Riv. Mat. Univ. Parma, Vol. **10** (2019), 63–84

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Measurable sequences

Abstract. The paper deals with the distribution functions of sequences with respect to asymptotic density and measure density. Furthermore also polyadicly continuous sequences and their extension to random variables are studied.

Keywords. Density, distribution function, uniform distribution.

Mathematics Subject Classification (2010): 11B05.

1 - Introduction

In the first part we study sequences having an asymptotic distribution function in the sense of Schoenberg [Sch]. The connection between independence and statistical independence is established in case of continuous distribution functions.

Later we develop relations between distribution functions of sequences and distribution functions of random variables. We study statistical independence and independence in the sense of probability theory. In the last part we transfer some probabilistic limit laws to certain types of deterministic sequences. This makes heavily use of methods developed in [CQ].

The "general" notion of uniform distribution was introduced by Hermann Weyl (1916) in his famous paper [WEY]: a sequence $\{v(n)\}, v(n) \in [0,1)$ is *uniformly distributed* if and only if for every $x \in [0,1)$

$$\lim_{N\to\infty} \frac{1}{N} |\{n \le N; v(n) < x\}| = x,$$

where $|A|$ denotes he cardinality of the set A. This can be equivalently formulated using the notion of asymptotic density. Let $\mathbb{N}$ be the set of positive

Received: November 22, 2018; accepted in revised form: May 7, 2019.
The first author supported by grant VEGA 02/0109/18.

integers. We say that a set $A \subset \mathbb{N}$ has an *asymptotic density* if and only if the limit

$$\lim_{N\to\infty} \frac{|A \cap [1,N)|}{N} := d(A)$$

exists, and in this case the value $d(A)$ is called the *asymptotic density* of A. Let $\mathcal{D}$ denote the system of all subsets of $\mathbb{N}$ having an asymptotic density. Then a sequence $\{v(n)\}, v(n) \in [0,1)$ is uniformly distributed if and only if for every $x \in [0,1)$ the set $\{n \in \mathbb{N}; v(n) < x\}$ belongs to $\mathcal{D}$ and $d(\{n \in \mathbb{N}; v(n) < x\}) = x$. Schoenberg [Sch] generalized this notion as follows: we say that a sequence $\{v(n)\}, v(n) \in [0,1)$ has an *asymptotic distribution function* if and only if for each real number x the set $\{n \in \mathbb{N}; v(n) < x\}$ belongs to $\mathcal{D}$. In this case the function $F(x) = d(\{n \in \mathbb{N}; v(n) < x\})$ is called the *asymptotic distribution function* of the sequence $\{v(n)\}$.

Our aim is to study distribution functions of sequences. The following statement is useful in this context.

Proposition 1. *If F is a non decreasing function defined on the real line then for each real numbers x_1, x_2 - the points of continuity of F-we have that for every $\varepsilon > 0$ there exist two continuous function g, g_1 such that*

$$g \leq \mathcal{X}_{[x_1,x_2]} \leq g_1$$

and

$$\int_{-\infty}^{\infty} (g_1(x) - g(x)) < \varepsilon,$$

$\mathcal{X}_{[x_1,x_2]}$ *denoting the indicator function of the interval* $[x_1, x_2]$.

The proof follows from a standard procedure, see [KN] page 54.

Another important notion of uniform distribution was introduce by Niven [NIV]. A sequence of positive integers $k = \{k_n\}$ is called *uniformly distributed in* $\mathbb{Z}$ if and only for each $m \in \mathbb{N}, r \in \mathbb{Z}$ we have that $\{n \in \mathbb{N}; k_n \equiv r \mod m\} \in \mathcal{D}$ and $d(\{n \in \mathbb{N}; k_n \equiv r \mod m\}) = \dfrac{1}{m}$. In sections 9 and 10 we will use this concept to prove structural properties concerning measurable sequences.

2 - Mean value, dispersion and Buck measurability

Let $v = \{v(n)\}$ be a sequence of real numbers. Set

$$E_N(v) = \frac{1}{N} \sum_{n=1}^{N} v(n)$$

for $N = 1, 2, 3, \dots$ and

$$\underline{E}(v) = \liminf_{N\to\infty} E_N(v), \overline{E}(v) = \limsup_{N\to\infty} E_N(v).$$

Definition 1. If $\underline{E}(v) = \overline{E}(v) := E(v)$ we say that v has a *mean value* and the number $E(v)$ will be called the *mean value of* v.

Clearly we have

Proposition 2. *If sequences v, w have mean values then for all numbers a, b the sequence $av + bw$ has a mean value and*

$$E(av + bw) = aE(v) + bE(w).$$

Proposition 3. *If v is bounded sequence with elements in the interval $[a, b]$ and having an asymptotic distribution function F, then v has a mean value and*

$$E(v) = \int_a^b x dF(x).$$

Definition 2. We say that a sequence v has a dispersion if v has a mean value and the sequence $(v - E(v))^2$ has a mean value; in this case the number

$$D^2(v) = E((v - E(v))^2)$$

is called the *dispersion* of v.

If a bounded sequence v has a dispersion then

$$D^2(v) = \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} (v(n) - E(v))^2.$$

In the following we introduce Buck measurability, weak measurability and weak distribution functions.

R. C. Buck [BUC] constructed a measure density via covering of sets by arithmetic progressions. Denote

$$r + (m) = \{r + jm; j = 0, 1, 2, \dots\}$$

for $r = 0, 1, 2, \dots$ and $m \in \mathbb{N}$. Then $r + (m)$ belongs to $\mathcal{D}$ and $d(r + (m)) = \dfrac{1}{m}$. If $S \subset \mathbb{N}$ then the value

$$\mu^*(S) = \inf \Big\{ \sum_{j=1}^{k} \frac{1}{m_k}; S \subset \bigcup_{j=1}^{k} r_j + (m_j) \Big\}$$

is called *Buck's measure density* of the set S.

The sets from the system

$$\mathcal{D}_\mu = \{S \subset \mathbb{N}; \mu^*(S) + \mu^*(\mathbb{N} \setminus S) = 1\}$$

are called *Buck measurable.*

The following trivial fact will be useful for us, (see [PAS3], page 39) :

P r o p o s i t i o n 4. a) *$\mathcal{D}_\mu$ is an algebra of sets, the restriction $\mu = \mu^*|_{\mathcal{D}_\mu}$ is a finitely additive probability measure on $\mathcal{D}_\mu$.*

b) *$\mathcal{D}_\mu \subset \mathcal{D}$ and $d(S) = \mu(S)$ for every $S \in \mathcal{D}_\mu$.*

c) *A set $S \subset \mathbb{N}$ belongs to $\mathcal{D}_\mu$ if and only if for each $\varepsilon > 0$ sets $S_1, S_2 \in \mathcal{D}_\mu$ exist such that $S_1 \subset S \subset S_2$ and $\mu(S_2) - \mu(S_1) < \varepsilon$.*

We say that a sequence of real numbers $\{v(n)\}$ is *Buck measurable* if and only if for every real number x the set $\{n \in \mathbb{N}; v(n) < x\}$ belongs to $\mathcal{D}_\mu$. In this case the function

$$F(x) = \mu(\{n \in \mathbb{N}; v(n) < x\})$$

is called *Buck's distribution function* (for short B-d.f.) of $\{v(n)\}$.

A Buck measurable sequence is called *Buck uniformly distributed (for short B- u.d.)* if and only if its Buck distribution function $F(x)$ satisfies

$$F(x) = 0, \text{for } x < 0, F(x) = x, \text{for } x \in [0,1], F(x) = 1, \text{for } x > 1. \tag{1}$$

Proposition 4 implies

P r o p o s i t i o n 5. *Each Buck measurable sequence of real numbers has an asymptotic distribution function which coincides with its Buck distribution function.*

D e f i n i t i o n 3. A real valued sequence $\{v(n)\}$ is called *weakly Buck measurable* if and only if the sets $\{n \in \mathbb{N}; v(n) < x\}$ are Buck measurable excluding at most a countable set of real numbers x. In this case the function

$$F(x) = \{n \in \mathbb{N}; v(n) < x\}$$

defined on the real line excluding at most a countable set is called a *weak Buck distribution function* of $\{v(n)\}$.

The standard procedure yields a variant of Chebyshev's inequality:

P r o p o s i t i o n 6. *If a bounded sequence v has weak distribution function then for each $\varepsilon > 0$ we have*

$$\overline{d}(\{n \in \mathbb{N}; |v(n) - E(v)| > \varepsilon\}) \leq \frac{D^2(v)}{\varepsilon^2}.$$

Using these concepts we obtain the following

Proposition 7. *If v is a bounded sequence having a weak distribution function and $D^2(v) = 0$ then there exists a set $A \in \mathcal{D}$ such that $d(A) = 1$ and $\lim_A v(n) = E(v)$, were the limit is taken along the set A.*

Definition 4. Let v, w be sequences having weak asymptotic distribution functions. Suppose moreover that the sequence vw has a mean value. The value

$$\rho(v, w) = \frac{|E(vw) - E(v)E(w)|}{D(v)D(w)}$$

will be called the *correlation coefficient* of the sequences v, w.

Definition 5. We say that the sequences v, w are *correlated* if and only if such values α, β exist that $\lim_A w(n) - \alpha v(n) - \beta = 0$ for some set A from $\mathcal{D}$ such that $d(A) = 1$, where the limit is taken along the set A.

In [P-T] the following result is proved

Proposition 8. *The sequences v, w are correlated if and only if vw has a mean value and $\rho(v, w) = 1$. In this case for α, β from Definition 5 we have*

$$\alpha = \frac{E(vw) - E(v)E(w)}{D^2(v)}, \beta = E(w) - \alpha E(v).$$

This has the following implication:

Corollary 1. *If v, w are sequences uniformly distributed modulo 1, then they are correlated if and only if $E(vw) = \dfrac{1}{3}$ or $E(vw) = \dfrac{1}{6}$.*

3 - Independent sequences

In the book [Ra] the following notion is defined :

Definition 6. Two bounded real valued sequences v, w are called *statistically independent* if and only if for every functions g, g_1, that are continuous on a closed interval containing the elements of both sequences

$$\lim_{N\to\infty} E_N(g(v))E_N(g_1(w)) - E_N(g(v)g_1(w)) = 0.$$

Properties of statistical independent sequences are studied in various papers. For a survey we refer to the monograph [SP].

Definition 7. Two sets $S, S_1 \in \mathcal{D}$ are called *independent* if and only if $S \cap S_1 \in \mathcal{D}$ and $d(S \cap S_1) = d(S)d(S_1)$. Bounded sequences v, w having asymptotic distribution functions are called *independent* if and only if for arbitrary intervals I, I_1 the sets $\{n \in \mathbb{N}; v(n) \in I\}$ and $\{n \in \mathbb{N}; w(n) \in I_1\}$ are independent.

We shall prove

Theorem 1. *Let v, w be bounded sequences having continuous asymptotic distribution functions. Then these sequences are independent if and only if they are statistically independent.*

We start with the following

Proposition 9. *Let $v_k, w_k, (k \in \mathbb{N})$ be two systems of sequences of elements from a certain closed interval $[a, b]$. Suppose that for each $k \in \mathbb{N}$ the sequences v_k, w_k are statistically independent. If v_k converges uniformly to v and w_k converges uniformly to w then the sequences v, w are statistically independent.*

Proof. Let g, g_1 be continuous functions defined on a closed interval containing the elements of both sequences. Then these functions are uniformly continuous. Thus $g(v_k)$ converges uniformly to $g(v)$, $g_1(w_k)$ converges uniformly to $g(w)$ and $g(v_k)g_1(w_k)$ converges uniformly to $g(v)g_1(w)$. Hence for given $\varepsilon > 0$ there exists k with

$$|E_N(g(v_k)) - E_N(g(v))| < \varepsilon, |E_N(g(w_k)) - E_N(g(w))| < \varepsilon$$

and

$$|E_N(g(v_k)g_1(w_k)) - E_N(g(v)g_1(w))| < \varepsilon.$$

Moreover there exists N_0 such that for $N \geq N_0$ we have

$$|E_N(g(v_k))E_N(g(w_k)) - E_N(g(v_k)g_1(w_k))| < \varepsilon.$$

From the first inequalities we derive

$$|E_N(g(v_k))E_N(g_1(w_k)) - E_N(g(v))E_N(g_1(w))| < 2M\varepsilon,$$

where M is an upper bound of $|v|, |w|$. This yields for $N \geq N_0$

$$|E_N(g(v)g_1(w)) - E_N(g(v))E_N(g_1(w))| < 2M\varepsilon + \varepsilon.$$

□

Definition 8. If $S_1, \ldots, S_k$ are disjoint sets belonging to $\mathcal{D}$ and $c_1, \ldots c_k \in \mathbb{R}$ then the sequence s defined by

$$s(n) = \sum_{j=1}^{k} c_j \mathcal{X}_{S_j}(n), n \in \mathbb{N}$$

is called a *simple sequence.*

It is easy to check :

Proposition 10. *If s is a simple sequence then s has a mean value and*

$$E(s) = \sum_{j=1}^{k} c_j d(S_j).$$

This leads to the following consequence:

Proposition 11. *Let $s = \sum_{j=1}^{k} c_j \mathcal{X}_{S_j}, r = \sum_{j=1}^{\ell} r_j \mathcal{X}_{R_j}$ be such simple sequences that the sets S_j, R_k, are independent for $j = 1, \ldots, k, k = 1, \ldots, \ell$. Then they are statistically independent.*

Proposition 12. *If v, w are bounded independent sequences then they are statistically independent.*

Proof. Let the values of v, w be contained in the interval $[a, b]$. Consider for $k \in \mathbb{N}$ the partition of $[a, b]$ into disjoint subintervals $I_j, j = 1, \ldots, m$ such that $|I_j| < \dfrac{1}{k}, j = 1, \ldots, m$. Then the sets

$$S_j = \{n \in \mathbb{N}; v(n) \in I_j\}, R_i = \{n \in \mathbb{N}; w(n) \in I_i\}, 1 \leq i, j \leq m$$

are independent. Thus the simple sequences

$$s_k = \sum_{j=1}^{m} c_j \mathcal{X}_{S_j}, r_k = \sum_{j=1}^{m} c_j \mathcal{X}_{R_j}, c_j \in I_j, j = 1, \ldots, m$$

are statistically independent. Since $|s_k(n) - v(n)| \leq \dfrac{1}{k}$ and $|r_k(n) - w(n)| \leq \dfrac{1}{k}$ for $n \in \mathbb{N}$ we obtain that s_k converges uniformly to v and r_k converges uniformly to w. Thus due to Proposition 9 v and w are statistically independent. $\square$

Proof of the second implication of Theorem 1. Consider statistically independent bounded sequences v, w having continuous asymptotic distribution functions F, F_1 respectively. Let $I_1 = [x_1, x_2], I_2 = [y_1, y_2]$. Since F, F_1, are continuous, Proposition 1 guarantees that for $\varepsilon > 0$ there exist positive continuous functions f, f_1, g, g_1 satisfying

$$f \le \mathcal{X}_{I_1} \le f_1, g \le \mathcal{X}_{I_2} \le g_1 \tag{2}$$

and

$$\int_a^b (f_1(x) - f(x))dF(x) < \varepsilon, \int_a^b (g_1(x) - g(x))dF_1(x) < \varepsilon. \tag{3}$$

From (2) we derive

$$E_N(f(v)g(w)) \le E_N(\mathcal{X}_{I_1}(v)\mathcal{X}_{I_2}(w)) \le E_N(f_1(v)g_1(w)),$$

moreover

$$E_N(f(v))E_N(g(w)) \le E_N(\mathcal{X}_{I_1}(v))E_N(\mathcal{X}_{I_2}(w)) \le E_N(f_1(v))E_N(g_1(w)).$$

If $N \to \infty$ we obtain for $\overline{E} = \overline{E}(\mathcal{X}_{I_1}(v)\mathcal{X}_{I_2}(w))$ and $\underline{E} = \underline{E}(\mathcal{X}_{I_1}(v)\mathcal{X}_{I_2}(w))$ the inequalities

$$\int_a^b f(x)dF(x) \int_b^a g(x)dF_1(x) \le \overline{E} \le \int_a^b f_1(x)dF(x) \int_b^a g_1(x)dF_1(x)$$

and

$$\int_a^b f(x)dF(x) \int_a^b g(x)dF_1(x) \le \underline{E} \le \int_a^b f_1(x)dF(x) \int_a^b g_1(x)dF_1(x).$$

Set $S_1 = \{n \in \mathbb{N}; v(n) \in I_1\}, S_2 = \{n \in \mathbb{N}; w(n) \in I_2\}$. Then

$$\lim_{N\to\infty} E_N(\mathcal{X}_{I_1}(v))E_N(\mathcal{X}_{I_2}(w)) = d(S_1)d(S_2).$$

This yields

$$|\overline{E} - d(S_1)d(S_2)| \le H\varepsilon, |\underline{E} - d(S_1)d(S_2)| \le H\varepsilon,$$

where H is suitable constant. Since $\varepsilon >$ is arbitrary we get $\overline{E} = \underline{E} = d(S_1)d(S_2)$. If we consider that $E = \overline{E} = \underline{E} = d(S_1 \cap S_2)$ the assertion follows. □

An immediate consequence of the definition is the following:

Proposition 13. *Let v, w be bounded sequences having continuous asymptotic distributions F, F_1, respectively. Suppose that these sequences are independent. Then for any intervals $I_1 = [x_1, x_2], I_2 = [y_1, y_2]$ the set $S = \{n \in \mathbb{N}; (v(n), w(n)) \in I_1 \times I_2\}$ belongs to $\mathcal{D}$ and*

$$d(S) = (F(x_2) - F(x_1))(F_1(y_1) - F_1(y_2)).$$

Using the above notation the standard method yields:

P r o p o s i t i o n 14. *Let A be a Riemann Stjeltjes measurable set with respect product measure $F \times F_1$ then the set $R = \{n \in \mathbb{N}; (v(n), w(n)) \in A\}$ belongs to $\mathcal{D}$ and*

$$d(R) = \int\int_R dF(t_1)dF_1(t_2).$$

Furthermore the following theorem holds (with the above notation).

T h e o r e m 2. *The sequence $v + w$ has an asymptotic distribution function F_2 given by*

$$F_2(x) = \int\int_{\{(t_1,t_2);t_1+t_2\leq x\}} dF(t_1)dF(t_2).$$

This leads after some calculation to:

C o r o l l a r y 2. *If v, w are two independent uniformly distributed sequences then the sequence $v + w$ has the distribution function G where $G(x) = 0, x \leq 0, G(x) = \dfrac{x^2}{2}, x \in [0, 1], G(x) = 2x - \dfrac{x^2}{2} - 1, x \in [1, 2], G(x) = 1, x > 2.$*

In following the more general notion of independence will be useful:

D e f i n i t i o n 9. If $v_1, \dots, v_k$ are bounded sequences having asymptotic distribution functions then they are called *independent* if and only if for all intervals $I_1, \dots, I_k$ the set $S = \{n \in \mathbb{N}; v_j(n) \in I_j, j = 1, \dots, k\}$ belongs to $\mathcal{D}$ and

$$d(S) = \prod_{j=1}^{k} d(\{n \in \mathbb{N}; v_j(n) \in I_j\}).$$

These sequences are called *statistically independent* if and only if

$$\lim_{N\to\infty} E_N(g_1(v_1) \dots g_k(v_k)) - E_N(g_1(v_1)) \dots E_N(g_k(v_k)) = 0$$

for any functions $g_1, \dots, g_k$ continuous on closed intervals containing all elements of the given sequences.

T h e o r e m 3. *Let $v_1, \dots, v_k$ be bounded sequences having continuous asymptotic distribution functions. Then they are independent if and only if they are statistically independent.*

P r o p o s i t i o n 15. *Let v_1, v_2, v_3 be bounded sequences having asymptotic distribution functions. If these sequences are independent then $v_1 + v_2, v_3$ are independent, too.*

From this we derive as above:

Theorem 4. *If $v_1, \dots, v_k$ are independent bounded sequences with continuous distribution functions, having the same mean value E and the same dispersion D^2. Then*

$$d\Big(\Big\{n \in \mathbb{N}; \Big|\frac{v_1 + \cdots + v_k}{k} - E\Big| \geq \varepsilon\Big\}\Big) \leq \frac{D^2}{n\varepsilon^2}.$$

4 - Polyadicly continuous sequences

Denote by Ω the compact metric ring of polyadic integers, (see [N], [N1], [PAS5], which is the completion of $\mathbb{N}$ with respect to the polyadic metric

$$\mathfrak{d}(a, b) = \sum_{n=1}^{\infty} \frac{\psi_n(a-b)}{2^n}, \tag{4}$$

where $\psi_n(x) = 0$ if n divides x and $\psi_n(x) = 1$ otherwise. For sequences $\{v(n)\}$ we shall use two synonymous expressions: sequences or arithmetic functions. A sequence $\{v(n)\}$ is called *polyadicly continuous* (for short: *p-continuous*) if and only if for each $\varepsilon > 0$ there is $m \in \mathbb{N}$ such that

$$\forall a, b \in \mathbb{N}; a \equiv b \pmod m \Rightarrow |v(a) - v(b)| < \varepsilon.$$

In [PAS2] it is proved:

Proposition 16. *Let $\{v(n)\}$ be a p-continuous sequence of elements of $[0,1]$. Suppose that F is a continuous function defined on $[0,1]$. Then $\{v(n)\}$ is Buck measurable with B-d.f. F if and only if*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} h(v(n)) = \int_0^1 h(x)dF(x)$$

for each continuous real valued function h defined on $[0,1]$.

This implies the following

Proposition 17. *If a p-continuous sequence of elements in $[0,1]$ has a continuous asymptotic distribution function then it is Buck measurable and its B-d.f. coincides with its asymptotic distribution function.*

The next result is due to P. Erdős [Er], see also [PAS3], p.32. In the following we use the notation

$$A(v(n), I) = |\ \{n \in \mathbb{N}; v(n) \in I\}\ |,$$

for sequences $v(n) \in [0,1]$ and intervals $I \subset [0,1]$.

Theorem 5. *Suppose that f is a non-negative additive arithmetic function such that for every prime p we have $f(p) = f(p^k), k = 1, 2, 3, \dots,$ and for distinct primes p_1, p_2 we have $f(p_1) \neq f(p_2)$. Assume that the infinite series $\sum_p \frac{f(p)}{p}$ (running over the primes) converges. Then for every interval I, there holds $A(\{f(n)\}, I) \in \mathcal{D}$. Moreover, in this case, the function*

$$g(x) = d(A(\{f(n)\}, [-\infty, x))$$

is continuous on the real line.

Corollary 3. *Let f be a non-negative additive arithmetic function such that for every prime p we have $f(p) = f(p^k), k = 1, 2, 3, \dots,$ for different primes $f(p_1) \neq f(p_2)$ and the series $\sum_p f(p)$ converges. Then the sequence $\{f(n)\}$ is Buck measurable with continuous Buck distribution function.*

Proof. Let $N \in \mathbb{N}$. If $n_1 \equiv n_2 \pmod{N!}$ then n_1, n_2 contain the same primes smaller than N in canonical decomposition and so in this case

$$|f(n_1) - f(n_2)| \leq 2 \sum_{p>N} f(p).$$

Thus the convergence of $\sum_p f(p)$ provides that $\{f(n)\}$ is a p-continuous sequence. This condition yields also the convergence of $\sum_p \frac{f(p)}{p}$, and the assertion follows. □

It is easy to check that each p-continuous sequence of real numbers is uniformly continuous with respect to the polyadic metric $\mathfrak{d}$, and so each p-continuous sequence of real numbers $\{v(n)\}$ can be extended in the natural way to a real valued continuous function $\tilde{v}$ defined on Ω such that

$$\tilde{v}(\alpha) = \lim_{j\to\infty} v(n_j),$$

where $\{n_j\}$ is a sequence of positive integers such that $n_j \to \alpha$ for $j \to \infty$ with respect the polyadic metric. The compact ring Ω is equipped with Haar probability measure P and so the function $\tilde{v}$ can be considered as random variable on the probability space (Ω, P). As usually h is a random variable on Ω and we denote $E(h) = \int h dP$, the mean value of h.

Let $m \in \mathbb{N}$ and $s = 0, 1, \dots m - 1$. Put

$$s + m\Omega = \{s + m\alpha; \alpha \in \Omega\}.$$

The ring Ω can be represented as disjoint union

$$\Omega = \bigcup_{s=0}^{m-1} s + m\Omega.$$

(see [N], [N1]). Thus for the Haar probability measure P we have

$$P(s + m\Omega) = \frac{1}{m} \tag{5}$$

for $m \in \mathbb{N}, s = 0, \ldots m - 1$.

In [PAS4] it is proven that

$$\mu^*(S) = P(cl(S)) \tag{6}$$

for each $S \subset \mathbb{N}$, where $cl(S)$ denote the topological closure of S in Ω.

Example 1. Let $\{Q_k\}$ be an increasing sequence of integers such that $Q_0 = 1$ and Q_k divides $Q_{k+1}, k = 1, 2, 3, \ldots$. Each positive integer n can be uniquely represented in the form

$$n = a_0 + a_1Q_1 + ... + a_kQ_k,$$

where $a_j < \dfrac{Q_{j+1}}{Q_j}, j = 1, \ldots, k$. To this n we associate an element $\gamma(n)$ in the unit interval of the form

$$\gamma(n) = \frac{a_0}{Q_1} + \cdots + \frac{a_k}{Q_{k+1}}.$$

The sequence $\{\gamma(n)\}$ is known as van der Corput sequence in base $\{Q_k\}$ and in [PAS2] it is proved that it is Buck uniformly distributed and p-continuous.

The following characterization allows us to apply results of probability theory to the distribution of p-continuous sequences:

Theorem 6. *Let $\{v(n)\}$ be a p-continuous sequence and F a continuous real valued function defined on the real line. Then the following the statements are equivalent:*

(i) *F is the distribution function of the random variable $\tilde{v}$.*

(ii) *$\{v(n)\}$ is a Buck measurable sequence and F is its B-d.f..*

(iii) *For each real number x we have*

$$\mu^*(\{n \in \mathbb{N}; v(n) < x\}) = F(x).$$

P r o o f. (i) $\Rightarrow$ (ii). The continuity of F yields

$$P(\tilde{v} < x) = F(x) = P(\tilde{v} \le x) \tag{7}$$

for each real number, x. From the inclusion

$$\{n \in \mathbb{N}; v(n) < x\} \subset \{\alpha \in \Omega; \tilde{v}(\alpha) \le x\}$$

we obtain

$$\mathrm{cl}\,(\{n \in \mathbb{N}; v(n) < x\}) \subset \{\alpha \in \Omega; \tilde{v}(\alpha) \le x\}.$$

Furthermore (7) yields

$$\mu^*(\{n \in \mathbb{N}; v(n) < x\}) \le F(x)$$

for every real number x. On the other hand

$$\mathbb{N} \setminus \{n \in \mathbb{N}; v(n) < x\} = \{n \in \mathbb{N}; v(n) \ge x\},$$

therefore

$$\mathrm{cl}\,(\mathbb{N} \setminus \{n \in \mathbb{N}; v(n) < x\}) \subset \{\alpha \in \Omega; \tilde{v}(\alpha) \ge x\}.$$

Hence

$$\mu^*(\mathbb{N} \setminus \{n \in \mathbb{N}; v(n) < x\}) \le 1 - F(x),$$

and so the set $\{n \in \mathbb{N}; v(n) < x\}$ is Buck measurable and its measure density is $F(x)$.

The implication (ii) $\Rightarrow$ (iii) is trivial.

(iii) $\Rightarrow$ (i).

Clearly

$$\{n \in \mathbb{N}; v(n) < x\} \subset \{\alpha \in \Omega; \tilde{v}(\alpha) \le x\}),$$

and so $F(x) \le P(\tilde{v} \le x)$. On the other hand

$$\{\alpha \in \Omega; \tilde{v}(\alpha) < x\}) \subset \mathrm{cl}(\{n \in \mathbb{N}; v(n) \le x\})$$

for $\varepsilon > 0$. This yields $F(x) \le P(\tilde{v} \le x) \le F(x+\varepsilon)$ for $\varepsilon > 0$. For $\varepsilon \to 0^+$ we obtain the assertion from the continuity of F. $\square$

5 - Independence and measurability

From our definitions in previous sections we immediately derive:

Proposition 18. *The Buck measurable sequences* $\{v_1(n)\}, \{v_2(n)\}, \dots, \{v_r(n)\}$ *are independent if and only if for every* $x_1, \dots, x_r \in \mathbb{R}$ *we have*

$$\mu\Big(\bigcap_{j=1}^{r}\{n \in \mathbb{N}; v_j(n) < x_j\}\Big) = \prod_{j=1}^{r} \mu(\{n \in \mathbb{N}; v_j(n) < x_j\}).$$

Example 2. We come back to Example 1. Consider the sequences $\{Q_k^{(j)}\}$ given such that $Q_0^{(j)} = 1, j = 1, \dots, r$ and $Q_k^{(j)} | Q_{k+1}^{(j)}$ for $j = 1, \dots, r$ and $k = 0, 1, 2 \dots$. Let $Q_k^{(j)}, Q_k^{(j_1)}$ be relatively prime for $j \neq j_1$. Denote by $\{\gamma_j(n)\}$ the van der Corput sequence with base $Q_k^{(j)}$ for $j = 1, \dots, r$. Then these sequences are independent (see [IPT]).

Theorem 7. *Let* $\{v_1(n)\}, \{v_2(n)\}, \dots, \{v_k(n)\}$ *be independent Buck measurable p-continuous sequences with continuous Buck distribution functions* $F_j, j = 1 \dots, k$. *Then the random variables* $\tilde{v_1}, \dots, \tilde{v_k}$ *are independent.*

Proof. For $x_1, \dots x_k \in \mathbb{R}$ we have

$$\{\alpha \in \Omega; \tilde{v_1}(\alpha) < x_1, \dots, \tilde{v_k}(\alpha) < x_k\}$$
$$\subset \mathrm{cl}(\{n \in \mathbb{N}; v_1(n) \leq x_1, \dots, v_k(n) \leq x_k\}).$$

Thus $P(\tilde{v_1} < x_1, \dots, \tilde{v_k} < x_k) \leq F_1(x) \dots F_k(x_k)$, and so from the above theorem we get $P(\tilde{v_1} < x_1, \dots, \tilde{v_k} <_k) \leq P(\tilde{v_1} < x_1) \dots P(\tilde{v_k} < x_k)$.

On the other hand we have

$$P(\tilde{v_1} \leq x_1) \dots P(\tilde{v_k} \leq x_k)$$
$$= \mu(\{n \in \mathbb{N}; v_1(n) \leq x\}) \dots \mu(\{n \in \mathbb{N}; v_k(n) \leq x_k\})$$
$$= P(\mathrm{cl}(\{n \in \mathbb{N}; v_1(n) \leq x_1, \dots, v_k(n) \leq x_k\}) \leq P(\tilde{v_1} \leq x_1, \dots, \tilde{v_k} \leq x_k).$$

$\square$

Let $F_1, ..., F_k$ be non-decreasing functions defined on $\mathbb{R}$, (k is a fixed positive integer). A set $B \subset \mathbb{R}^k$ is called *Jordan Stieltjes measurable* with respect to the functions $F_1, ..., F_k$ if and only the Riemann Stieltjes integral

$$\int\int ... \int \mathcal{X}_B dF_1 ... dF_k$$

exists; $\mathcal{X}_B$ denoting the indicator function of B.

Theorem 8. *Let $\{v_1(n)\}, ..., \{v_k(n)\}$ be independent Buck measurable p-continuous sequences with continuous Buck distribution functions $F_1, ..., F_k$. Suppose that a set $B \subset \mathbb{R}^k$ is Jordan Stieltjes measurable with respect to the functions $F_1, ..., F_k$. Then the set $\{n \in \mathbb{N}; (v_1(n), ..., v_k(n)) \in B\}$ is Buck measurable and its Buck measure density is*

$$\int\int ... \int \mathcal{X}_B dF_1 ... dF_k. \tag{8}$$

Proof. If $B = [a_1, b_1] \times ... \times [a_k, b_k]$ is a cylinder set then (8) follows directly from independence of $\{v_1(n)\}, ..., \{v_k(n)\}$ and Theorem 6. Proposition 4 then implies the assertion. □

6 - Integral and mean value

Let $h : \Omega \to (-\infty, \infty)$ be a continuous function. Since Ω is a compact space, it is uniformly continuous. Consider $m \in \mathbb{N}$. To the function h we can associate a periodic function h_m with period m in the following way:

$$\alpha \in s + m\Omega \Longleftrightarrow h_m(\alpha) = h(s).$$

Clearly,

$$\int h_m dP = \frac{1}{m} \sum_{s=0}^{m-1} h(s). \tag{9}$$

Clearly $\lim_{N\to\infty} \mathfrak{d}(N!, 0) = 0$, and so uniform continuity of h implies that $h_{N!}$ converges uniformly to h. From (9) we obtain

$$\int h dP = \lim_{N\to\infty} \frac{1}{N!} \sum_{s=0}^{N!-1} h(s). \tag{10}$$

The function h restricted on $\mathbb{N}$ is p-continuous. Thus there exists the limit $\lim_{m\to\infty} \dfrac{1}{m} \sum_{s=0}^{m-1} h(s)$. From (10) we conclude

$$\int h dP = \lim_{m\to\infty} \frac{1}{m} \sum_{s=0}^{m-1} h(s). \tag{11}$$

R e m a r k 1. If the random variable $\tilde{v}$ has a continuous distribution function F then

$$\int \tilde{v} dP = \int_{-\infty}^{\infty} x dF(x) = E(v).$$

The central limit theorem immediately yields:

P r o p o s i t i o n 19. *Let $\{v_k(n)\}, k = 1, 2, 3, \dots$ be a sequence of p-continuous sequences such that for every $k = 1, 2, 3, \dots$ the sequences $\{v_j(n)\}, j = 1, \dots, k$ are independent and have the same continuous Buck distribution function. Then for every $x \in \mathbb{R}$ we have*

$$\lim_{k\to\infty} \mu\Big(\Big\{n \in \mathbb{N}; \frac{v_1(n) + \cdots + v_k(n) - kE}{\sqrt{k}D} \le x\Big\}\Big) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{\frac{-t^2}{2}} dt.$$

We conclude this section with the following metric result:

T h e o r e m 9. *Let $v_k, k = 1, 2, 3, \dots$ be a system of independent p-continuous uniformly distributed sequences. Then the sequence $\{\tilde{v}_n(\alpha)\}$ is uniformly distributed for almost all $\alpha \in \Omega$.*

P r o o f. Denote

$$S_N(h, \alpha) = \frac{1}{N} \sum_{n=1}^{N} e^{2\pi i h \tilde{v}_n(\alpha)}$$

for $h \in \mathbb{Z} \setminus \{0\}$ and $\alpha \in \Omega$. Put

$$A_h = \{\alpha \in \Omega; \lim_{N\to\infty} S_N(h, \alpha) = 0\}$$

for $h \neq 0$. For every $n \in \mathbb{N}$ we have $E(e^{2\pi i h \tilde{v}_n}) = 0$. Therefore the strong low of large numbers implies that $P(A_h) = 1$. Thus $P(\cap_{h\neq 0} A_h) = 1$ and the assertion follows. □

7 - Weak Buck measurability

D e f i n i t i o n 10. Let $v = \{v(n)\}$ be a real valued sequence. We say that v is *weakly polyadicly continuous* if and only if for every $\varepsilon > 0, \delta > 0$ there exists a set $A \in \mathcal{D}_\mu$ with $\mu(A) < \delta$ such that

$$n_1 \equiv n_2 \pmod{m} \Rightarrow |v(n_1) - v(n_2)| < \varepsilon$$

for all $n_1, n_2 \in \mathbb{N} \setminus A$ and a suitable $m \in \mathbb{N}$.

Our aim is to prove the following equivalence :

Theorem 10. *A bounded sequence of real numbers is weakly Buck measurable if and only if it is weakly polyadicly continuous.*

We start by the proof of the first implication. We recall the following notion:

Definition 11. A real valued sequence v is called *almost polyadicly continuous* if and only if for each $\delta > 0$ there exists a set $A \in \mathcal{D}_\mu$ with $\mu(A) < \delta$ such that v is polyadicly continuous on the set $\mathbb{N} \setminus A$.

Directly from the definition we get

Proposition 20. *A set $S \subset \mathbb{N}$ is Buck measurable if and only if its indicator function $\mathcal{X}_S$ is almost polyadicly continuous.*

Proposition 21. *If v_1, v_2 are two almost polyadicly continuous sequences and c_1, c_2 are real numbers then the sequence $c_1v_1 + c_2v_2$ is almost polyadicly continuous.*

Proposition 22. *If v is a real valued sequence such that for each $\varepsilon > 0$ there exists an almost polyadicly sequence v_0 such that $|v(n) - v_0(n)| < \varepsilon$ for $n \in \mathbb{N}$ then v is weakly polyadicly continuous.*

Proposition 23. *Each bounded weakly Buck measurable sequence is weakly polyadicly continuous.*

Proof. Let v be a weakly Buck measurable sequence of elements in the interval $[a, b], a < b$. Consider $\varepsilon > 0$. Then there exists a partition $x_0, \ldots, x_k$ of $[a, b]$ such that the sets

$$S_i = \{n \in \mathbb{N}; v(n) \in [x_i, x_{i+1})\},\ i = 0, \ldots, k - 2$$

and

$$S_{k-1} = \{n \in \mathbb{N}; v(n) \in [x_{k-1}, b]\}$$

are Buck measurable and $x_{i+1} - x_i < \varepsilon$. Then the sequence

$$v_0(n) = \sum_{i=0}^{k-1} x_i \mathcal{X}_{S_i}(n),\ n \in \mathbb{N}$$

is almost polyadicly continuous and $|v_0(n) - v(n)| < \varepsilon$. The assertion follows from Proposition 22. ☐

Now we prove the second implication.

If $v = \{v(n)\}$ is a real valued sequence and $k = \{k_n\}$ is a sequence of positive integers then we shall denote $v(k) = \{v(k_n)\}$.

P r o p o s i t i o n 24. *A set $S \subset \mathbb{N}$ is Buck measurable if and only if for each sequence of positive integers k the sequence $\mathcal{X}_S(k)$ has a mean value and in this case*

$$\mu(S) = E(\mathcal{X}_S(k)).$$

This proposition is an easy reformulation of Theorem 7 in [PAS3] page 51 or Theorem 50 in [PAS5] page 113.

P r o p o s i t i o n 25. *If v is a bounded weakly polyadicly continuous sequence then it has a mean value and for each sequence of positive integers k which is uniformly distributed in $\mathbb{Z}$ we have*

$$E(v(k)) = E(v).$$

P r o o f. Consider $\delta > 0, \varepsilon > 0$. Let A, m be as in Definition 10. Suppose that $r_1, \dots, r_s$ is the maximal finite sequence of elements of $\mathbb{N} \setminus A$ incongruent modulo m and $r_{s+1}, \dots, r_m$ its completion with respect to a complete residue system modulo m. Define the periodic sequence $v_m(n) = v(r_j)$ if and only if $n \equiv r_j \pmod{m}$ for $j = 1, \dots, m$ and $n \in \mathbb{N}$. Then for each $n \in \mathbb{N} \setminus A$ we have

$$|v_m(n) - v(n)| < \varepsilon. \tag{12}$$

For $N = 1, 2, 3, \dots$ we obtain

$$E_N(v_m(k)) - E_N(v(k)) = \frac{1}{N} \sum_{n=1}^{N} (v_m(k_n) - v(k_n))$$

$$= \frac{1}{N} \sum_{n \le N, k_n \in A} (v_m(k_n) - v(k_n)) + \frac{1}{N} \sum_{n \le N, k_n \notin A} (v_m(k_n) - v(k_n)).$$

And so from (12) we device

$$|E_N(v_m(k)) - E_N(v(k))| < 2H E_N(\mathcal{X}_A(k)) + \varepsilon$$

where H is upper bound of $\{|v(n)|\}$. If k is uniformly distributed in $\mathbb{Z}$ we get for $N \to \infty$

$$|E(v_m) - \overline{E}(v(k))| < 2H\delta + \varepsilon \tag{13}$$

and

$$|E(v_m) - \underline{E}(v(k))| < 2H\delta + \varepsilon. \tag{14}$$

Therefore

$$\overline{E}(v(k)) - \underline{E}(v(k)) < 4H\delta + 2\varepsilon.$$

Since δ, ε are arbitrary we have $\overline{E}(v(k)) = \underline{E}(v(k)) = E(v(k))$. If in the inequalities (13) and (14) we substitute the sequence $\{n\}$ instead of k we conclude $E(v) = E(v(k))$. □

Proposition 26. *If v is a weakly polyadicly continuous sequence of elements in $[a, b]$ and f is a continuous real function defined on this interval then the sequence $f(v)$ is weakly polyadicly continuous, too.*

Proof. The assertion follows immediately from the fact that a continuous function on a compact interval is uniformly continuous. □

Proposition 27. *Each bounded weakly polyadic continuous real valued sequence is weakly Buck measurable.*

Proof. Let v be a weakly polyadic continuous real valued sequence of elements in $[a, b]$. Then for every continuous function f defined on $[a, b]$ the sequence $f(v)$ has a mean value and for every sequence of positive integers k uniformly distributed in $\mathbb{Z}$ we have

$$E(f(v)) = E(f(v(k)).$$

We define a positive linear functional

$$\Phi(f) = E(f(v))$$

on the linear space of all continuous real functions defined on $[a, b]$ such that $\Phi(1) = 1$.

Thus Riesz representation theorem provides that a non decreasing function F exists such $F(a) = 0, F(b) = 1$ and

$$E(f(v(k)) = \Phi(f) = \int_a^b f(x)dF(x) \tag{15}$$

holds for each sequence k uniformly distributed in $\mathbb{Z}$. If the function F is continuous in x_0 then by Proposition 1 we can construct for every $\varepsilon > 0$ two continuous functions f_1, f_2 defined on $[a, b]$ satistying

$$\int_a^b (f_2(x) - f_1(x))dF(x) < \varepsilon$$

and

$$f_1 \leq \mathcal{X}_{[0,x_0)} \leq f_2.$$

Hence for each sequence k uniformly distributed in $\mathbb{Z}$ we have

$$E(\mathcal{X}_{[0,x_0)}(v(k))) = F(x_0).$$

Proposition 24 implies that the set $\{n \in \mathbb{N}; v(n) < x_0\}$ is Buck measurable. Since every non-decreasing function has at most a countable set of discontinuities, the proof is complete. □

Let $\mathcal{B}_\mu$ be the set of all bounded weakly measurable sequences. Theorem 3 implies

Proposition 28. *Define the norm*

$$||v|| = \sup\{|v(n)|; n \in \mathbb{N}\}$$

for $v \in \mathcal{B}_\mu$. *Then* $(\mathcal{B}_\mu, +, \cdot, ||\cdot||)$ *is a Banach algebra.*

8 - Statistical independence

If $v = \{v(n)\}$ is a sequence and g is a function defined on the set containing the elements of v then we denote by $g(v)$ the sequence $\{g(v(n))\}$. The following theorem relates p-continuous independent sequences to the concept of uniform distribution in $\mathbb{Z}$.

Theorem 11. *Let* $\{v_1(n)\}, \ldots, \{v_k(n)\}$ *be* p*-continuous independent sequences. Then for arbitrary functions* $g_1, \ldots, g_k$ *continuous on the real line we have*

$$E\Big(\prod_{j=1}^{k} g_j(v_j(k))\Big) = \prod_{j=1}^{k} E\Big(g_j(v_j(k))\Big)$$

for each sequence $k = \{k_n\}$ *which is uniformly distributed in* $\mathbb{Z}$.

Proof. If $\{v(n)\}$ is a p-continuous function, then it is bounded. Every continuous function g defined on the real line is uniformly continuous on the interval $[b_1, b_2]$ where b_1 is a lower bound of the sequence $\{v(n)\}$ and b_2 its upper bound. Thus the sequence $\{g(v(n))\}$ is p-continuous, too.

Let us consider $\{v_1(n)\}, \ldots, \{v_k(n)\}$ - polyadicly continuous independent sequences. Then the random variables $\tilde{v_1}, \ldots, \tilde{v_k}$ are independent and so the random variables $g_1(\tilde{v_1}), \ldots, g_k(\tilde{v_k})$ are independent, too. Thus

$$E(g_1(v_1)\ldots g_k(v_k)) = E(g_1(v_1))\ldots E(g_k(v_k))$$

and the assertion follows from Proposition 25. □

Theorem 3 and Proposition 25 imply

Theorem 12. *Let* $\{v_1(n)\}, \ldots, \{v_k(n)\}$ *Buck measurable independent sequences having continuous Buck distribution functions. Then for arbitrary functions* $g_1, \ldots, g_k$ *continuous on the real line*

$$E\Big(\prod_{j=1}^{k} g_j(v_j(k)\Big) = \prod_{j=1}^{k} E\Big(g_j(v_j(k))\Big)$$

for each sequence $k = \{k_n\}$ *which is uniformly distributed in* $\mathbb{Z}$.

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