Riv. Mat. Univ. Parma, Vol. **10** (2019), 85–97

Mohammad Ashraf and Bilal Ahmad Wani

On derivations of rings and Banach algebras involving anti-commutator

Abstract. In the present paper we obtain commutativity of a semiprime ring $\mathcal{R}$ which admits a derivation d such that either $(d(x^m \circ y^n))^\ell \pm (x^p \circ_k y^q) = 0$ for all $x, y \in \mathcal{R}$ or $(d(x^m) \circ d(y^n))^\ell \pm (x^p \circ_k y^q) = 0$ for all $x, y \in \mathcal{R}$, where m, n, p, q, k, ℓ are fixed positive integers. Finally, we apply the above purely ring theoretic results to Banach algebras and obtain a noncommutative version of the Singer-Wermer theorem. In particular, we prove that if $\mathfrak{B}$ is a noncommutative Banach algebra which admits a continuous linear derivation $d : \mathfrak{B} \to \mathfrak{B}$ such that either $(d(x^m \circ y^n))^\ell \pm (x^p \circ_k y^q) \in rad(\mathfrak{B})$ or $(d(x^m) \circ d(y^n))^\ell \pm (x^p \circ_k y^q) \in rad(\mathfrak{B})$ for all $x, y \in \mathfrak{B}$, where m, n, p, q, k, ℓ are fixed positive integers, then $d(\mathfrak{B}) \subseteq rad(\mathfrak{B})$.

Keywords. Semiprime ring, derivation, maximal right ring of quotient, Banach algebra.

Mathematics Subject Classification (2010): 46J10, 16N20, 16N60, 16W25.

1 - Introduction

In all that follows, unless stated otherwise, $\mathcal{R}$ will be an associative ring, $Z(\mathcal{R})$ the center of $\mathcal{R}$, Q the Martindale quotient ring of $\mathcal{R}$ and $\mathfrak{U}$ the Utumi quotient ring of $\mathcal{R}$. The center of $\mathfrak{U}$, denoted by $\mathcal{C}$, is called the extended centroid of $\mathcal{R}$ (we refer the reader to [**4**] for these objects). By a Banach algebra we shall mean complex normed algebra $\mathfrak{B}$ whose underlying vector space is a Banach space (see [**8**]). The Jacobson radical $rad(\mathfrak{B})$ of $\mathfrak{B}$ is the intersection of all primitive ideals. If the Jacobson radical reduces to the zero element, $\mathfrak{B}$ is called semisimple.

Received: December 3, 2018; accepted in revised form: March 18, 2019.

For any $x, y \in \mathcal{R}$, the symbol $[x, y]$ and $x \circ y$ stand for the commutator $xy - yx$ and anti-commutator $xy + yx$, respectively. Given $x, y \in \mathcal{R}$, set $x \circ_0 y = x$, $x \circ y = x \circ_1 y = xy + yx$ and inductively $x \circ_k y = (x \circ_{k-1} y) \circ y$ for $k > 1$. The ring $\mathcal{R}$ is said to satisfy the Engel condition if for all $x, y \in \mathcal{R}$ there exists a positive integer k such that $[x, y]_k = 0$, where $[x, y]_k = [[x, y]_{k-1}, y]$ for $k > 1$. Recall that a ring $\mathcal{R}$ is prime if for any $a, b \in \mathcal{R}$, $a\mathcal{R}b = \{0\}$ implies $a = 0$ or $b = 0$, and is semiprime if for any $a \in \mathcal{R}$, $a\mathcal{R}a = \{0\}$ implies $a = 0$. An additive mapping $d : \mathcal{R} \to \mathcal{R}$ is said to be a derivation on $\mathcal{R}$ if $d(xy) = d(x)y + xd(y)$ holds for all $x, y \in \mathcal{R}$. For a fixed $a \in \mathcal{R}$, the mapping $\mathcal{I}_a : \mathcal{R} \to \mathcal{R}$ given by $\mathcal{I}_a(x) = [a, x]$ for all $x \in \mathcal{R}$ is a derivation which is called an inner derivation determined by a in $\mathcal{R}$. A mapping $f : \mathcal{R} \to \mathcal{R}$ is said to be commutativity preserving on $\mathcal{R}$ if $[x, y] = 0$ implies that $[f(x), f(y)] = 0$ for all $x, y \in \mathcal{R}$. The mapping f is called strong commutativity preserving (scp) on $\mathcal{R}$ if $[f(x), f(y)] = [x, y]$ holds for all $x, y \in \mathcal{R}$.

A classical problem of ring theory is to find combinations of properties that force a ring to be commutative. Posner [**27**] connected commutativity and derivations in 1957, proving that if a prime ring $\mathcal{R}$ admitting a nonzero derivation d such that $[d(x), x] \in Z(\mathcal{R})$ for all $x \in \mathcal{R}$, must be commutative. Since then several authors have studied this kind of Engel type identities with derivations acting on ideals and Lie ideals of prime and semiprime rings (see [**6**], [**13**], [**16**] for a partial bibliography). In the year 1992, Daif and Bell [**12**, Theorem 3] showed that if in a semiprime ring $\mathcal{R}$ there exists a nonzero ideal $\mathcal{I}$ of $\mathcal{R}$ and a derivation $d : \mathcal{R} \to \mathcal{R}$ such that $d([x, y]) = [x, y]$ holds for all $x, y \in \mathcal{I}$, then $\mathcal{I} \subseteq Z(\mathcal{R})$. If $\mathcal{R}$ is a prime ring, this implies that $\mathcal{R}$ is commutative. In the year 2002, Ashraf and Rehman [**1**, Theorem 4.1] showed that if $\mathcal{R}$ is a prime ring, $\mathcal{I}$ a nonzero ideal of $\mathcal{R}$ and d a derivation on $\mathcal{R}$ such that either $d(x \circ y) = x \circ y$ or $d(x) \circ d(y) = x \circ y$ holds for all $x, y \in \mathcal{I}$, then $\mathcal{R}$ is commutative. Very recently, this result was further extended by Argaç and Inceboz [**3**, Theorem 3] who proved that if a semiprime ring $\mathcal{R}$ admits a derivation d such that $(d(x \circ y))^n = x \circ y$ holds for all $x, y \in \mathcal{R}$ and n a fixed positive integer, then $\mathcal{R}$ is commutative.

In view of the latter result due to Argaç and Inceboz, it is natural to explore the commutativity of a ring $\mathcal{R}$ which satisfies the identity $(d(x^m \circ y^n))^\ell = \pm(x^p \circ_k y^q)$ for all $x, y \in \mathcal{R}$. In this paper we investigate this identity and obtain the commutativity of $\mathcal{R}$. In fact, we also prove the commutativity of a semiprime $\mathcal{R}$, which allows us to obtain a commutativity theorem in the setting of Banach algebras.

2 - The Result on prime and semiprime rings

We shall use the fact that any derivation of a semiprime ring $\mathcal{R}$ can be uniquely extended to a derivation of its Utumi quotient ring $\mathfrak{A}$ (maximal right ring of quotient), and so any derivation of $\mathcal{R}$ can be defined on the whole $\mathfrak{A}$ (Beidar et al., [**4**, Proposition 2.5.1]). Moreover, if $\mathcal{R}$ is a semiprime ring then so is its Utumi quotient ring. The extended centroid $\mathcal{C}$ of a semiprime ring $\mathcal{R}$ coincides with the center of its Utumi quotient ring (Chuang, [**11**, pp.38]). Also, if $\mathcal{B}$ is the set of all the idempotents in $\mathcal{C}$, one may assume that $\mathcal{R}$ is a $\mathcal{B}$-algebra which is orthogonal complete. For any maximal ideal $\mathcal{P}$ of $\mathcal{B}$, $\mathcal{PR}$ forms a minimal prime ideal of $\mathcal{R}$, which is invariant under any nonzero derivation of $\mathcal{R}$ (Chuang, [**11**, pp.42]). We use the theory of differential identities and the fact that any semiprime ring $\mathcal{R}$ and its maximal right ring of quotient satisfy the same differential identities (for the explanation of differential identities we refer the reader to Beidar et al. [**4**], Chuang [**11**], Kharchenko [**19**], Lee [**22**]).

For the proof of our main results, we need the following facts, which might be of some independent interest.

Fact 2.1 ([**22**]). *If $\mathfrak{I}$ is a two-sided ideal of $\mathcal{R}$, then $\mathcal{R}$, $\mathfrak{I}$ and $\mathfrak{A}$ satisfy the same generalized polynomial identities with coefficient in $\mathfrak{A}$.*

Fact 2.2 ([**4**, Proposition 2.5.1]). *Any derivation of a semiprime ring $\mathcal{R}$ can be uniquely extended to a derivation of its left Utumi quotient ring $\mathfrak{A}$, and so any derivation of $\mathcal{R}$ can be defined on the whole $\mathfrak{A}$.*

Fact 2.3 ([**19**]). *Let $\mathcal{R}$ be a prime ring, d a nonzero derivation of $\mathcal{R}$ and $\mathfrak{I}$ a nonzero two-sided ideal of $\mathcal{R}$. Let $f(x_1, \ldots, x_n, d(x_1), \ldots, d(x_n))$ be a differential identity in $\mathfrak{I}$, that is*

$$f(r_1, \ldots, r_n, d(r_1), \ldots, d(r_n)) = 0 \text{ for all } r_1, \ldots, r_n \in \mathfrak{I}.$$

Then one of the following holds:

1. *Either d is an inner derivation in Q, the Martindale quotient ring of $\mathcal{R}$, in the sense that there exists $b \in Q$ such that $d(x) = [b, x]$ for all $x \in \mathcal{R}$, and $\mathfrak{I}$ satisfies the generalized polynomial identity*

$$f(r_1, \ldots, r_n, [b, r_1], \ldots, [b, r_n])$$

   or

2. *$\mathfrak{I}$ satisfies the generalized polynomial identity*

$$f(x_1, \ldots, x_n, y_1, \ldots, y_n).$$

We begin with the following result which are crucial for developing the proof of our main theorem. The proof of Lemma 2.1 can be seen in [**5**].

Lemma 2.1. *Let $\mathcal{R}$ be a ring satisfying an identity $q(X) = 0$, where $q(X)$ is the polynomial in the finite number of non-commuting indeterminates, its coefficients being integers with highest common factor 1. If there exists no prime p for which the ring of 2×2 matrices over $GF(p)$ satisfies $q(X) = 0$, then $\mathcal{R}$ has nil commutator ideal and the nilpotent elements of $\mathcal{R}$ form an ideal.*

Lemma 2.2. *Let $\mathcal{R}$ be a ring and $a \in \mathcal{R}$ such that $a^2 = 0$. Then $(ax)^m \circ_k (xa)^n = (ax)^m (xa)^{kn}$ for all $x \in \mathcal{R}$ and fixed positive integers m, n, k.*

Proof. We proceed by induction on k. For $k = 1$, we have $(ax)^m \circ (xa)^n = (ax)^m (xa)^n + (xa)^n (ax)^m$. Using the given hypothesis we get $(ax)^m \circ (xa)^n = (ax)^m (xa)^n$ for all $x, y \in \mathcal{R}$. Thus the result is true for $k = 1$. Now for $k > 1$ assume that the result is true for $k - 1$ i.e.;

$$(ax)^m \circ_{k-1} (xa)^n = (ax)^m (xa)^{(k-1)n}$$

for all $x, y \in \mathcal{R}$. Now,

$$(ax)^m \circ_k (xa)^n = ((ax)^m \circ_{k-1} (xa)^n) \circ (xa)^n$$

for all $x, y \in \mathcal{R}$. Then by the induction hypothesis, we find that

$$(ax)^m \circ_k (xa)^n = (ax)^m (xa)^{(k-1)n} \circ (xa)^n = (ax)^m (xa)^{kn}$$

$x, y \in \mathcal{R}$. Thus the result is true for k also. Hence the result is true for all positive integer k. $\square$

Lemma 2.3. *Let $m \geq 1$, $n \geq 1$, $k \geq 1$ be fixed integers and let $\mathcal{R}$ be a semiprime ring satisfying $x^m \circ_k y^n = 0$ for all $x, y \in \mathcal{R}$. Then $\mathcal{R}$ is commutative.*

Proof. Suppose that

$$x^m \circ_k y^n = 0 \text{ for all } x, y \in \mathcal{R}. \tag{2.1}$$

First we show that $\mathcal{R}$ has no nonzero nilpotent element. Let $a \in \mathcal{R}$ such that $a^2 = 0$. Using our hypothesis, we find that

$$(ax)^m \circ_k (xa)^n = 0, \text{ for all } x \in \mathcal{R}.$$

By Lemma 2.2, we have $(ax)^m(xa)^{kn} = 0$ for all $x \in \mathcal{R}$. Now, one can see that

$$\begin{aligned}(ax)^{m+kn+1} &= \{(ax)^{m+kn}a + (ax)^m(xa)^{kn}\}x \\ &= \{a(xa+x)\}^m\{(xa+x)a\}^{kn}x \\ &= 0.\end{aligned}$$

Therefore $(ax)^{m+kn+1} = 0$ for all $x \in \mathcal{R}$. If $a\mathcal{R} \neq 0$, then $a\mathcal{R}$ is a nil right ideal satisfying the identity $z^{m+kn+1} = 0$ for all $z \in a\mathcal{R}$. Application of Lemma 1.1 of Herstein [**15**], yields that $a\mathcal{R} = 0$ and hence we find that $a = 0$.

Since $\mathcal{R}$ is a semiprime ring satisfying (2.1), $\mathcal{R}$ is isomorphic to a subdirect sum of prime rings $\mathcal{R}\alpha$ each of which as a homomorphic image of $\mathcal{R}$ satisfies the hypothesis placed on $\mathcal{R}$. Hence we can assume that $\mathcal{R}$ is a prime ring which satisfies the identity (2.1). Now if we consider $x = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $y = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$, then we find that no ring of 2×2 matrices over $GF(2)$, satisfies the identity (2.1). Hence by Lemma 2.1, $\mathcal{R}$ has a nil commutator ideal. But since $\mathcal{R}$ has no nonzero nilpotent element, $\mathcal{R}$ has no nonzero nil ideal and $\mathcal{R}$ is commutative. □

Theorem 2.1. *Let $\mathcal{R}$ be a prime ring and m, n, p, q, k, ℓ be fixed positive integers. If $\mathcal{R}$ admits a derivation d such that $(d(x^m \circ y^n))^\ell = \pm(x^p \circ_k y^q)$ for all $x, y \in \mathcal{R}$, then $\mathcal{R}$ is commutative.*

Proof. If $d = 0$, then $x^p \circ_k y^q = 0$ for all $x, y \in \mathcal{R}$. By Lemma 2.3 we see that $\mathcal{R}$ is commutative. Now we assume d is a nonzero derivation satisfying $(d(x^m \circ y^n))^\ell = \pm(x^p \circ_k y^q)$ for all $x, y \in \mathcal{R}$ which can be rewritten as

$$\{d(x^m)y^n + x^m d(y^n) + d(y^n)x^m + y^n d(x^m)\}^\ell = \pm(x^p \circ_k y^q).$$

By Kharchenko [**19**] we divide the proof into two cases:

Case 1. If d is outer derivation, then $\mathcal{R}$ satisfies the polynomial identity

$$\begin{aligned}(x^p \circ_k y^q) = &\Bigg(\Big(\sum_{s=0}^{m-1} x^s z x^{m-1-s}\Big)y^n + x^m\Big(\sum_{s=0}^{n-1} y^s w y^{n-1-s}\Big) \\ &+\Big(\sum_{s=0}^{n-1} y^s w y^{n-1-s}\Big)x^m + y^n\Big(\sum_{s=0}^{m-1} x^s z x^{m-1-s}\Big)\Bigg)^\ell\end{aligned}$$

for all $x, y, z, w \in \mathcal{R}$. In particular, for $z = w = 0$, we obtain the identity $x^p \circ_k y^q = 0$ for all $x, y \in \mathcal{R}$. Hence $\mathcal{R}$ is commutative by Lemma 2.3.

Case 2. Let d be Q-inner derivation induced by an element $b \in Q$, that is, $d(x) = [b, x]$ for all $x \in \mathcal{R}$. It follows that $\big([b, x^m]y^n + x^m[b, y^n] + [b, y^n]x^m + y^n[b, x^m]\big)^\ell = \pm(x^p \circ_k y^q)$ for all $x, y \in \mathcal{R}$. By Chuang [**10**, Theorem 2], $\mathcal{R}$ and Q satisfy the same generalized polynomial identities (GPIs), and hence we have,

$$([b, x^m]y^n + x^m[b, y^n] + [b, y^n]x^m + y^n[b, x^m])^\ell = \pm(x^p \circ_k y^q) \text{ for all } x, y \in Q.$$

In case the center $\mathcal{C}$ of Q is infinite, we have

$$([b, x^m]y^n + x^m[b, y^n] + [b, y^n]x^m + y^n[b, x^m])^\ell = \pm(x^p \circ_k y^q) \text{ for all } x, y \in Q \bigotimes_{\mathcal{C}} \overline{\mathcal{C}},$$

where $\overline{\mathcal{C}}$ is the algebraic closure of $\mathcal{C}$. Since both Q and $Q \bigotimes_{\mathcal{C}} \overline{\mathcal{C}}$ are prime and centrally closed [**14**, Theorem 2.5 and 3.5], we may replace $\mathcal{R}$ by Q or $Q \bigotimes_{\mathcal{C}} \overline{\mathcal{C}}$ according as $\mathcal{C}$ is finite or infinite. Thus we may assume that $\mathcal{R}$ is centrally closed over $\mathcal{C}$ (i.e., $\mathcal{R}\mathcal{C} = \mathcal{R}$) which is either finite or algebraically closed and hence

$$([b, x^m]y^n + x^m[b, y^n] + [b, y^n]x^m + y^n[b, x^m])^\ell = \pm(x^p \circ_k y^q) \text{ for all } x, y \in \mathcal{R}.$$

By Martindale [**24**, Theorem 3], $\mathcal{R}\mathcal{C}$ (and so $\mathcal{R}$) is a primitive ring having nonzero socle H with $\mathcal{C}$ as the associated division ring. Hence by Jacobson's theorem [**17**, pp.75], $\mathcal{R}$ is isomorphic to a dense ring of linear transformations of some vector space V over $\mathcal{C}$ and H consists of the finite rank linear transformations in $\mathcal{R}$. Assume that $dim_{\mathcal{C}} V \geq 2$, otherwise we are done.

Our aim is to show that for any $v \in V$, v and bv are linearly $\mathcal{C}$-dependent. If $bv = 0$, then v, bv is $\mathcal{C}$-dependent. Thus we may assume that $bv \neq 0$. If v and bv are linearly $\mathcal{C}$-independent for some $v \in V$. By the density of $\mathcal{R}$ there exist $x, y \in \mathcal{R}$ such that

$$xv = v, \quad xbv = v;$$

$$yv = 0, \quad ybv = bv.$$

We can easily see that

$$0 = (([b, x^m]y^n + x^m[b, y^n] + [b, y^n]x^m + y^n[b, x^m])^\ell \pm (x^p \circ_k y^q))v = (-1)^\ell v \neq 0,$$

a contradiction. So we conclude that v and bv are linearly C-dependent for all $v \in V$. Hence for each $v \in V$, $bv = v\alpha_v$ for some $\alpha_v \in C$. Now we prove α_v is not depending on the choice of $v \in V$. Since $dim_C V \geq 2$ there exists $w \in V$ such that v and w are linearly independent over C. Now there exist $\alpha_v, \alpha_w, \alpha_{v+w} \in C$ such that

$$bv = v\alpha_v, bw = w\alpha_w, b(v + w) = (v + w)\alpha_{v+w},$$

and hence,

$$v\alpha_v + w\alpha_w = b(v+w) = (v+w)\alpha_{v+w}.$$

This implies that $v(\alpha_v - \alpha_{v+w}) + w(\alpha_v - \alpha_{v+w}) = 0$. Since v and w are linearly independent over C, it follows $\alpha_v = \alpha_{v+w} = \alpha_v$. Therefore there exists $\alpha \in C$ such that $bv = v\alpha$ for all $v \in V$.

Now let $r \in \mathcal{R}$, $v \in V$. Since $bv = v\alpha$,

$$[b,r]v = (br)v - (rb)v = b(rv) - r(bv) = (rv)\alpha - r(v\alpha) = 0,$$

that is, $[b,\mathcal{R}]V = 0$. Since V is a faithful irreducible $\mathcal{R}$-module, $[b,\mathcal{R}] = 0$, i.e.; $b \in Z(\mathcal{R})$, and hence $d = 0$, a contradiction. □

Theorem 2.2. *Let $\mathcal{R}$ be a prime ring and m,n,p,q,k,ℓ be fixed positive integers. If $\mathcal{R}$ admits a derivation d such that $(d(x^m) \circ d(y^n))^\ell = \pm(x^p \circ_k y^q)$ for all $x,y \in \mathcal{R}$, then $\mathcal{R}$ is commutative.*

Proof. If $d = 0$, then $x^p \circ_k y^q = 0$ for all $x,y \in \mathcal{R}$. By Lemma 2.3 we see that $\mathcal{R}$ is commutative. Now we assume d is a nonzero derivation satisfying $(d(x^m) \circ d(y^n))^\ell = \pm(x^p \circ_k y^q)$ for all $x,y \in \mathcal{R}$ which can be rewritten as

$$\{d(x^m)d(y^n) + d(x^m)d(y^n)\}^\ell = \pm(x^p \circ_k y^q).$$

By Kharchenko [**19**] we divide the proof into two cases:

Case 1. If d is outer derivation, then $\mathcal{R}$ satisfies the polynomial identity

$$\begin{aligned}(x^p \circ_k y^q) \;=\; &\Bigg(\Big(\sum_{s=0}^{m-1} x^s z x^{m-1-s}\Big)\Big(\sum_{s=0}^{n-1} y^s w y^{n-1-s}\Big)\\ &+\Big(\sum_{s=0}^{n-1} y^s w y^{n-1-s}\Big)\Big(\sum_{s=0}^{m-1} x^s z x^{m-1-s}\Big)\Bigg)^\ell\end{aligned}$$

for all $x,y,z,w \in \mathcal{R}$. In particular, for $z = w = 0$, we obtain the identity $x^p \circ_k y^q = 0$ for all $x,y \in \mathcal{R}$. Then $\mathcal{R}$ is commutative by Lemma 2.3.

Case 2. Let d be Q-inner derivation induced by an element $b \in Q$, that is $d(x) = [b,x]$ for all $x \in \mathcal{R}$, then the proof runs exactly parallel to that given in the proof of the Theorem 2.1(Case 2) except the case $dim_{\mathcal{C}}V \geq 2$. We omit the details of the proof just to avoid repetition. Assume that $dim_{\mathcal{C}}V \geq 2$, otherwise we are done.

Our aim is to show that for any $v \in V$, v and bv are linearly $\mathcal{C}$-dependent. If $bv = 0$, then v, bv is $\mathcal{C}$-dependent. Thus we may assume that $bv \neq 0$. If v and

bv are linearly $\mathcal{C}$-independent for some $v \in V$. then we consider the following cases:

If $b^2v \notin Span_{\mathcal{C}}\{v, bv\}$ then the set $\{v, bv, b^2v\}$ is linearly $\mathcal{C}$-independent. By the density of $\mathcal{R}$ there exist $x, y \in \mathcal{R}$ such that

$$xv = 0,\ xbv = bv,\ xb^2v = b^2v$$

$$yv = v,\ ybv = 0,\ yb^2v = v.$$

We can easily see that $0 = (([b, x^m][b, y^n] + [b, y^n][b, x^m])^\ell \pm (x^p \circ_k y^q))v = v \neq 0$, a contradiction.

On the other hand if $b^2v \in Span_{\mathcal{C}}\{v, bv\}$ then $b^2v = v\alpha + bv\beta$ for some $0 \neq \alpha, \beta \in \mathcal{C}$. In view of the density of $\mathcal{R}$, there exist $x, y \in \mathcal{R}$ such that

$$xv = 0, xbv = bv;$$

$$yv = v, ybv = 0.$$

Hence we find that $0 = (([b, x^m][b, y^n]+[b, y^n][b, x^m])^\ell \pm (x^p \circ_k y^q))v = 2^\ell v\alpha^\ell \neq 0$, a contradiction. So we conclude that v and bv are linearly C-dependent for all $v \in V$. Hence for each $v \in V$, $bv = v\alpha_v$ for some $\alpha_v \in C$. Now we prove α_v is not depending on the choice of $v \in V$. Since $dim_C V \geq 2$ there exists $w \in V$ such that v and w are linearly independent over C. Now there exist $\alpha_v, \alpha_w, \alpha_{v+w} \in C$ such that

$$bv = v\alpha_v, bw = w\alpha_w, b(v + w) = (v + w)\alpha_{v+w},$$

and hence,

$$v\alpha_v + w\alpha_w = b(v + w) = (v + w)\alpha_{v+w}.$$

Which implies

$$v(\alpha_v - \alpha_{v+w}) + w(\alpha_v - \alpha_{v+w}) = 0.$$

Since v and w are linearly independent over C, it follows $\alpha_v = \alpha_{v+w} = \alpha_v$. Therefore there exists $\alpha \in C$ such that $bv = v\alpha$ for all $v \in V$.

Now let $r \in \mathcal{R}$, $v \in V$. Since $bv = v\alpha$,

$$[b, r]v = (br)v - (rb)v = b(rv) - r(bv) = (rv)\alpha - r(v\alpha) = 0,$$

that is, $[b, \mathcal{R}]V = 0$. Since V is a faithful irreducible $\mathcal{R}$-module, $[b, \mathcal{R}] = 0$, i.e.; $b \in Z(\mathcal{R})$, and hence $d = 0$, a contradiction. $\square$

Theorem 2.3. *Let $\mathcal{R}$ be a semiprime ring and m, n, p, q, k, ℓ be fixed positive integers. If $\mathcal{R}$ admits a derivation d such that $(d(x^m \circ y^n))^\ell = \pm(x^p \circ_k y^q)$ for all $x, y \in \mathcal{R}$, then $\mathcal{R}$ is commutative.*

Proof. If $d = 0$, then $x^p \circ_k y^q = 0$ for all $x, y \in \mathcal{R}$. Thus we are done by Lemma 2.3. So we assume that d is a nonzero derivation such that $(d(x^m \circ y^n))^\ell = \pm(x^p \circ_k y^q)$ for all $x, y \in \mathcal{R}$. By Chuang [**11**, pp.38], $Z(\mathfrak{A}) = \mathcal{C}$, the extended centroid of $\mathcal{R}$, and by Beidar et al. [**4**, Proposition 2.5.1], derivation d can be uniquely extended on $\mathfrak{A}$, the maximal right ring of quotient of $\mathcal{R}$. In view of Lee [**22**], $\mathcal{R}$ and $\mathfrak{A}$ satisfy the same differential identities, hence

$$(d(x^m \circ y^n))^\ell = \pm(x^p \circ_k y^q) \text{ for all } x, y \in \mathfrak{A}.$$

Let $\mathcal{B}$ be the complete Boolean algebra of idempotents in $\mathcal{C}$ and let M be any maximal ideal of $\mathcal{B}$. Due to Chuang [**11**, pp.42], $\mathfrak{A}$ is an orthogonal complete $\mathcal{B}$-algebra and $M\mathfrak{A}$ is a prime ideal of $\mathfrak{A}$, which is d-invariant. Denote $\overline{\mathfrak{A}} = \mathfrak{A}/M\mathfrak{A}$ and let $\overline{d}$ be the derivation induced by d on $\overline{\mathfrak{A}}$, i.e., $\overline{d}(\overline{u}) = \overline{d(u)}$ for all $u \in \mathfrak{A}$. Therefore $\overline{d}$ has in $\overline{\mathfrak{A}}$ the same property as d on $\mathfrak{A}$. In particular, $\overline{\mathfrak{A}}$ is prime and hence by Theorem 2.1 $\overline{\mathfrak{A}}$ is commutative. This implies that, for any maximal ideal M of $\mathcal{B}$, $[\mathfrak{A}, \mathfrak{A}] \subseteq M\mathfrak{A}$ and hence $[\mathfrak{A}, \mathfrak{A}] \subseteq \cap_M M\mathfrak{A} = 0$, where $M\mathfrak{A}$ runs over all prime ideals of $\mathfrak{A}$. In particular, $[\mathcal{R}, \mathcal{R}] = 0$ and hence $\mathcal{R}$ is commutative. □

By arguments similar to those used in the proof of the above theorem, one can prove the following:

Theorem 2.4. *Let $\mathcal{R}$ be a semiprime ring and m, n, p, q, k, ℓ be fixed positive integers. If $\mathcal{R}$ admits a derivation d such that $(d(x^m) \circ d(y^n))^\ell = \pm(x^p \circ_k y^q)$ for all $x, y \in \mathcal{R}$, then $\mathcal{R}$ is commutative.*

We close this section by the following corollary.

Corollary 2.1 ([**3**, Theorem 3]). *Let $\mathcal{R}$ be a semiprime ring and n a fixed positive integer. If $\mathcal{R}$ admits a derivation d such that $(d(x)y + xd(y) + d(y)x + yd(x))^n = xy + yx$ for all $x, y \in \mathcal{R}$, then $\mathcal{R}$ is a commutative ring.*

3 - The Result on prime Banach algebra

Let us introduce the background of our investigation. Singer and Wermer [**29**] obtained a fundamental result which started investigation into the ranges of derivations on Banach algebras. In [**29**] , Singer and Wermer proved that any continuous derivation on a commutative Banach algebra has the range in the Jacobson radical of the algebra. In their paper they conjectured that the continuity is not necessary. Thomas [**30**] established this conjecture. It is clear that the same result of Singer and Wermer does not hold in noncommutative Banach algebras because of inner derivations. Hence in this context a very interesting question arises that how to obtain noncommutative version of Singer-Wermer theorem. The first answer to this problem was obtained by Sinclair in **[28]**, who proved that every continuous derivation of a Banach algebra leaves primitive ideals of the algebra invariant. In **[20]**, Kim proved that if a noncommutative Banach algebra $\mathfrak{B}$ admits a continuous linear Jordan derivation d such that $d(x)[d(x), x]d(x) \in rad(\mathfrak{B})$ for all $x \in \mathfrak{B}$ then $d(\mathfrak{B}) \subseteq rad(\mathfrak{B})$. More recently, Park **[26]** proved that if d is a linear continuous derivation of a noncommutative Banach algebra $\mathfrak{B}$ such that $[[d(x), x], d(x)] \in rad(\mathfrak{B})$ for all $x \in \mathfrak{B}$ then $d(\mathfrak{B}) \subseteq rad(\mathfrak{B})$. In the meanwhile many authors obtained more information about derivations satisfying certain suitable conditions in Banach algebra (see **[31]**, **[32]** where further references can be found). Motivated by these results, in this section, we use the above ring theoretic results and prove that if a non commutative Banach algebra admits a continuous linear derivation d such that $(d(x^m \circ y^n))^\ell \pm (x^p \circ_k y^q) \in rad(\mathfrak{B})$ for all $x, y \in \mathfrak{B}$ and m, n, p, q, k, ℓ, fixed positive integers, then $d(\mathfrak{B}) \subseteq rad(\mathfrak{B})$.

Theorem 3.1. *Let $\mathfrak{B}$ be a noncommutative Banach algebra and m, n, p, q, k, ℓ be fixed positive integers. If there exists a continuous linear derivation $d : \mathfrak{B} \to \mathfrak{B}$ such that $(d(x^m \circ y^n))^\ell \pm (x^p \circ_k y^q) \in rad(\mathfrak{B})$ for all $x, y \in \mathfrak{B}$, then $d(\mathfrak{B}) \subseteq rad(\mathfrak{B})$.*

Proof. Following the result of A. M. Sinclair [**28**, Theorem 2.2] that every continuous linear derivation on a Banach algebra leaves the primitive ideals of $\mathfrak{B}$ invariant, for every primitive ideal $P \subseteq \mathfrak{B}$, we can define a linear derivation $d_P : \mathfrak{B}/P \to \mathfrak{B}/P$, where $\mathfrak{B}/P$ is a factor Banach algebra, by $d_P(\hat{x}) = d(x)+P$, $\hat{x} = x + P$ for all $x \in \mathfrak{B}$. Since P is a primitive ideal, the factor algebra $\mathfrak{B}/P$ is prime and so it is semiprime. The hypothesis $(d(x^m \circ y^n))^\ell \pm (x^p \circ_k y^q) \in rad(\mathfrak{B})$ yields that $(d(\overline{x}^m \circ \overline{y}^n))^\ell \pm \overline{x^p} \circ_k \overline{y^q} = 0$ for all $\overline{x}, \overline{y} \in \overline{\mathfrak{B}}$. We also see that d_P is continuous since $\mathfrak{B}/P$ is semisimple **[18]**. Thus we obtain that $d_P(\mathfrak{B}/P) \subseteq rad(\mathfrak{B}/P)$. Again using the semisimplicity of $\mathfrak{B}/P$, we see that $d_P = 0$ on $\mathfrak{B}/P$. In case $\mathfrak{B}/P$ is commutative, we can conclude that $d_P = 0$ on $\mathfrak{B}/P$ as well since $\mathfrak{B}/P$ is semisimple and we know that there are no nonzero linear derivations on commutative semisimple Banach algebras. In both the cases, we obtain $d(\mathfrak{B} \subseteq P$ for any primitive ideal P. Since the intersection of all primitive ideals is the Jacobson radical $rad(\mathfrak{B})$, it follows that $d(\mathfrak{B}) \subseteq rad(\mathfrak{B})$. This completes the proof of the theorem. □

By arguments similar to those used in the proof of Theorem 3.1 (use Theorem 2.4 instead of Theorem 2.3), we get the following theorem. We omit the details of the proof just to avoid the repetition.

Theorem 3.2. *Let $\mathfrak{B}$ be a noncommutative Banach algebra and m, n, p, q, k, ℓ be fixed positive integers. If there exists a continuous linear derivation $d : \mathfrak{B} \to \mathfrak{B}$ such that $(d(x^m) \circ d(y^n))^\ell \pm (x^p \circ_k y^q) \in rad(\mathfrak{B})$ for all $x, y \in \mathfrak{B}$, then $d(\mathfrak{B}) \subseteq rad(\mathfrak{B})$.*

Acknowledgments. The authors are highly indebted to the referee for his/her valuable comments which have improved the paper immensely.

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Mohammad Ashraf
Department of Mathematics
Aligarh Muslim University
Aligarh-202002 India
e-mail: mashraf80@hotmail.com

Bilal Ahmad Wani
Department of Mathematics
Aligarh Muslim University
Aligarh-202002 India
e-mail: bilalwanikmr@gmail.com