\mathbb{R}^N$, we can infer that $u = |u|$ a.e. in $\mathbb{R}^N$, and applying the maximum principle we get that u is a positive solution to (1.1).

It remains to prove that (3.1) holds true. Combining the boundedness of $\{u_n\}_{n \in \mathbb{N}} \subset \mathbb{E}$ and $\langle \mathcal{I}'(u_n), u_n \rangle = 0$, we have

$$\int_{\mathbb{R}^N} f u_n \, dx = [u_n]_s^2 + |u_n|_r^r - |u_n|_q^q > (\alpha - 1) \, |u_n|_q^q$$

and using (3.2) we can infer that

$$\int_{\mathbb{R}^N} f u \, dx \ge (\alpha - 1) \liminf_{n \to \infty} |u_n|_q^q \ge (\alpha - 1) \, |u|_q^q.$$

Therefore we can conclude that

$$[u]_s^2 + |u|_r^r - |u|_q^q = \int_{\mathbb{R}^N} f u \, dx \ge (\alpha - 1) \, |u|_q^q.$$

□

Finally, we deal with the continuity of solutions in the perturbation parameter f at 0.

Theorem 3.2. *Under the assumptions of Theorem* 1.1, *let* u_f *be the solution of* (1.1) *given by Lemma* 3.1. *If* $f \to 0$ *in* $L^{\frac{q}{q-1}}(\mathbb{R}^N)$, *then* $u_f \to 0$ *in* $\mathbb{E}$.

Proof. Let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence such that $|f_n|_{\frac{q}{q-1}} \to 0$ and let u_{f_n} be a solution to (1.1) given by Lemma 3.1. Taking into account that $u_{f_n} \in \overline{\mathcal{N}}_\alpha$, we can infer that

$$|u_{f_n}|_q^q + \int_{\mathbb{R}^N} f_n u_{f_n}\, dx = [u_{f_n}]_s^2 + |u_{f_n}|_r^r \geq \alpha |u_{f_n}|_q^q, \tag{3.11}$$

and recalling that u_{f_n} is positive, we deduce

$$(\alpha - 1)|u_{f_n}|_q^q \leq |f_n|_{\frac{q}{q-1}} |u_{f_n}|_q^q,$$

that is $|u_{f_n}|_q \to 0$ as $|f_n|_{\frac{q}{q-1}} \to 0$. Combining this and (3.11) it follows that $u_{f_n} \to 0$ in $\mathbb{E}$. □

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