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Multiple sequences of entire solutions for critical polyharmonic equations

Abstract. In this paper we study the critical polyharmonic equation in $\mathbb{R}^d$. By exploiting some algebraic–theoretical arguments developed in [**2**, **13**, **20**], we prove the existence of a finite number ζ_d of sequences of infinitely many finite energy nodal solutions which are unbounded in the classical higher order Sobolev space $\mathcal{D}^{m,2}(\mathbb{R}^d)$, associated to the polyharmonic operator $(-\Delta)^m$, with $m \in \mathbb{N}$. Taking into account the recent results contained in [**20**], an explicit expression of ζ_d is given in terms of the number of unrestricted partitions of the Euclidean dimension d, given by the celebrated Rademacher formula. Furthermore, the asymptotic behavior of the number ζ_d obtained here is a direct consequence of the classical Hardy–Ramanujan analyis based on the circle method. The main multiplicity result represents a more precise form of Theorem 1.1 of [**2**] for polyharmonic problems settled in higher dimensional Euclidean spaces. Finally, an explicit numerical comparison with Theorem 4.8 of [**20**] is presented.

Keywords. Variational methods, principle of symmetric criticality, nodal solutions.

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Contents

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1 - Introduction

Since the appearance of the celebrated paper of W.Y. Ding [**7**] on the conformally invariant scalar field equation in $\mathbb{R}^d$, concerning the existence of infinitely many conformally inequivalent changing sign solutions, with finite energy, the method of pulling back the problem into the unit sphere $\mathbb{S}^d$ of $\mathbb{R}^{d+1}$ by means of a stereographic projection and then into its variational formulation has been having a large use in literature for different problems, involving critical nonlinearities in the sense of Sobolev.

For instance, inspired by [**7**], T. Bartsch, M. Schneider, and T. Weth in [**2**] prove for the critical polyharmonic equation

$$\begin{cases} (-\Delta)^m u = |u|^{2^*_m - 2} u \quad \text{in } \mathbb{R}^d, \quad u \in \mathcal{D}^{m,2}(\mathbb{R}^d), \\ d > 2m, \quad 2^*_m = \dfrac{2d}{d-2m}, \end{cases} \tag{1.1}$$

the existence of a sequence of infinitely many finite energy nodal solutions which are unbounded in the Sobolev space $\mathcal{D}^{m,2}(\mathbb{R}^d)$. After a careful analysis of suitable subgroups of the orthogonal group $O(d+1)$, in [**2**] the authors obtain crucial compact Sobolev embeddings, which are essential in order to use variational methods. More recently, A. Maalaoui and V. Martino in [**18**] and A. Maalaoui, V. Martino, and G. Tralli in [**19**], motivated again by the original paper of Ding, establish the existence of changing sign solutions for the Yamabe problem on the Heisenberg group $\mathbb{H}^d$. Finally, for the sake of completeness, we cite the paper [**14**], in which A. Kristály proves a more general multiplicity existence theorem of changing sign solutions for the fractional Yamabe problem on the Heisenberg group $\mathbb{H}^d$ via a nonlocal version of the Ding–Hebey–Vaugon compactness result on the Cauchy–Riemann unit sphere $\mathbb{S}^{2d+1}$ and an algebraic theoretical approach on suitable subgroups of the unitary group $U(d+1)$.

In this spirit, starting from the pioneristic paper [**2**] and encouraged by a wide interest on the current literature on polyharmonic problems, in the present paper we prove that equation (1.1) admits at least a finite number ζ_d of sequences of infinitely many finite energy nodal solutions which are unbounded in the Sobolev space $\mathcal{D}^{m,2}(\mathbb{R}^d)$. More precisely we study (1.1) from the point of view of the $O(d+1)$ symmetry theory, as in [**2**, **13**, **20**]. This approach presents new and challenging features in the higher–order case. The main result of the paper is

Theorem 1.1. *Let m and d be two positive integers, with $d > 2m$. Set $\zeta_d = \max\{1, s_d, \kappa_d\}$, where*

$$s_d = \begin{cases} 0 & \text{if either } d = 3 \text{ or } d = 4 \\ [d/2] + (-1)^{d+1} - 1 & \text{if } d \geq 5, \end{cases}$$

$$\kappa_d = \begin{cases} 0 & \text{if } d = 4 \\ p\left(\dfrac{d+1}{4}\right) & \text{if } d = 4n - 1 \\ p\left(\dfrac{d-4}{4}\right) & \text{if } d = 4n + 4 \\ p\left(\dfrac{d-1}{4}\right) & \text{if } d = 4n + 1 \\ p\left(\dfrac{d-2}{4}\right) & \text{if } d = 4n + 2, \end{cases} \qquad (\text{for some } n \in \mathbb{N}) \tag{1.2}$$

and $p : \mathbb{N} \to \mathbb{N}$ denotes the unrestricted partition function.

Then, the critical polyharmonic equation (1.1) *admits al least ζ_d sequences of infinitely many finite energy nodal weak solutions, which are unbounded in $\mathcal{D}^{m,2}(\mathbb{R}^d)$ and mutually symmetrically distinct. Moreover, the constructed weak solutions are of class $C^{2m}(\mathbb{R}^d)$ and classical.*

The tables below show the behavior of the key numbers s_d, κ_d and ζ_d as d increases in $\mathbb{N}$. The main properties of the unrestricted partition function $n \mapsto p(n)$ show rigorously that $s_d < \kappa_d$ for $d \geq 29$, by (1.2) and the fact that $p(n) > 2n$ for all $n \geq 7$, see Table 3 of Section 2.

Table 1. $3 \leq d \leq 19$

d	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
s_d	0	0	2	1	3	2	4	3	5	4	6	5	7	6	8	7	9
κ_d	1	0	1	1	2	1	2	2	3	2	3	3	5	3	5	5	7
ζ_d	1	1	2	1	3	2	4	3	5	4	6	5	7	6	8	7	9

Table 2. $20 \leq d \leq 35$

d	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35
s_d	8	10	9	11	10	12	11	13	12	14	13	15	14	16	15	17
κ_d	5	7	7	11	7	11	11	15	11	15	15	22	15	22	22	30
ζ_d	8	10	9	11	10	12	11	15	12	15	15	22	15	22	22	30

As the numerical computations show, there is a natural competition between the parameters κ_d and s_d due to their analytical definitions. For small dimensions, that is when $5 \leq d \leq 28$ and $d \neq 27$, then $s_d \geq \kappa_d$. Therefore, Theorem 1.1 gives a number of solutions of (1.1) which is greater or equal to the one found in [**20**]. Moreover, for the above dimensions the proof of Theorem 1.1 produces the exact symmetry of each family of solutions, constructed as in [**13**].

For the sake of clarity, we present a consequence of Theorem 1.1, when $m = 2$, that is when the biharmonic operator in equation (1.1), transformed into the sphere $\mathbb{S}^d$ of $\mathbb{R}^{d+1}$, reduces to the celebrated Paneitz operator, introduced by Paneitz himself in [**22**] for smooth Riemannian manifolds. For further details in this context we refer to [**11**] and the references therein. Indeed, the polyharmonic operator $\mathfrak{D}^m$ on the unit sphere $\mathbb{S}^d = (\mathbb{S}^d, h)$ of $\mathbb{R}^{d+1}$ is expressed by

$$\mathfrak{D}^m = \prod_{k=1}^{m} \left(-\Delta_h + \frac{1}{4}(d-2k)(d+2k-2)\mathbb{I}_{L^2(\mathbb{S}^d)} \right), \tag{1.3}$$

where Δ_h denotes the usual Laplace–Beltrami operator on $\mathbb{S}^d$. In the case $m = 2$ the operator $\mathfrak{D}^2$ has the form

$$\Delta_h^2 - \alpha\, \Delta_h + a\, \mathbb{I}_{L^2(\mathbb{S}^d)},$$

where $\alpha = (d^2 - 2d - 4)/2$ and $a = d(d^2 - 4)(d-4)/16$. Thus, $\alpha > 0$ and $a > 0$ for all $d \geq 5$.

Corollary 1.2. *Let $d \geq 5$. Then, the critical Paneitz equation*

$$\Delta_h^2 v - \alpha\, \Delta_h v + a\, v = |v|^{2_2^*-2} v \qquad \text{in } \mathbb{S}^d,$$

admits al least $s_d = [d/2] + (-1)^{d+1} - 1$ sequences $(v_k^{(i)})_k$ of infinitely many finite energy nodal weak solutions $v_k^{(i)} \in H^2(\mathbb{S}^d)$, $i = 1, \dots, s_d$, which are unbounded in $H^2(\mathbb{S}^d)$ and mutually symmetrically distinct.

More precisely, for each i the unbounded sequence $(v_k^{(i)})_k$ lies in the subspace $H^2_{G^{\tau_i}_{d,i}}(\mathbb{S}^d)$ of the $G^{\tau_i}_{d,i}$–invariant functions of $H^2(\mathbb{S}^d)$, with respect to the action

$$\widehat{\circledast}_i : G^{\tau_i}_{d,i} \times H^2(\mathbb{S}^d) \to H^2(\mathbb{S}^d), \qquad (\widetilde{g}, v) \mapsto \widetilde{g}\widehat{\circledast}_i v,$$

defined pointwise by

$$(\widetilde{g}\widehat{\circledast}_i v)(\sigma) = \begin{cases} v(g^{-1}\cdot\sigma), & \text{if } \widetilde{g} = g \in G_{d,i} \\ -v(g^{-1}\tau_i^{-1}\cdot\sigma), & \text{if } \widetilde{g} = \tau_i\, g \in G^{\tau_i}_{d,i} \setminus G_{d,i},\ g \in G_{d,i}, \end{cases}$$

where $G^{\tau_i}_{d,i}$ is the compact group of $O(d+1)$ generated by the compact subgroup

$$G_{d,i} = \begin{cases} O(i+1)\times O(d-2i-1)\times O(i+1), & \text{if } i \neq \dfrac{d-1}{2}, \\ O(i+1)\times O(i+1), & \text{if } i = \dfrac{d-1}{2}, \end{cases}$$

of $O(d+1)$ and by an involution $\tau_i : \mathbb{S}^d \to \mathbb{S}^d$, with the properties that

$$\tau_i \notin G_{d,i}, \quad \tau_i G_{d,i} \tau_i^{-1} = G_{d,i} \quad \text{and} \quad \tau_i^2 = \mathbb{I}_{\mathbb{R}^{d+1}}$$

for every $i = 1, \dots, s_d$.

The mutual symmetry difference comes from the fact that $H^2_{G^{\tau_i}_{d,i}}(\mathbb{S}^d) \cap H^2_{G^{\tau_j}_{d,j}}(\mathbb{S}^d) = \{0\}$ for all i, $j = 1, \dots, s_d$, with $i \neq j$.

The paper is organized as follows. Section 2 contains some notations and useful preliminaries. In Section 3 we discuss the reduction of (1.1) to an equivalent equation on the unit sphere $\mathbb{S}^d$ of $\mathbb{R}^{d+1}$. In the same section, in order to prove the multiplicity Theorem 1.1, we describe the construction of the s_d subspaces $H^m_{G^{\tau_i}_{d,i}}(\mathbb{S}^d)$ of the Sobolev space $H^m(\mathbb{S}^d)$ related to certain subgroups $G^{\tau_i}_{d,i}$ of the orthogonal group $O(d+1)$. Some tools, which are useful along the paper as, for instance, the geometrical profile of the subspaces $H^m_{G^{\tau_i}_{d,i}}(\mathbb{S}^d)$ given in Proposition 3.6, are also presented. Finally, the last Section 4 is dedicated to the proof of Theorem 1.1.

2 - Preliminaries

This section is devoted to some notations and preliminary results: the reader familiar with these topics may skip it.

2.1 - *The main problem in the Euclidean space* $\mathbb{R}^d$

The natural solution space of (1.1) is the Beppo Levi higher order space $\mathcal{D}^{m,2}(\mathbb{R}^d)$, $d > 2m$, which is the completion of $C_c^\infty(\mathbb{R}^d)$, with respect to the norm $\|\cdot\|$ induced by the inner product

$$\langle \varphi, \psi \rangle = \begin{cases} \displaystyle\int_{\mathbb{R}^d} \Delta^k \varphi \, \Delta^k \psi \, dx & \text{if } m = 2k \text{ is even} \\ \displaystyle\int_{\mathbb{R}^d} \nabla \Delta^k \varphi \cdot \nabla \Delta^k \psi \, dx & \text{if } m = 2k+1 \text{ is odd,} \end{cases} \tag{2.1}$$

for every φ, $\psi \in C_c^\infty(\mathbb{R}^d)$.

Following [**23**], we say that a function $u \in \mathcal{D}^{m,2}(\mathbb{R}^d)$ is a *weak solution* of (1.1) if

$$\langle u, \varphi \rangle = \int_{\mathbb{R}^d} |u|^{2_m^*-2} u \varphi \, dx \tag{2.2}$$

for every $\varphi \in \mathcal{D}^{m,2}(\mathbb{R}^d)$.

It is known that the only finite energy positive solutions of (1.1) are given by the family of functions

$$u_{\varepsilon,\xi}(x) = \varepsilon^{-\frac{d-2m}{2}} U\big((x-\xi)/\varepsilon\big), \quad \text{where } U(x) = P_{m,d}^{\frac{d-2m}{4m}} (1+|x|^2)^{-\frac{d-2m}{2}}, \tag{2.3}$$

$\varepsilon > 0$, $\xi \in \mathbb{R}^d$ and $P_{m,d} = \Pi_{k=-m}^{m}(d+2k)$. We refer to [**3**, **10**] for further details. Conversely, if the polyharmonic equation with the nonlinear term $f(u)$ possesses a positive solution, then $d > 2m$ and $f(u) = c\,|u|^{2_m^*-2}u$, provided that f is locally Lipschitz in $\mathbb{R}_0^+$, $f \geq 0$ and nondecreasing in $\mathbb{R}_0^+$, $u^{-q} f(u)$ is nonincreasing in $\mathbb{R}^+$, and $f(u) = o(u^q)$ as $u \to \infty$, where $1 < q \leq 2_m^* - 1$, as shown in Theorem 1.5 of [**27**]. See also [**4**] and references therein.

Finally, we recall that every nontrivial nonnegative solution $u \in \mathcal{D}^{m,2}(\mathbb{R}^d)$ of (1.1) is positive in $\mathbb{R}^d$ and has the form given in (2.3). See the quoted paper [**2**] for additional remarks and comments, as well as [**24**, **25**] for related topics.

For historical details and a wide list of recent contributions on semilinear problems involving the biharmonic or polyharmonic operator as principal part we refer to the modern excellent monograph [**9**] and the references therein.

2.2 - *Number of unrestricted partitions*

In number theory there are several ways to write an integer n as a sum of positive integers, when the order of addends is not considered significant. This is denoted by $p(n)$, and is called the *number of unrestricted partitions.* Partitions can be graphically visualized with Young diagrams or Ferrers diagrams. They occur in many branches of mathematics and physics, including the study of symmetric polynomials, the symmetric group and in group representation theory in general. A classical and celebrated result due to Hardy and Ramanujan in 1918 is dedicated to the asymptotic behavior of the function p. More precisely, they showed that

$$p(n) \sim \frac{1}{4n\sqrt{3}} e^{\pi\sqrt{2n/3}} \quad \text{as } n \to \infty. \tag{2.4}$$

Lately, in 1937 Rademacher in a celebrated paper proved that the unrestricted partition function $p(n)$, for any fixed $n \in \mathbb{N}$, has the form

$$p(n) = \frac{1}{\pi\sqrt{2}} \sum_{k=1}^{\infty} A_k(n)\sqrt{k}\,\phi'(n), \tag{2.5}$$

where

$$\phi(x) = \frac{\sinh\left(\dfrac{\pi}{k}\sqrt{\dfrac{2}{3}\left(x - \dfrac{1}{24}\right)}\right)}{\sqrt{x - \dfrac{1}{24}}},$$

as well as

$$A_k(n) = \sum_{\substack{h=0 \\ (h,k)=1}}^{k-1} e^{\pi i\left(s(h,k) - 2n\frac{h}{k}\right)} \quad \text{and} \quad s(h,k) = \sum_{j=1}^{k-1} \frac{j}{k}\left(\frac{hj}{k} - \left[\frac{hj}{k}\right] - \frac{1}{2}\right).$$

As usual, the notation $(h,k) = 1$ means that the two integers h and k are co–prime.

For the sake of clarity we present in Table 3 the first 10 values of $p(n)$, which we have used to prove that $s_d < \kappa_d$ for $d \geq 29$ in the Introduction.

Thanks to (1.2) and (2.4), the number ζ_d of sequences of solutions in Theorem 1.1 grows exponentially as $d \to \infty$.

Table 3. *Values of $p(n)$ for $1 \le n \le 10$*

n	1	2	3	4	5	6	7	8	9	10
$p(n)$	1	2	3	5	7	11	15	22	30	42

3 - Reduction to unit sphere $\mathbb{S}^d$ of $\mathbb{R}^{d+1}$

Let $\mathbb{S}^d \hookrightarrow \mathbb{R}^{d+1}$ be the Euclidean unit sphere endowed by the induced Riemannian metric h. Consider the following problem

$$\begin{cases} \mathfrak{D}^m v = |v|^{2^*_m-2}v & \text{in } \mathbb{S}^d, \\ v \in H^m(\mathbb{S}^d), \quad d > 2m, \end{cases} \tag{3.1}$$

where $\mathfrak{D}^m$ is the elliptic differential operator defined in (1.3).

Let $H^m(\mathbb{S}^d)$ be the standard higher order Sobolev space $W^{m,2}(\mathbb{S}^d)$, whose Hilbertian structure is given by the scalar product

$$\langle v, w\rangle_{H^m(\mathbb{S}^d)} = \begin{cases} \displaystyle\int_{\mathbb{S}^d} \Big(\Delta_h^k v\, \Delta_h^k w + vw\Big)\, d\sigma_h & \text{if } m = 2k \text{ is even} \\ \displaystyle\int_{\mathbb{S}^d} \Big(\nabla_h \Delta_h^k v \cdot \nabla_h \Delta_h^k w + vw\Big)\, d\sigma_h & \text{if } m = 2k+1 \text{ is odd} \end{cases} \tag{3.2}$$

for every $v, w \in H^m(\mathbb{S}^d)$. We denote by $\|\cdot\|_{H^m(\mathbb{S}^d)}$ the norm induced by the scalar product in (3.2).

In the case of $m = 1$, equation (3.1) reduces to

$$-\Delta_h v + \frac{d(d-2)}{2}\, v = |v|^{2^*_m-2}v \quad \text{in } \mathbb{S}^d.$$

The search of positive solution is the well known Yamabe problem, which arises from the conformal geometry. For details we refer to Chapter 7 of the monograph [**1**].

In order to handle the variational formulation of problem (3.1) we introduce a different Hilbertian norm $\|\cdot\|_*$ on the Sobolev space $H^m(\mathbb{S}^d)$, which is equivalent to the norm $\|\cdot\|_{H^m(\mathbb{S}^d)}$. This equivalence will be more readable if we express (3.2) in a convenient form given in terms of the Fourier coefficients of the functions v and w. To this aim, let $L^2(\mathbb{S}^d)$ be the standard Lebesgue space of square–summable functions on $\mathbb{S}^d$ endowed by the natural inner product

$$\langle v, w\rangle_{L^2(\mathbb{S}^d)} = \int_{\mathbb{S}^d} v\, w d\sigma_h \quad \text{for every } v, w \in L^2(\mathbb{S}^d).$$

Clearly, $L^2(\mathbb{S}^d)$ can be decomposed as direct sum of the orthogonal eigenspaces connected with the eigenfunctions of $-\Delta_h$ on $H^1(\mathbb{S}^d)$, that is

$$L^2(\mathbb{S}^d) = \bigoplus_{\ell=0}^{\infty} E_\ell, \tag{3.3}$$

where for every $\ell \in \mathbb{N}_0$ the ℓ–th eigenspace $E_\ell = \operatorname{Ker}(-\Delta_h - \lambda_\ell \mathbb{I}_{L^2(\mathbb{S}^d)})$ is generated by the ℓ degree orthonormal (real valued) spherical harmonics Y_ℓ^j, with $j = 1, ..., c_\ell$ and

$$c_\ell = \binom{\ell + d}{d} - \binom{\ell + d - 2}{d}.$$

More precisely, the ℓ–th graded component of $L^2(\mathbb{S}^d)$ is generated by harmonic polynomial maps $P : \mathbb{R}^{d+1} \to \mathbb{R}$ restricted to $\mathbb{S}^d$ that are homogeneous of degree ℓ. Moreover, the representation of the orthogonal group $O(d+1)$ on the linear space E_ℓ is irreducible, in the sense of the representation theory, see Chapter IV of the celebrated monograph [**26**] by E.M. Stein and G. Weiss.

By (3.3) every function $v \in L^2(\mathbb{S}^d)$ admits a unique Fourier decomposition

$$v = \sum_{\ell=0}^{\infty} \sum_{j=1}^{c_\ell} \widehat{v}\,(\ell, j)\, Y_\ell^j, \tag{3.4}$$

where $\widehat{v}(\ell, j)$ denotes the Fourier coefficient of v given by

$$\widehat{v}(\ell, j) = \langle v, Y_\ell^j \rangle_{L^2(\mathbb{S}^d)}$$

for every $\ell \in \mathbb{N}_0$ and $j = 1, ..., c_\ell$. In other words, (3.4) has the expected expression

$$v = \sum_{\ell=0}^{\infty} \sum_{j=1}^{c_\ell} \langle v, Y_\ell^j \rangle_{L^2(\mathbb{S}^d)} Y_\ell^j$$

for every $v \in L^2(\mathbb{S}^d)$. Accordingly to (3.4), we can rewrite the inner product given in (3.2) as

$$\langle v, w \rangle_{H^m(\mathbb{S}^d)} = \sum_{\ell=0}^{\infty} (b_\ell^m + 1) \sum_{j=1}^{c_\ell} \widehat{v}\,(\ell, j)\, \widehat{w}\,(\ell, j) \quad \text{for every } v, w \in H^m(\mathbb{S}^d), \tag{3.5}$$

where $b_\ell = \ell(\ell + d - 1)$ denotes the ℓ–th eigenvalue of $-\Delta_h$ in $H^1(\mathbb{S}^d)$, that is

$$-\Delta_h Y_\ell^j = b_\ell Y_\ell^j \quad \text{in } \mathbb{S}^d \tag{3.6}$$

for all $j = 1, \dots, c_\ell$. Moreover, as it is well–known, by (3.5) the inner product on $H^m(\mathbb{S}^d)$, defined for every v, $w \in H^m(\mathbb{S}^d)$ by

$$\langle v, w\rangle_* = \sum_{\ell=0}^{\infty} \gamma_\ell(d,m) \sum_{j=1}^{c_\ell} \widehat{v}(\ell,j)\widehat{w}(\ell,j), \quad \gamma_\ell(d,m) = \frac{\Gamma\left(\dfrac{d}{2}+m+\ell\right)}{\Gamma\left(\dfrac{d}{2}-m+\ell\right)}, \tag{3.7}$$

induces the norm

$$\|v\|_* = \left(\sum_{\ell=0}^{\infty} \gamma_\ell(d,m) \sum_{j=1}^{c_\ell} \left|\widehat{v}(\ell,j)\right|^2\right)^{1/2} \qquad \text{for every } v \in H^m(\mathbb{S}^d),$$

which is equivalent to $\|\cdot\|_{H^m(\mathbb{S}^d)}$.

3.1 - *Variational formulation of problem* (3.1)

We first claim that

$$\mathfrak{D}^m Y_\ell^j = \gamma_\ell(d,m) Y_\ell^j \tag{3.8}$$

for every $\ell \in \mathbb{N}_0$ and $j = 1, ..., c_\ell$. Fix $\ell \in \mathbb{N}_0$ and $j = 1, ..., c_\ell$. Then, by (3.6) and (3.7)

$$\begin{aligned}\mathfrak{D}^m Y_\ell^j &= \prod_{k=1}^{m}\left(-\Delta_h + \frac{1}{4}(d-2k)(d+2k-2)\mathbb{I}_{L^2(\mathbb{S}^d)}\right) Y_\ell^j \\ &= \prod_{k=1}^{m}\left(b_\ell + \frac{1}{4}(d-2k)(d+2k-2)\right) Y_\ell^j \\ &= \gamma_\ell(d,m) Y_\ell^j,\end{aligned}$$

as claimed.

Let us now prove

$$\langle v, w\rangle_* = \int_{\mathbb{S}^d} (\mathfrak{D}^m v) w \, d\sigma_h \tag{3.9}$$

for every $v, w \in H^m(\mathbb{S}^d)$. To see this fix $v, w \in H^m(\mathbb{S}^d)$. By (3.4) clearly

$$\int_{\mathbb{S}^d} (\mathfrak{D}^m v) w \, d\sigma_h = \sum_{\ell=0}^{\infty}\sum_{j=1}^{c_\ell} \widehat{v}(\ell,j) \int_{\mathbb{S}^d} \left(\mathfrak{D}^m Y_\ell^j\right) w \, d\sigma_h.$$

On the other hand, (3.8) yields

$$\int_{\mathbb{S}^d} \left(\mathfrak{D}^m Y_\ell^j\right) w d\sigma_h = \gamma_\ell(d,m) \int_{\mathbb{S}^d} Y_\ell^j w d\sigma_h.$$

Thus

$$\int_{\mathbb{S}^d} (\mathfrak{D}^m v) w d\sigma_h = \sum_{\ell=0}^{\infty} \gamma_\ell(d,m) \sum_{j=1}^{c_\ell} \widehat{v}(\ell,j) \int_{\mathbb{S}^d} Y_\ell^j w d\sigma_h. \tag{3.10}$$

By (3.3) it follows that

$$\begin{aligned} \int_{\mathbb{S}^d} Y_\ell^j w d\sigma_h &= \sum_{\widetilde{\ell}=0}^{\infty} \sum_{\widetilde{j}=1}^{c_\imath} \widehat{w}(\widetilde{\ell},\widetilde{j}) \int_{\mathbb{S}^d} Y_\ell^j Y_{\widetilde{\ell}}^{\widetilde{j}} d\sigma_h = \sum_{\widetilde{\ell}=0}^{\infty} \sum_{\widetilde{j}=1}^{c_\imath} \widehat{w}(\widetilde{\ell},\widetilde{j}) \delta_{\widetilde{\ell},\ell} \delta_{\widetilde{j},j} \\ &= \widehat{w}(\ell,j). \end{aligned} \tag{3.11}$$

Then, (3.10) and (3.11) give

$$\int_{\mathbb{S}^d} (\mathfrak{D}^m v) w d\sigma_h = \sum_{\ell=0}^{\infty} \gamma_\ell(d,m) \sum_{j=1}^{c_\ell} \widehat{v}(\ell,j) \widehat{w}(\ell,j),$$

i.e. (3.9) is verified.

In conclusion, we have shown that the main transformed problem (3.1) has a variational nature. Consequently, we say that a function $v \in H^m(\mathbb{S}^d)$ is a *weak solution* of (3.1) if

$$\langle v, \varphi \rangle_* = \int_{\mathbb{S}^d} |v|^{2_m^* - 2} v \varphi d\sigma_h \tag{3.12}$$

for every $\varphi \in H^m(\mathbb{S}^d)$.

As shown in [**2**], there exists an explicit correspondence between the weak solutions of (3.1) and (1.1). To this end, let us introduce the classical stereographic projection $\pi : \mathbb{S}^d \setminus \{\sigma_o\} \to \mathbb{R}^d$ from the south pole $\sigma_o = (0, ..., 0, -1) \in \mathbb{S}^d \subset \mathbb{R}^{d+1}$, which is defined for every $\sigma = (x_1, ..., x_d, x_{d+1}) \in \mathbb{S}^d$ by

$$\pi(\sigma) = \left(\frac{x_1}{1 + x_{d+1}}, ..., \frac{x_d}{1 + x_{d+1}} \right).$$

It is well–known that the stereographic projection π is a conformal diffeomorphism whose inverse map $\pi^{-1} : \mathbb{R}^d \to \mathbb{S}^d \setminus \{\sigma_o\}$ is given for all $x = (x_1, ..., x_d) \in \mathbb{R}^d$ by

$$\pi^{-1}(x) = \left(\frac{2x_1}{1 + \sum_{i=1}^d x_i^2}, ..., \frac{2x_d}{1 + \sum_{i=1}^d x_i^2}, \frac{1 - \sum_{i=1}^d x_i^2}{1 + \sum_{i=1}^d x_i^2} \right). \tag{3.13}$$

We are now in a position to state the following key result.

Proposition 3.1. *If* $v \in L^{2^*_m}(\mathbb{S}^d)$, *then the function* u, *defined for all* $x = (x_1, ..., x_d) \in \mathbb{R}^d$ *by*

$$u(x) = U(x)v(\pi^{-1}(x)), \quad \text{where } U(x) = \left(\frac{2}{1+|x|}\right)^{d/2^*_m},$$

is of class $L^{2^*_m}(\mathbb{R}^d)$ *and*

$$\int_{\mathbb{S}^d} |v(\sigma)|^{2^*_m} d\sigma_h = \int_{\mathbb{R}^d} |u(x)|^{2^*_m} dx.$$

Moreover, if $v \in H^m(\mathbb{S}^d)$ *is a weak solution of* (3.1), *then* $u = Uv \circ \pi^{-1}$ *is of class* $\mathcal{D}^{m,2}(\mathbb{R}^d)$ *and solves* (1.1) *in the weak sense. Conversely, if* $u \in \mathcal{D}^{m,2}(\mathbb{R}^d)$ *is a weak solution of* (1.1), *then*

$$v = \frac{u \circ \pi}{U \circ \pi} \quad \text{in } \mathbb{S}^d$$

is of class $H^m(\mathbb{S}^d)$ *and solves* (3.1) *in the weak sense.*

For a detailed proof of the above result we refer to Proposition 2.2 and Lemma 2.3 of [**2**], as well as Proposition 3.1 of [**14**].

3.2 - *Groups actions and embedding results*

Let $\diamond : G \times W^{m,l}(\mathbb{S}^d) \to W^{m,l}(\mathbb{S}^d)$ be an action of a topological group G on the Sobolev space $W^{m,l}(\mathbb{S}^d)$, where m and l in $\mathbb{N}$. Set

$$W^{m,l}_G(\mathbb{S}^d) = \{v \in W^{m,l}(\mathbb{S}^d) \, : \, g \diamond v = v \text{ for all } g \in G\}.$$

If G is a subgroup of $O(d+1)$, the orbit $G\sigma$ of an element $\sigma \in \mathbb{S}^d$ is given by

$$G\sigma = \{g \cdot \sigma \, : \, \text{for all } g \in G\},$$

where $\cdot : G \times \mathbb{S}^d \to \mathbb{S}^d$ is the natural multiplicative action.

Proposition 3.2. *Let* G *be a closed subgroup of the orthogonal group* $O(d+1)$ *and let*

$$d_G = \min_{\sigma \in \mathbb{S}^d} \dim(G\sigma)$$

be the minimal dimension of the orbits in $\mathbb{S}^d$. *Then the Sobolev embedding*

$$W^{m,l}_G(\mathbb{S}^d) \hookrightarrow L^q(\mathbb{S}^d)$$

is compact for every $q \in [1, q_G)$, *where*

$$q_G = \begin{cases} \dfrac{l(d - d_G)}{d - d_G - lm} & \text{if } d > ml + d_G, \\ \infty & \text{if } d \le ml + d_G. \end{cases}$$

If $d > ml + d_G$, *then the space* $W_G^{m,l}(\mathbb{S}^d)$ *is continuously embedded in* $L^{q_G}(\mathbb{S}^d)$.

If G is a connected algebraic group, which acts on a variety Y (not necessarily affine), then for each $y \in Y$ the orbit Gy is an irreducible variety, that is Gy is open in its closure. Moreover, its boundary, $\partial Gy = \overline{Gy} \setminus Gy$, is the union of orbits of strictly smaller dimension. Finally, in this case orbits of minimal dimension are closed.

Corollary 3.3. *Let* m *and* l *be two positive integers. Let*

$$\odot : O(d+1) \times W^{m,l}(\mathbb{S}^d) \to W^{m,l}(\mathbb{S}^d)$$

be the action induced by the natural multiplicative action $\cdot : O(d+1) \times \mathbb{S}^d \to \mathbb{S}^d$ *of the orthogonal group* $O(d+1)$ *on the unit sphere* $\mathbb{S}^d$. *If* $G \subset O(d+1)$ *is a closed subgroup and*

$$W_G^{m,l}(\mathbb{S}^d) = \{w \in W^{m,l}(\mathbb{S}^d) \,:\, v(g^{-1} \cdot \sigma) = v(\sigma) \quad \text{for each } \sigma \in \mathbb{S}^d \text{ and for all } g \in G\},$$

then the embedding

$$W_G^{m,l}(\mathbb{S}^d) \hookrightarrow L^q(\mathbb{S}^d) \text{ is compact for every } q \in [1, \infty).$$

In particular, if $d > ml$ *then the embedding*

$$W_G^{m,l}(\mathbb{S}^d) \hookrightarrow L^{l_m^*}(\mathbb{S}^d), \quad l_m^* = \frac{ld}{d - lm},$$

is compact.

Proof. Since the orthogonal group $O(d+1)$ acts transitively on $\mathbb{S}^d$, then the orbit of each element of the sphere $\mathbb{S}^d$ is the whole sphere itself, so that $\dim O(d+1)\sigma = d$ for every $\sigma \in \mathbb{S}^d$. Thus, the minimal dimension

$$d_{O(d+1)} = \min_{\sigma \in \mathbb{S}^d} \dim O(d+1)\sigma = d,$$

and Proposition 3.2 yields that the embedding

$$W_{O(d+1)}^{m,l}(\mathbb{S}^d) \hookrightarrow L^q(\mathbb{S}^d)$$

is compact for every $q \in [1, \infty)$. Consequently, if $\odot : O(d+1) \times W^{m,l}(\mathbb{S}^d) \to W^{m,l}(\mathbb{S}^d)$ is the action induced by the natural multiplicative action $\cdot$, then the embedding

$$W^{m,l}_G(\mathbb{S}^d) \hookrightarrow W^{m,l}_{O(d+1)}(\mathbb{S}^d)$$

is continuous for every closed subgroup $G \subset O(d+1)$. In particular, the embedding

$$W^{m,l}_G(\mathbb{S}^d) \hookrightarrow L^q(\mathbb{S}^d) \text{ is compact for every } q \in [1, \infty)$$

along the closed subgroup $G \subset O(d+1)$. This completes the proof. □

Due to the usual role of the critical exponent, the Sobolev space $H^m(\mathbb{S}^d)$ cannot be compactly embedded into the Lebesgue space $L^{2^*_m}(\mathbb{S}^d)$. In order to prove Theorem 1.1 we recover compactness for suitable symmetric subspaces of $H^m(\mathbb{S}^d)$ thanks to the validity of Proposition 3.2. Indeed, Proposition 3.2 produces compact embeddings in higher order Lebesgue spaces $L^q(\mathbb{S}^d)$, $q \geq 2^*_m$. Such properties have been observed in specific contexts by several authors, see [**12, 13**] and references therein for related topics. This approach is fruitful in the study of a wide class of variational elliptic problems in the presence of a suitable group action on the Sobolev space, thanks to the famous principle of symmetric criticality due to R.S. Palais in [**21**].

Let $d \geq 5$ and $s_d = [d/2] + (-1)^{d+1} - 1$. For every $i \in J_d = \{1, ..., s_d\}$, let us define

$$G_{d,i} = \begin{cases} O(i+1) \times O(d-2i-1) \times O(i+1), & \text{if } i \neq \dfrac{d-1}{2}, \\ O(i+1) \times O(i+1), & \text{if } i = \dfrac{d-1}{2}, \end{cases}$$

and for all i, $j \in J_d$, with $i \neq j$. Denote by $G^d_{i,j}$ the group $\langle G_{d,i}; G_{d,j} \rangle$ generated by $G_{d,i}$ and $G_{d,j}$. The following result, proved in Proposition 3.2 of [**13**], will be crucial in the sequel.

Proposition 3.4. *For every $i, j \in J_d$, with $i \neq j$, the group $G^d_{i,j}$ acts transitively on $\mathbb{S}^d$, i.e. there exists $\sigma_0 \in \mathbb{S}^d$ such that $G^d_{i,j}\sigma_0 = \mathbb{S}^d$.*

Fix $d \geq 5$ and $G_{d,i}$ for some $i \in J_d$. Let $\tau_i : \mathbb{S}^d \to \mathbb{S}^d$ be the function associated to $G_{d,i}$ and defined for all $\sigma = (\sigma_1, \sigma_2, \sigma_3) \in \mathbb{S}^d$ by

$$\tau_i(\sigma) = \begin{cases} (\sigma_3, \sigma_2, \sigma_1), & \text{if } i \neq \dfrac{d-1}{2} \text{ and } \sigma_1, \sigma_3 \in \mathbb{R}^{i+1},\ \sigma_2 \in \mathbb{R}^{d-2i-1}, \\ (\sigma_3, \sigma_1), & \text{if } i = \dfrac{d-1}{2} \text{ and } \sigma_1, \sigma_3 \in \mathbb{R}^{i+1}. \end{cases}$$

By construction,

$$\tau_i \notin G_{d,i}, \quad \tau_i G_{d,i} \tau_i^{-1} = G_{d,i} \quad \text{and} \quad \tau_i^2 = \mathbb{I}_{\mathbb{R}^{d+1}}$$

for every $i \in J_d$.

For small dimensions $d > 2m$ the explicit form of the groups $G_{d,i}$ and of the functions τ_i are summarized in Chapter 10 of [**16**] and we report the table here for clarity purposes.

Table 4. *Some explicit constructions for low dimensions*

d	s_d	$G_{d,i};\ i \in \{1,...,s_d\}$	$\tau_i;\ i \in \{1,...,s_d\}$
5	2	$G_{5,1} = O(2) \times O(2) \times O(2)$	$\tau_1(\sigma_1,\sigma_2,\sigma_3) = (\sigma_3,\sigma_2,\sigma_1)$; $\sigma_1,\sigma_2,\sigma_3 \in \mathbb{R}^2$
		$G_{5,2} = O(3) \times O(3)$	$\tau_2(\sigma_1,\sigma_2) = (\sigma_2,\sigma_1)$; $\sigma_1,\sigma_2 \in \mathbb{R}^3$
6	1	$G_{6,1} = O(2) \times O(3) \times O(2)$	$\tau_1(\sigma_1,\sigma_2,\sigma_3) = (\sigma_3,\sigma_2,\sigma_1)$; $\sigma_1,\sigma_3 \in \mathbb{R}^2, \sigma_2 \in \mathbb{R}^3$
7	3	$G_{7,1} = O(2) \times O(4) \times O(2)$	$\tau_1(\sigma_1,\sigma_2,\sigma_3) = (\sigma_3,\sigma_2,\sigma_1)$; $\sigma_1,\sigma_3 \in \mathbb{R}^2, \sigma_2 \in \mathbb{R}^4$
		$G_{7,2} = O(3) \times O(2) \times O(3)$	$\tau_2(\sigma_1,\sigma_2,\sigma_3) = (\sigma_3,\sigma_2,\sigma_1)$; $\sigma_1,\sigma_3 \in \mathbb{R}^3, \sigma_2 \in \mathbb{R}^2$
		$G_{7,3} = O(4) \times O(4)$	$\tau_3(\sigma_1,\sigma_2) = (\sigma_2,\sigma_1)$; $\sigma_1,\sigma_2 \in \mathbb{R}^4$
8	2	$G_{8,1} = O(2) \times O(5) \times O(2)$	$\tau_1(\sigma_1,\sigma_2,\sigma_3) = (\sigma_3,\sigma_2,\sigma_1)$; $\sigma_1,\sigma_3 \in \mathbb{R}^2, \sigma_2 \in \mathbb{R}^5$
		$G_{8,2} = O(3) \times O(3) \times O(3)$	$\tau_2(\sigma_1,\sigma_2,\sigma_3) = (\sigma_2,\sigma_1,\sigma_3)$; $\sigma_1,\sigma_2,\sigma_3 \in \mathbb{R}^3$
9	4	$G_{9,1} = O(2) \times O(6) \times O(2)$	$\tau_1(\sigma_1,\sigma_2,\sigma_3) = (\sigma_3,\sigma_2,\sigma_1)$; $\sigma_1,\sigma_3 \in \mathbb{R}^2, \sigma_2 \in \mathbb{R}^6$
		$G_{9,2} = O(3) \times O(4) \times O(3)$	$\tau_2(\sigma_1,\sigma_2,\sigma_3) = (\sigma_3,\sigma_2,\sigma_1)$; $\sigma_1,\sigma_3 \in \mathbb{R}^3, \sigma_2 \in \mathbb{R}^4$
		$G_{9,3} = O(4) \times O(2) \times O(4)$	$\tau_3(\sigma_1,\sigma_2,\sigma_3) = (\sigma_3,\sigma_2,\sigma_1)$; $\sigma_1,\sigma_3 \in \mathbb{R}^4, \sigma_2 \in \mathbb{R}^2$
		$G_{9,4} = O(5) \times O(5)$	$\tau_4(\sigma_1,\sigma_2) = (\sigma_2,\sigma_1)$; $\sigma_1,\sigma_2 \in \mathbb{R}^5$

d	s_d	$G_{d,i};\ i \in \{1,...,s_d\}$	$\tau_i;\ i \in \{1,...,s_d\}$
10	3	$G_{10,1} = O(2) \times O(7) \times O(2)$	$\tau_1(\sigma_1,\sigma_2,\sigma_3) = (\sigma_3,\sigma_2,\sigma_1)$; $\sigma_1,\sigma_3 \in \mathbb{R}^2, \sigma_2 \in \mathbb{R}^7$
		$G_{10,2} = O(3) \times O(5) \times O(3)$	$\tau_2(\sigma_1,\sigma_2,\sigma_3) = (\sigma_3,\sigma_2,\sigma_1)$; $\sigma_1,\sigma_3 \in \mathbb{R}^3, \sigma_2 \in \mathbb{R}^5$
		$G_{10,3} = O(4) \times O(3) \times O(4)$	$\tau_3(\sigma_1,\sigma_2,\sigma_3) = (\sigma_3,\sigma_2,\sigma_1)$; $\sigma_1,\sigma_3 \in \mathbb{R}^4, \sigma_2 \in \mathbb{R}^3$
11	5	$G_{11,1} = O(2) \times O(8) \times O(2)$	$\tau_1(\sigma_1,\sigma_2,\sigma_3) = (\sigma_3,\sigma_2,\sigma_1)$; $\sigma_1,\sigma_3 \in \mathbb{R}^2, \sigma_2 \in \mathbb{R}^8$
		$G_{11,2} = O(3) \times O(6) \times O(3)$	$\tau_2(\sigma_1,\sigma_2,\sigma_3) = (\sigma_3,\sigma_2,\sigma_1)$; $\sigma_1,\sigma_3 \in \mathbb{R}^3, \sigma_2 \in \mathbb{R}^6$
		$G_{11,3} = O(4) \times O(4) \times O(4)$	$\tau_3(\sigma_1,\sigma_2,\sigma_3) = (\sigma_2,\sigma_1,\sigma_3)$; $\sigma_1,\sigma_2,\sigma_3 \in \mathbb{R}^4$
		$G_{11,4} = O(5) \times O(2) \times O(5)$	$\tau_4(\sigma_1,\sigma_2,\sigma_3) = (\sigma_3,\sigma_2,\sigma_1)$; $\sigma_1,\sigma_3 \in \mathbb{R}^5,\ \sigma_2 \in \mathbb{R}^2$
		$G_{11,5} = O(6) \times O(6)$	$\tau_5(\sigma_1,\sigma_2) = (\sigma_2,\sigma_1);\ \sigma_1,\sigma_2 \in \mathbb{R}^6$
12	4	$G_{12,1} = O(2) \times O(9) \times O(2)$	$\tau_1(\sigma_1,\sigma_2,\sigma_3) = (\sigma_3,\sigma_2,\sigma_1)$; $\sigma_1,\sigma_3 \in \mathbb{R}^2, \sigma_2 \in \mathbb{R}^9$
		$G_{12,2} = O(3) \times O(7) \times O(3)$	$\tau_2(\sigma_1,\sigma_2,\sigma_3) = (\sigma_3,\sigma_2,\sigma_1)$; $\sigma_1,\sigma_3 \in \mathbb{R}^3, \sigma_2 \in \mathbb{R}^7$
		$G_{12,3} = O(4) \times O(5) \times O(4)$	$\tau_3(\sigma_1,\sigma_2,\sigma_3) = (\sigma_3,\sigma_2,\sigma_1)$; $\sigma_1,\sigma_3 \in \mathbb{R}^4, \sigma_2 \in \mathbb{R}^5$
		$G_{12,4} = O(5) \times O(3) \times O(5)$	$\tau_4(\sigma_1,\sigma_2,\sigma_3) = (\sigma_3,\sigma_2,\sigma_1)$; $\sigma_1,\sigma_3 \in \mathbb{R}^5, \sigma_2 \in \mathbb{R}^3$

As in [**13**], recalling that $d > 2m$ throughout the paper, we define for all $i \in J_d$ an action $\widehat{\circledast}_i$ of the compact group

$$G_{d,i}^{\tau_i} = \langle G_{d,i}, \tau_i \rangle \subset O(d+1) \tag{3.14}$$

on the Sobolev space $H^m(\mathbb{S}^d)$.

More precisely, we consider the action $\widehat{\circledast}_i : G_{d,i}^{\tau_i} \times H^m(\mathbb{S}^d) \to H^m(\mathbb{S}^d)$, $(\widetilde{g}, v) \mapsto \widetilde{g}\widehat{\circledast}_i v$, which is defined pointwise for every $\sigma \in \mathbb{S}^d$ by

$$(\widetilde{g}\widehat{\circledast}_i v)(\sigma) = \begin{cases} v(g^{-1} \cdot \sigma), & \text{if } \widetilde{g} = g \in G_{d,i} \\ -v(g^{-1}\tau_i^{-1} \cdot \sigma), & \text{if } \widetilde{g} = \tau_i\, g \in G_{d,i}^{\tau_i} \setminus G_{d,i},\ g \in G_{d,i}. \end{cases} \tag{3.15}$$

This can be done by the properties of τ_i. Therefore, $\widehat{\circledast}_i$ is well defined, linear and continuous.

Let us consider for every $i \in J_d$ the subspace $H^m_{G^{\tau_i}_{d,i}}(\mathbb{S}^d)$ of $H^m(\mathbb{S}^d)$ given by

$$H^m_{G^{\tau_i}_{d,i}}(\mathbb{S}^d) = \{v \in H^m(\mathbb{S}^d) \,:\, \tilde{g}\widehat{\circledast}_i v = v \text{ for all } \tilde{g} \in G^{\tau_i}_{d,i}\}.$$

Clearly, $H^m_{G^{\tau_i}_{d,i}}(\mathbb{S}^d)$ contains all the functions $v \in H^m(\mathbb{S}^d)$, which are symmetric with respect to the action $\widehat{\circledast}_i$ of the compact group $G^{\tau_i}_{d,i}$.

Moreover, for every $i \in J_d$ we also introduce

$$H^m_{G_{d,i}}(\mathbb{S}^d) = \{v \in H^m(\mathbb{S}^d) \,:\, g \circledast_i v = v \text{ for all } g \in G_{d,i}\},$$

where the action $\circledast_i : G_{d,i} \times H^m(\mathbb{S}^d) \to H^m(\mathbb{S}^d)$ of the compact group $G_{d,i}$ on $H^m(\mathbb{S}^d)$, $(g, v) \mapsto g \circledast_i v$, is defined pointwise for all $\sigma \in \mathbb{S}^d$ by

$$(g \circledast_i v)(\sigma) = v(g^{-1} \cdot \sigma). \tag{3.16}$$

Remark 3.5. Every $v \in H_{G^{\tau_i}_{d,i}}(\mathbb{S}^d) \setminus \{0\}$ does not have constant sign. Indeed, $v(\sigma) = -v(\tau_i^{-1} \cdot \sigma)$ for every $\sigma \in \mathbb{S}^d$, since v is $G^{\tau_i}_{d,i}$–invariant by (3.16). The conclusion follows immediately from the fact that v is not zero.

By Proposition 3.4, arguing as in the proof of Theorem 3.1 of [**13**], the following result holds.

Proposition 3.6. *Let $d > 2m$, with $m \geq 2$. Then the following statements hold for any fixed $i \in J_d$.*

(*i*) *The Sobolev space $H^m_{G_{d,i}}(\mathbb{S}^d)$ is compactly embedded into $L^q(\mathbb{S}^d)$, whenever $q \in [1, q_i^\star)$, where*

$$q_i^\star = \begin{cases} \dfrac{2(d-1)}{d-2m-1} & \text{if } d > 2m+1, \\ \infty & \text{if } d = 2m+1; \end{cases}$$

(*ii*) $H^m_{G_{d,i}}(\mathbb{S}^d) \cap H^m_{G_{d,j}}(\mathbb{S}^d) = \{\text{constant functions on } \mathbb{S}^d\}$ *for every $j \in J_d$, with $j \neq i$;*

(*iii*) $H^m_{G^{\tau_i}_{d,i}}(\mathbb{S}^d) \cap H^m_{G^{\tau_j}_{d,j}}(\mathbb{S}^d) = \{0\}$ *for every $j \in J_d$, with $j \neq i$.*

Proof. Part (i) – A careful analysis of the definition of $G_{d,i}$ shows that the $G_{d,i}$–orbit of every point $\sigma \in \mathbb{S}^d$ has at least dimension 1, i.e., $\dim(G_{d,i}\sigma) \geq 1$ for every $\sigma \in \mathbb{S}^d$, and

$$d_{G_{d,i}} = \min_{\sigma \in \mathbb{S}^d} \dim(G_{d,i}\sigma) \geq 1.$$

Hence, by Proposition 3.2 the space $H^m_{G_{d,i}}(\mathbb{S}^d)$ is compactly embedded into $L^q(\mathbb{S}^d)$ for every $q \in [1, q_i^\star)$. Since $d > 2m$, then

$$q_i^\star > 2^*_m = \frac{2d}{d-2m}$$

and $H_{G^{\tau_i}_{d,i}}(\mathbb{S}^d) \subset H_{G_{d,i}}(\mathbb{S}^d)$, so that the embedding

$$H_{G^{\tau_i}_{d,i}}(\mathbb{S}^d) \hookrightarrow L^{2^*_m}(\mathbb{S}^d)$$

is compact for every $i \in J_d$.

Part (ii) – Fix $j \in J_d$, with $j \neq i$, and $v \in H^m_{G_{d,i}}(\mathbb{S}^d) \cap H^m_{G_{d,j}}(\mathbb{S}^d)$. Since v is both $G_{d,i}$ and $G_{d,j}$–invariant, then v is also $G^d_{i,j}$–invariant, i.e., $v(g \cdot \sigma) = v(\sigma)$ for every $g \in G^d_{i,j}$ and $\sigma \in \mathbb{S}^d$. According to Proposition 3.4, the group $G^d_{i,j}$ acts transitively on the sphere $\mathbb{S}^d$, i.e., $G^d_{i,j}\sigma = \mathbb{S}^d$ for each $\sigma \in \mathbb{S}^d$. Thus, v is a constant function.

Part (iii) – Fix $j \in J_d$, with $j \neq i$, and $v \in H^m_{G^{\tau_i}_{d,i}}(\mathbb{S}^d) \cap H^m_{G^{\tau_j}_{d,j}}(\mathbb{S}^d)$. The second relation of (3.15) shows that $v(\sigma) = -v(\tau_i^{-1} \cdot \sigma) = -v(\tau_j^{-1} \cdot \sigma)$ for every $\sigma \in \mathbb{S}^d$. But, Part (ii) shows that v is constant. Thus, v must be identically zero in $\mathbb{S}^d$. □

Problem (3.1) has a variational nature and its Euler–Lagrange functional $\mathcal{J}$ is given by

$$\mathcal{J}(v) = \frac{1}{2}\|v\|_*^2 - \int_{\mathbb{S}^d} |v|^{2^*_m} d\sigma_h, \quad v \in H^m(\mathbb{S}^d). \tag{3.17}$$

Clearly, the functional $\mathcal{J}$ is well–defined in $H^m(\mathbb{S}^d)$ and it is of class $C^1(H^m(\mathbb{S}^d))$. Moreover, for each $v \in H^m(\mathbb{S}^d)$

$$\langle \mathcal{J}(v), \varphi \rangle = \langle v, \varphi \rangle_* - \int_{\mathbb{S}^d} |v|^{2^*_m - 2} v \varphi d\sigma_h \tag{3.18}$$

for every $\varphi \in H^m(\mathbb{S}^d)$. Hence, the critical points of $\mathcal{J}$ in $H^m(\mathbb{S}^d)$ are exactly the weak solutions of (3.1).

Let G be a topological group. We say that $u \in H^m_G(\mathbb{S}^d)$ is a *weak solution of* (3.1) *only in the $H^m_G(\mathbb{S}^d)$ sense* if

$$\langle \mathcal{J}(v), \varphi\rangle = \langle v, \varphi\rangle_* - \int_{\mathbb{S}^d} |v|^{2^*_m - 2} v\varphi d\sigma_h \tag{3.19}$$

for any $\varphi \in H^m_G(\mathbb{S}^d)$. Then, $u \in H^m_G(\mathbb{S}^d)$ is a weak solution of (3.1) in the whole space $H^m(\mathbb{S}^d)$, that is in sense of definition (3.18), if the *principle of symmetric criticality* of Palais given in [**21**] holds. To prove this let us recall the well known principle of symmetric criticality of Palais stated in the general form proved in [**6**] for reflexive strictly convex Banach spaces. For details and comments we refer to Section 5 of [**5**].

More precisely, let $X = (X, \|\cdot\|_X)$ be a reflexive strictly convex Banach space. Suppose that $\mathcal{G}$ is a subgroup of isometries $g : X \to X$, that is g is linear and $\|g(u)\|_X = \|u\|_X$ for all $u \in X$. Consider the $\mathcal{G}$–invariant closed subspace of X

$$\Sigma_{\mathcal{G}} = \{u \in X : g(u) = u \text{ for all } g \in \mathcal{G}\}.$$

By Proposition 3.1 of [**6**] we have

Lemma 3.7. *Let X, $\mathcal{G}$ and Σ be as before and let I be a C^1 functional defined on X such that the composition $I \circ g = I$ for all $g \in G$. Then $u \in \Sigma_{\mathcal{G}}$ is a critical point of I in X if and only if u is a critical point of $I|_{\Sigma_{\mathcal{G}}}$.*

Bartsch, Schneider, and Weth observed in Remark 1.2 – Part a) of [**2**] that a careful choice of a subgroup of $O(d + 1)$ in certain dimensions assures the existence of infinitely many solutions of equation (1.1). Their abstract approach, based on groups symmetries, gives additional information on the nodal structure of the solutions. In the same paper the authors also point out that several unbounded sequences of changing sign solutions distinguished by their symmetry properties can be obtained. The number of such sequences increases with the number of partitions of the Euclidean dimension d. In the case $d \geq 5$, exploiting the topological group arguments developed in [**13**], a key ingredient used along our proof is based on the explicit construction of the subgroups $G^{\tau_i}_{d,i} \subset O(d+1)$ such that the energy functional $\mathcal{J}$ is invariant under a subgroup action of $O(d + 1)$ and whose restriction to the subspaces $H^m_{G^{\tau_i}_{d,i}}(\mathbb{S}^d)$ of $G^{\tau_i}_{d,i}$–invariant functions admits a sequence of critical points. Due to the principle of symmetric criticality of Palais recalled in Lemma 3.7, these points will be also critical points of the original functional $\mathcal{J}$, depending on the choice of the subgroup of $O(d + 1)$. According to Proposition 3.6 the subspaces $H^m_{G^{\tau_i}_{d,i}}(\mathbb{S}^d)$ have different symmetry structures for every $i, j \in J_d$, with $i \neq j$. Consequently, (1.1) admits at least s_d sequences of infinitely many finite energy nodal weak solutions, with mutually different symmetry structure. We emphasize that the invariance of $\mathcal{J}$, with respect to translations and dilations, implies that the functional $\mathcal{J}$ does not satisfy the Palais–Smale condition. However, as observed in Proposition 3.1 of [**2**], the symmetric mountain pass theorem in addition to Lemma 3.7 yields the following critical point result.

Theorem 3.8. *Let G be a compact group. Let*

$$\diamond : G \times H^m(\mathbb{S}^d) \to H^m(\mathbb{S}^d), \quad (g, w) \mapsto g \diamond w,$$

be a linear and isometric action of G on $H^m(\mathbb{S}^d)$ and denote by

$$H^m_G(\mathbb{S}^d) = \{v \in H^m(\mathbb{S}^d) \,:\, g \diamond v = v \text{ for all } g \in G\}$$

the subspace of $H^m(\mathbb{S}^d)$ containing all the symmetric functions with respect to the group G. Let $\mathcal{J}$ be the energy functional associated to (3.1) *and assume that*

(*i*) *$\mathcal{J}$ is G–invariant*;

(*ii*) *the embedding $H^m_G(\mathbb{S}^d) \hookrightarrow L^{2^*_m}(\mathbb{S}^d)$ is compact*;

(*iii*) *$H^m_G(\mathbb{S}^d)$ has infinite dimension*;

hold. Then, the functional $\mathcal{J}$ admits a sequence of critical points $(v_k)_k \subset H^m_G(\mathbb{S}^d)$ such that

$$\int_{\mathbb{S}^d} |v_k|^{2^*_m} d\sigma_h \to \infty$$

as $k \to \infty$.

More recently, in [**20**] the author describes a group theoretical scheme, which arises in previous papers on $O(d+1)$–invariant variational problems, as a method to show the existence of several geometrically different sequences of solutions, distinguished by their symmetry properties. The abstract approach developed in [**20**] can be applied to $H^m(\mathbb{S}^d)$ in order to find a finite family $\{H^m_{K_i}(\mathbb{S}^d)\}_{i=1}^{\kappa_d}$ of subspaces $H^m_{K_i}(\mathbb{S}^d) \subset H^m(\mathbb{S}^d)$ such that for every K_l, $K_s \subset O(d+1)$, with $l \neq s$, we have

(*i*) $H^m_{K_l}(\mathbb{S}^d) \cap H^m_{K_s}(\mathbb{S}^d) = \{0\}$;

(*ii*) $\big(O(d+1)v\big) \cap H^m_{K_l}(\mathbb{S}^d) = \emptyset$ for every $v \in H^m_{K_s}(\mathbb{S}^d) \setminus \{0\}$.

The above construction in addition to the Palais symmetry principle, recalled in Lemma 3.7, as well as Proposition 3.1, ensure that equation (1.1) admits at least κ_d geometrically different sequences of solutions distinguished by their symmetry properties. This result is summarized in Theorem 4.8 of [**20**] in a more general form. In order to prove Theorem 4.8 W. Marzantowicz studies the intrinsic linking between orthogonal Borel subgroups in $O(d+1)$ with partial and orthogonal flags in $\mathbb{R}^{d+1}$. The key tool is the use of the number $p(d+1)$ of the unrestricted partitions of the Euclidean dimension $d+1$.

4 - Proof of the main result

We divide the proof of Theorem 1.1 in three steps.

Step 1: *Euclidean dimension* $d = 4$ *and order of differentiability* $m = 1$. In such a case the main result is a direct consequence of Theorem 1.1 of [**2**]. Indeed, the functional $\mathcal{J} : H^1(\mathbb{S}^4) \to \mathbb{R}$ reduces to

$$\mathcal{J}(v) = \frac{1}{2}\int\limits_{\mathbb{S}^4} |\nabla_h v|_h^2 d\sigma_h - \int\limits_{\mathbb{S}^4} |v|^4 d\sigma_h$$

for every $v \in H^1(\mathbb{S}^4)$.

From now on we argue as in the proof of Theorem 1.1 of [**2**] and take as compact group

$$G = O(2) \times O(3) \subset O(5).$$

Let $\cdot : G \times \mathbb{S}^4 \to \mathbb{S}^4$ be the standard action of the group G on the sphere $\mathbb{S}^4$ and denote by $\natural : G \times H^1(\mathbb{S}^4) \to H^1(\mathbb{S}^4)$ the induced linear, continuous and isometric action of the group G on the space $H^1(\mathbb{S}^4)$, $(g, v) \mapsto g\natural v$, defined pointwise for all $\sigma \in \mathbb{S}^4$ by

$$(g\natural v)(\sigma) = v(g^{-1} \cdot \sigma).$$

Now, an easy calculation ensures that the functional $\mathcal{J}$ is G–invariant, that is

$$\mathcal{J}(g\natural v) = \mathcal{J}(v)$$

for every $(g, v) \in G \times H^1(\mathbb{S}^4)$. Moreover, the subspace

$$H^1_G(\mathbb{S}^4) = \{v \in H^1(\mathbb{S}^4) \, : \, g\natural v = v \text{ for all } g \in G\}$$

of $H^1(\mathbb{S}^4)$, which consists of the G–invariant functions of $H^1(\mathbb{S}^4)$, has infinite dimension.

Since $\dim(G\sigma) \geq 1$ for every $\sigma \in \mathbb{S}^4$, then $q_G = 6 > 4$. Thus, Proposition 3.2 ensures that the embedding

$$H^1_G(\mathbb{S}^4) \hookrightarrow L^4(\mathbb{S}^4)$$

is compact.

Hence, by Theorem 3.8 the functional $\mathcal{J}$ admits a sequence of critical points $(v_k)_k$ in $H^1_G(\mathbb{S}^4)$ such that

$$\int_{\mathbb{S}^4} |v_k|^4 d\sigma_h \to \infty,$$

as $k \to \infty$. Lemma 3.7 and Proposition 3.1 yield that the equation

$$\begin{cases} -\Delta u = |u|^2 u & \text{in } \mathbb{R}^4, \\ u \in \mathcal{D}^{1,2}(\mathbb{R}^4), \end{cases}$$

admits a sequence $(u_k)_k \subset \mathcal{D}^{1,2}(\mathbb{R}^4)$ of solutions such that

$$\int_{\mathbb{R}^4} |\nabla u_k|^4 dx \to \infty, \text{ that is } \|u_k\| \to \infty,$$

as $k \to \infty$. Finally, Remark 1.2 (b) of [**2**] implies that there is $k_0 \in \mathbb{N}$ such that u_k changes sign for every $k \geq k_0$.

Step 2: Euclidean dimension $d \geq 5$ and order of differentiability $m < d/2$. Let $\mathcal{J}$ be the energy functional associated to (3.1) and given in (3.17). Fix $i \in J_d$ and consider the compact group

$$G^{\tau_i}_{d,i} \subset O(d+1),$$

given in (3.14) and let $\widehat{\circledast}_i : G^{\tau_i}_{d,i} \times H^m(\mathbb{S}^d) \to H^m(\mathbb{S}^d)$ be the action defined in (3.15).

Thanks to the definition of $\widehat{\circledast}_i$, the functional $\mathcal{J}$ is $G^{\tau_i}_{d,i}$–invariant, that is

$$\mathcal{J}(\widetilde{g}\widehat{\circledast}_i v) = \mathcal{J}(v)$$

for every $(\widetilde{g}, v) \in G^{\tau_i}_{d,i} \times H^m(\mathbb{S}^d)$. Then, the subspace

$$H^m_{G^{\tau_i}_{d,i}}(\mathbb{S}^d) = \{v \in H^m(\mathbb{S}^d) \,:\, \tilde{g}\widehat{\circledast}_i v = v \text{ for all } \tilde{g} \in G^{\tau_i}_{d,i}\}$$

of $H^m(\mathbb{S}^d)$, which consists of $G^{\tau_i}_{d,i}$–invariant functions, has infinite dimension.

Since $\dim\left(G^{\tau_i}_{d,i}\sigma\right) \geq 1$ for every $\sigma \in \mathbb{S}^d$, then $q_i^\star > 2^*_m$. Thus, Proposition 3.2 ensures that the embedding

$$H^m_{G^{\tau_i}_{d,i}}(\mathbb{S}^d) \hookrightarrow L^{2^*_m}(\mathbb{S}^d)$$

is compact.

Hence, by Theorem 3.8 the functional $\mathcal{J}$ admits a sequence of critical points $(v_k^{(i)})_k$ in $H^m_{G^{\tau_i}_{d,i}}(\mathbb{S}^d)$ such that

$$\int_{\mathbb{S}^d} |v_k^{(i)}|^{2^*_m} d\sigma_h \to \infty,$$

as $k \to \infty$. Lemma 3.7 and Proposition 3.1 imply that (1.1) admits a sequence $(u_k^{(i)})_k \subset \mathcal{D}^{m,2}(\mathbb{R}^d)$ of solutions such that

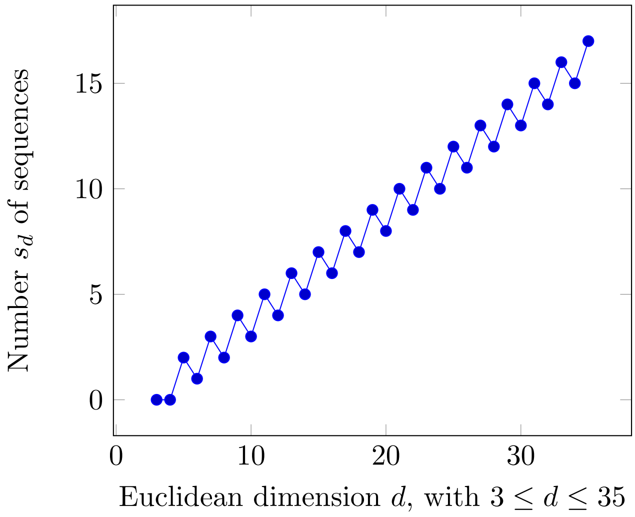
$$\|u_k^{(i)}\| \to \infty \quad \text{as } k \to \infty. \tag{4.1}$$

Consequently, Proposition 3.6 – Part (ii) gives that (1.1) admits at least

$$s_d = [d/2] + (-1)^{d+1} - 1$$

sequences $(u_k^{(i)})_k \subset \mathcal{D}^{m,2}(\mathbb{R}^d)$ of weak solutions, satisfying (4.1).

Finally, Remark 3.5 yields that the solutions $u_k^{(i)}$ for every $k \in \mathbb{N}$ and $i \in J_d$ are changing sign.



Step 3: *Euclidean dimension* $d \geq 3$, *with* $d \neq 4$, *and order of differentiability* $m < d/2$. In this case the assertion is a consequence of the general result proved in Theorem 4.8 of [**20**]. Indeed, the C^1 functional $\mathcal{J}$, associated to (3.1) and given in (3.17), is $O(d+1)$–invariant, with respect to the linear isometric action $\odot : O(d+1) \times H^m(\mathbb{S}^d) \to H^m(\mathbb{S}^d)$ induced by the natural multiplicative action $\cdot$ of the orthogonal group $O(d+1)$ on $\mathbb{S}^d$. By Corollary 3.3 and Theorem 3.8 the following facts hold for every closed subgroup $G \subset O(d+1)$.

(i) $\mathcal{J}$ is G–invariant;

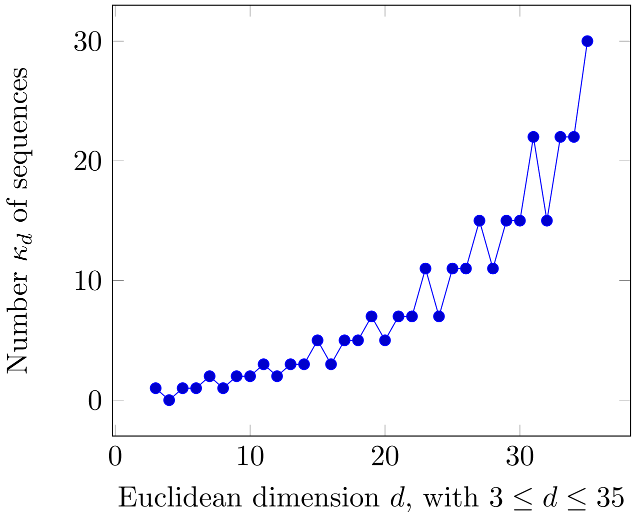
(ii) The Sobolev space $H^m_G(\mathbb{S}^d)$ has infinite dimension;

(iii) The restriction $\mathcal{J}\big|_{H^m_G(\mathbb{S}^d)}$ of $\mathcal{J}$ on $H^m_G(\mathbb{S}^d)$ has infinitely many critical points $(v_k)_k$ such that $\|v_k\|_{L^{2^*_m}(\mathbb{S}^d)} \to \infty$ as $k \to \infty$.

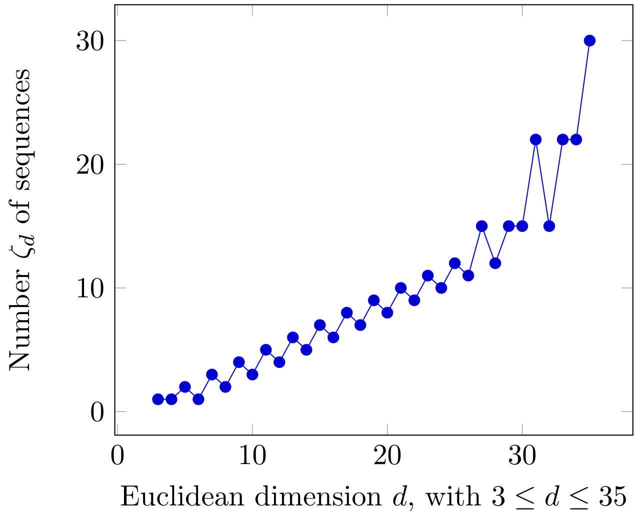
Then, by Theorem 4.8 of [**20**] there exist κ_d sequences of changing sign solutions, where κ_d is defined in (1.2). In particular, for each $i = 1, \ldots, \kappa_d$ there exists a sequence $(v^{(i)}_k)_k$ of changing sign solutions $v^{(i)}_k$ in $H^m(\mathbb{S}^d)$, which are mutually symmetrically distinct in k and such that

$$\int_{\mathbb{S}^d} |v^{(i)}_k|^{2^*_m} d\sigma_h \to \infty$$

as $k \to \infty$. Proposition 3.1 implies that (1.1) admits κ_d sequences $(u^{(i)}_k)_k \subset \mathcal{D}^{m,2}(\mathbb{R}^d)$ of solutions such that (4.1) is satisfied. Moreover, in this case the solutions $u^{(i)}_k$ are changing sign and mutually symmetrically distinct in k.



In summary, Steps 1–3 show that (1.1) admits at least $\zeta_d = \max\{1, s_d, \kappa_d\}$ sequences $(u_k^{(i)})_k$ of changing sign solutions $u_k^{(i)}$ of class $\mathcal{D}^{m,2}(\mathbb{R}^d)$, satisfying (4.1) for each $i = 1, \dots, \zeta_d$ and mutually symmetrically distinct in k.



Finally, the constructed weak solutions are of class $C^{2m}(\mathbb{R}^d)$ and classical for (1.1), by the regularity result of S. Luckhaus [**17**]. Even if in [**17**] only the Dirichlet problem on a bounded domain is considered, the methods used there also yield interior regularity for arbitrary boundary conditions. This completes the proof of Theorem 1.1. $\square$

We emphasize that the strategy adopted along this paper can be performed in order to investigate the existence of entire solutions for Schrödinger–Hardy systems involving two nonlocal fractional operators; see [**8**] for related topics.

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