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# On the integral representation of the nonnegative superharmonic functions in a balayage space

**Abstract.** In this paper we study the integral representation of nonnegative superharmonic functions in a balayage space $(X, \mathcal{W})$ by using Choquet's method. When the space $X$ has a Green kernel $G$, we show that if a sequence of potentials in $X$ are representable by $G$ and majorized by some potential converges in the natural topology to a superharmonic function $p$ on $X$, then $p$ is representable by $G$. If in addition of the existence of the Green kernel, the potentials of harmonic support reduced to a single point are proportional, then any potential on $X$ can be represented by the function $G$ and reciprocally.

**Keywords.** Balayage space, Green function, potential, axiom of proportionality, extreme element, integral representation.

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## 1 - Introduction

The integral representation of potentials in the classical Potential Theory was established by F. Riesz [**21**] in 1925. The result of Riesz affirms that if $p$ is a potential (that is, a nonnegative superharmonic function whose the only nonnegative harmonic minorant is 0) in a Green domain $\Omega$ of $\mathbb{R}^n$, $n \geq 2$, then $p$ has a representation of the form $p = G\mu := \int G(\centerdot, y)d\mu(y)$, where $G$ is the Green kernel of $\Omega$ and $\mu$ is a nonnegative Radon measure on $\Omega$. A similar representation was given by R. S. Martin [**18**] in 1941 for the nonnegative harmonic functions on $\Omega$ by means of the boundary $\Delta(\Omega)$ of $\Omega$ (the Martin boundary of $\Omega$) in a suitable compactification $\widehat{\Omega}$ of $\Omega$ and a kernel $K : \Omega \times \Delta(\Omega) \to ]0, +\infty]$, called the Martin kernel of $\Omega$. Any nonnegative harmonic

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function $h$ on $\Omega$ has a representation $h = K\mu := \int K(.,y)d\mu(y)$, where $\mu$ is a measure on $\Delta(\Omega)$. This representation is unique if the measure $\mu$ is supposed to be carried by $\Delta_1(\Omega)$, the minimal Martin boundary of $\Omega$ (in general $\neq \Delta(\Omega)$).

Choquet [**8**] established in 1956 that any point of a metrizable compact convex $K$ of a locally convex topological vector space is the barycenter of a probability measure $\mu$ carried by the extreme elements of $K$ [**8**, Theor. 30.20]. This representation is not unique in general, but if the cone $S = \mathbb{R}_+.K$ is a lattice in its own order (that is, the order on $S$ defined by: $u \prec v$ if and only if there exists $w \in S$ such that $v = u + w$) then $\mu$ is unique. Since a convex compact appears as a compact base of a convex cone, this leads to the study the integral representation in the convex cones with compact bases of locally convex topological vector spaces by Choquet. The integral representation in the cones with compact base was extended by Choquet to the more general framework of weakly complete convex cones which do not have necessarily a compact base by means of the concept of conical measure.

The Choquet integral representation theorem in a convex cone with compact base allows to retrieve the result of Riesz. Indeed, there is on the cone $\mathcal{S}^+(\Omega)$ of the nonnegative superharmonic functions on a Green domain $\Omega$ of $\mathbb{R}^n$ (with Green kernel $G$) a topology for which it has a compact base and whose extreme elements are nonnegative harmonic functions (minimal nonnegative harmonic function), or the functions of the form $\alpha(y)G(\centerdot, y)$, $y \in \Omega$, where $\alpha$ is a nonnegative continuous function on $\Omega$. It was also used by R.-M Hervé in 1960 to establish in her thesis ([**16**, Chap. IV]) the integral representation for the nonnegative superharmonic functions in the setting of the Brelot axiomatic theory by means of the extreme elements of the cone $\mathcal{S}^+(\Omega)$ of the nonnegative superharmonic functions on a locally compact space $\Omega$ with countable base, endowed with a suitable topology for which it has a compact base. Mokobodzki [**19**] obtained this integral representation in the cone $\mathcal{S}^+(\Omega)$ endowed with the topology of convergence in graph (which coincides with that of R.-M Hervé) by using the "reduite" functions.

These methods and results apply to the standard H-cones of Boboc, Bucur and Cornea [**5**] and it seems that they can be extended to the nonnegative superharmonic functions in a balayage spaces introduced by Bliedtner and Hansen at the early 1980's.

In a balayage space $(X, \mathcal{W})$ admitting a Green kernel $G$, Hansen and Netuka recently established in [**15**] that if a sequence of nonnegative Radon measures $(\mu_n)$ on $X$ is such that the functions $G\mu_n = \int G(\centerdot, y)d\mu_n$ are bounded by some potential $q$ on $X$ and converges (pointwise) to a potential $p$ outside a polar set (a semi-polar set if $(X, \mathcal{W})$ is a harmonic space), then the potential $p$ admits an integral representation of the form $p = G\mu$ (the measure $\mu$ is the (weak) limit of the sequence $(\mu_n)$).

Our purpose in this paper is to study the question of the integral representation in the framework of balayage spaces $(X, \mathcal{W})$ by using the Choquet method. This will allow us to get the integral representation of the nonnegative superharmonic functions and that of the nonnegative harmonic functions in $X$. As an application, we prove that, if the space $X$ has a Green kernel $G$, then any potential which is a limit in the topology of the cone $\mathcal{S}^+(X)$ of nonnegative superharmonic $\geq 0$ on $X$ (the natural topology) or almost everywhere (relatively to a resolvent whose cone of excessive functions is equal to $\mathcal{W}$) of a sequence of potentials representable by $G$, can be represented by $G$. This result is stronger than a recent result of Hansen-Netuka [**15**, Theor. 1.1].

We also show that if the cone $\mathcal{S}^+(X)$ is closed in the cone of the excessive functions which are finite $\mathbb{V}$-a.e. of a resolvent family $\mathbb{V}$ of (Borel) kernels on $X$ and whose cone of excessive functions is equal to $\mathcal{W}$ and if the space $X$ has a Green kernel $G$ and the potentials of support superharmonic reduced to a single point are proportional, then any potential on $X$ is the Green potential of a nonnegative Radon measure on $X$.

**Notation and definitions**. Let $X$ be a locally compact space with countable base, we denote $\mathcal{C}(X)$ (resp. $\mathcal{C}^+(X)$, resp. $\mathcal{C}_0(X)$) the set of all real continuous (resp. nonnegative and continuous, resp. continuous and vanishing at infinity) functions on $X$, and by $\mathcal{B}(X)$ (resp. $\mathcal{B}^+(X)$, resp. $\mathcal{B}_b^+(X)$) the set of all (resp. nonnegative, resp. nonnegative and bounded) Borel measurable functions on $X$ with values in $\overline{\mathbb{R}}$. We denote by $\mathcal{K}^+(X)$ and $\mathcal{M}^+(X)$ respectively the set of all nonnegative real continuous functions with compact support and that of all nonnegative Radon measures on $X$. If $\mu \in \mathcal{M}^+(X)$ we denote by $Supp(\mu)$ the support of $\mu$. For any function $f : U \to \overline{\mathbb{R}}$, where $U$ is an open subset of $X$, we denote by $\widehat{f}$ the lower semi-continuous (l.s.c. in abbreviated form) regularized of $f$. Recall that $\widehat{f}$ is defined by $\widehat{f}(x) = \liminf_{y \to x} f(y)$ for any $x \in U$ and that $\widehat{f}$ is the greatest l.s.c. minorant of the function $f$ on $U$. If $(u_i)$ is a family of functions on an open subset $U$ of $X$ (with values in $\overline{\mathbb{R}}$), we denote $\widehat{\inf} u_i$ the function $\widehat{\inf u_i}$.

## 2 - Some background of the theory of balayage spaces

### 2.1 - *Balayage spaces*

Let $X$ be a locally compact space with countable base, and $\mathcal{W}$ a convex cone of l.s.c. functions on $X$ with values in $[0, +\infty]$. The coarsest topology on $X$ which is at least as fine as the initial topology and for which all functions of $\mathcal{W}$

are continuous will be called the $\mathcal{W}$-fine topology or simply the fine topology if there is no risk of confusion. We will use the prefix "fine(ly)" or "f-" to distinguish topological concepts related to the fine topology from those relative to the initial topology.

Definition 2.1. *The pair $(X, \mathcal{W})$ is called a balayage space if the following axioms are satisfied:*

$(b_1)$ $\sup v_n \in \mathcal{W}$ *for any increasing sequence* $(v_n)$ *of elements of* $\mathcal{W}$.

$(b_2)$ $\widehat{\inf \mathcal{V}}^f \in \mathcal{W}$ *for every non-empty subset* $\mathcal{V}$ *of* $\mathcal{W}$ *(where* $\widehat{g}^f$ *denotes the finely l.s.c. regularized function of* $g$*).*

$(b_3)$ *If* $u, v', v'' \in \mathcal{W}$ *are such that* $u \leq v' + v''$, *then there exist* $u', u'' \in \mathcal{W}$ *such that* $u = u' + u''$, $u' \leq v'$ *and* $u'' \leq v''$ *(the Riesz decomposition property).*

$(b_4)$ *(i) The functions of* $\mathcal{W}$ *are linearly separating (i.e., for any* $x, y \in X$, $x \neq y$ *and for any* $\lambda \in [0, +\infty[$, *there exists* $v \in \mathcal{W}$ *such that* $v(x) \neq \lambda v(y)$*).*

*(ii) For any* $w \in \mathcal{W}$, *we have* $w = \sup\{v \in \mathcal{W} \cap \mathcal{C}(X) : v \leq w\}$.

*(iii) There exist* $u_0, v_0 \in \mathcal{W} \cap \mathcal{C}(X)$ *and* $> 0$ *such that* $u_0/v_0 \in \mathcal{C}_0(X)$.

The elements of $\mathcal{W}$ will be called the *nonnegative hyperharmonic functions* on $X$, the set of *continuous real potentials* on $X$ is defined by

$$\mathcal{P} := \{p \in \mathcal{W} \cap \mathcal{C}(X) : \exists v \in \mathcal{W} \cap \mathcal{C}(X), v > 0 \text{ and } \frac{p}{v} \in \mathcal{C}_0(X)\}.$$

A potential $p \in \mathcal{P}$ is called *strict* if the measures $\mu, \nu \in \mathcal{M}^+(X)$ coincide provided that $\int p d\mu = \int p d\nu < +\infty$ and $\int u d\mu \leq \int u d\nu$, for all $u \in \mathcal{W}$.

A convex cone $\mathcal{F} \subset \mathcal{C}^+(X)$ is called a *function cone* if $\mathcal{F}$ is linearly separating and if, for any function $f \in \mathcal{F}$, there exists $g \in \mathcal{F}$ such that $g > 0$ and $f/g \in \mathcal{C}_0(X)$. A function $f \in \mathcal{F}$ is called *strict*, if measures $\mu, \nu \in \mathcal{M}^+(X)$ coincide provided that $\mu(f) = \nu(f) < +\infty$ and $\mu(g) = \nu(g)$, for any function $g \in \mathcal{W}$. Given an $\wedge$-stable function cone $\mathcal{F}$ and an increasing additive functional $T : \mathcal{F} \to [0, +\infty)$, there exists a unique measure $\mu \in \mathcal{M}^+(X)$ such that $T(f) = \int f d\mu$ for any function $f \in \mathcal{F}$. If $(X, \mathcal{W})$ is a balayage space, then $\mathcal{P}$ is an $\wedge$-stable function cone and

$$\mathcal{W} = \mathcal{S}(\mathcal{P}) := \{\sup p_n : (p_n) \subset \mathcal{P} \text{ increasing}\},$$

$\mathcal{P}$ is the greatest function cone in $\mathcal{W}$ and there exists a strict $p \in \mathcal{P}$ (obtained by taking sums $\sum_{n=1}^{\infty} \alpha_n p_n$, where $(p_n)$ is a sequence in $\mathcal{P}$ separating $\mathcal{M}^+(X)$ and $(\alpha_n)$ is a suitable sequence of real numbers $> 0$).

Remark 2.2. If the axiom $(b_2)$ in the above definition is satisfied, then $\widehat{\inf}\mathcal{V} = \widehat{\inf}^f\mathcal{V} \in \mathcal{W}$, moreover, the set $\{\widehat{\inf}\mathcal{V} < \inf\mathcal{V}\}$ is finely meager for any family $\mathcal{V} \subset \mathcal{W}$ ([**4**, Chap. II, Prop. 4.3, p. 58]).

We define the *reduite* of a function $u \in \mathcal{W}$ on a subset $A$ of $X$ by

$$R_u^A := R_{u\mathbf{1}_A} = \inf\{v \in \mathcal{W}, v \geq u \text{ on A}\};$$

its l.s.c regularized function $\widehat{R}_u^A := \widehat{R_u^A}$ is called the *balayage* of $u$ on $A$.

The following properties are immediate:

1. $R_u^A \leq u$, $R_u^A = u$ on $A$.
2. $R_{\alpha u}^A = \alpha R_u^A$ for all $\alpha \in \mathbb{R}^+$.
3. If $u \leq v$ and $A \subset B$ then $R_u^A \leq R_v^B$ and $R_{u+v}^A \leq R_u^A + R_v^A$.
4. If $(A_n)$ is an increasing sequence of subsets of $X$ and $(s_n)$ is an increasing sequence in $\mathcal{W}$, then $\sup_n R_{s_n}^{A_n} = R_s^A$ and $\sup_n \widehat{R}_{s_n}^A = \widehat{R}_s^A$, where $s = \sup_n s_n$ and $A = \bigcup_n A_n$.

It follows from the above results and from the property $(b_3)$ (the Riesz decomposition property) of the above definition of a balayage space that, for any subset $A$ of $X$, the mappings $R_{\cdot}^A : \mathcal{W} \to [0, +\infty]^X$, $u \mapsto R_u^A$ and $\widehat{R}_{\cdot}^A : \mathcal{W} \to \mathcal{W}$, $u \mapsto \widehat{R}_u^A$ are affine, that is,

5. for any $u_1, u_2 \in \mathcal{W}$ and $\alpha \in \mathbb{R}_+$, we have $R_{\alpha u_1 + u_2}^A = \alpha R_{u_1}^A + R_{u_2}^A$ and $\widehat{R}_{\alpha u_1 + u_2}^A = \alpha \widehat{R}_{u_1}^A + \widehat{R}_{u_2}^A$ (see [**4**, Ch. VI, Prop. 1.1, p. 243 ]), where it is understood that $0.u = 0$ for any $u \in \mathcal{W}$.

We denote by $\mathcal{M}^+(\mathcal{P})$ the set of measures $\mu \in \mathcal{M}^+(X)$ such that $\int p d\mu < +\infty$ for any potential $p \in \mathcal{P}$, $p > 0$. If $\mu \in \mathcal{M}^+(X)$ is of compact support, then $\mu \in \mathcal{M}^+(\mathcal{P})$. Given a measure $\mu \in \mathcal{M}^+(\mathcal{P})$ and a set $A \subset X$, then, according to ([**4**, Chap. VI, p. 256]), there exists a unique measure $\mu^A \in \mathcal{M}^+(X)$ such that for any function $u \in \mathcal{W}$, we have

$$\int \widehat{R}_u^A(x) d\mu = \int u d\mu_x^A.$$

The measure $\mu^A$ is called the *balayage measure* of $\mu$ on $A$.

The balayage of measures allows us to characterize the hyperharmonic functions, namely: a l.s.c. function $u : X \to \overline{\mathbb{R}}^+$ belongs to $\mathcal{W}$ if and only if, for any $x \in X$ and for any neighborhood $U$ of $x$, there exists a set $V \subset U$ such that $x \in V$, $\varepsilon_x^{X \setminus V} \neq \varepsilon_x$ and $\int u \varepsilon_x^{X \setminus V} \leq u(x)$ ([**4**, Chap. II, Prop. 5.5, p. 68]).

Let $A \subset X$, the *base* of $A$ is the set $b(A) = \{x \in X : \varepsilon_x^A = \varepsilon_x\}$. The set $A$ is called polar if $\varepsilon_x^A = 0$ for any $x \in X$. It is called *totally thin*, respectively *semi-polar*, if $b(A) = \emptyset$, respectively $A$ is the countable union of totally thin sets. Then $A$ is polar $\Rightarrow$ $A$ is totally thin $\Rightarrow$ $A$ is semi-polar.

One can show that every point of $X$ has a fundamental system of fine neighborhoods which are compact in the initial topology. In particular, $X$ endowed with the fine topology is a Baire space.

**2.2 -** *Resolvent associated with a balayage space.*

Let $X$ be a locally compact space with countable base. A *kernel* (or a Borel kernel) on $X$ is a function $N : X \times \mathcal{B}(X) \to [0, +\infty]$ such that

1. For any $A \in \mathcal{B}(X)$, the function $x \mapsto N(x, A)$ is $\mathcal{B}(X)$-measurable.
2. For any $x \in X$, the function of sets $A \ni \mathcal{B}(X) \mapsto N(x, A)$ is a (nonnegative) measure on $(X, \mathcal{B}(X))$.

If $N$ is a Borel kernel on $X$ and $f \in \mathcal{B}^+(X)$, we denote $Nf$ or $N(f)$ the function defined on $X$ by

$$Nf(x) = \int f(y) N(x, dy), \ \forall x \in X,$$

where the integral is taken relatively to the measure $N(x, \cdot)$.

Let $\mathbb{V} = (V_\lambda)_{\lambda > 0}$ be a family of (Borel) kernels on $X$. The family $\mathbb{V}$ is called a *resolvent* (family) if,

$$V_\alpha = V_\beta + (\alpha - \beta) V_\alpha V_\beta, \forall \alpha \geq \beta > 0.$$

The resolvent $\mathbb{V} = (V_\lambda)$ it is said to be *sub-markovian* if $\lambda V_\lambda 1 \leq 1, \forall \lambda > 0$.

The potential kernel of the resolvent $\mathbb{V} = (V_\lambda)_{\lambda > 0}$ is the kernel on $X$ defined by $V_0 := \sup_{\lambda > 0} V_\lambda$. A (Borel) kernel $V$ on $X$ is said to be *proper* if $V 1_K$ is bounded for every compact subset $K$ of $X$. We say that a function $u \in \mathcal{B}^+(X)$ is $\mathbb{V}$-*supermedian* if $\sup_{\lambda > 0} \lambda V_\lambda u \leq u$. A $\mathbb{V}$-supermedian function is said to be $\mathbb{V}$-*excessive* if $\sup_{\lambda > 0} \lambda V_\lambda u = u$. We denote by $\mathbb{S}_V$ and $\mathbb{E}_V$ respectively, the set of all $\mathbb{V}$-supermedian functions and that of all $\mathbb{V}$-excessive functions associated with $\mathbb{V}$.

A semigroup $\mathbb{P}$ of Borel kernels on $X$ is a family $\mathbb{P} = (P_t)_{t > 0}$ of Borel kernels $P_t$ on $X$ such that $P_s P_t = P_{s+t}$, $\forall s, t > 0$. The set of all the $\mathbb{P}$-excessive functions associated with $\mathbb{P}$ is

$$\mathbb{E}_{\mathbb{P}} := \{u \in \mathcal{B}^+(X) : \sup_{t > 0} P_t u = u\}.$$

If $\mathbb{P} = (P_t)$ is a semigroup of Borel kernels on $X$, the family $\mathbb{V} = (V_\lambda)$ of kernels defined by

$$V_\lambda f(x) = \int_0^{+\infty} e^{-\lambda t} P_t f(x) dt$$

for any $\lambda > 0$, any function $f \in \mathcal{B}^+(X)$ on $X$ and any $x \in X$, is a resolvent, called *resolvent associated with the semigroup* $\mathbb{P}$.

If $\mathbb{V}$ is the resolvent associated with the semigroup of kernels $\mathbb{P} = (P_t)$, then $\mathbb{E}_\mathbb{P} = \mathbb{E}_\mathbb{V}$ and, in this case, the potential kernel $V_0$ of $\mathbb{V}$ is given by $V_0 f(x) = \int_0^{+\infty} P_t f(x) dt$, for any $f \in \mathcal{B}^+(X)$.

Given a sub-markovian resolvent $\mathbb{V} = (V_\lambda)_{\lambda>0}$, resp. a sub-markovian semigroup $\mathbb{P} = (P_t)_{t>0}$, on $X$ such that $\lim_{\lambda\to 0} \lambda V_\lambda f = f$, resp. $\lim_{t\to 0} P_t f = f$, for any $f \in \mathcal{K}^+(X)$ and that there exists functions $u, v \in \mathbb{E}_\mathbb{V}$, resp. $u, v \in \mathbb{E}_\mathbb{P}$, such that $u, v > 0$ and $u/v \in \mathcal{C}_0(X)$, then $(X, \mathbb{E}_\mathbb{V})$, resp. $(X, \mathbb{E}_\mathbb{P})$, is a balayage space if $\mathbb{E}_\mathbb{V}$, resp. $\mathbb{E}_\mathbb{P}$, separates the points of $X$ or if the potential kernel $V_0$ of $\mathbb{V}$, resp. $\mathbb{P}$, is proper ([**4**, Chap. II, Sections 7 et 8]).

Conversely, if $p$ is a bounded continuous potential in a balayage space $(X, \mathcal{W})$ satisfying $1 \in \mathcal{W}$ (which is not really a restriction since we can always be reduced to this case by considering the $f$-hyperharmonic functions, that is, functions of the form $u/f$, $u \in \mathcal{W}$, where $f$ is a fixed continuous and finite function $> 0$ on $X$), then there exists a unique sub-markovian resolvent $\mathbb{V}$, resp. a sub-markovian semigroup $\mathbb{P}$, such that $\mathbb{E}_\mathbb{V} \subset \mathcal{W} \subset \mathbb{S}_\mathbb{V}$, resp. $\mathbb{E}_\mathbb{P} \subset \mathcal{W} \subset \mathbb{S}_\mathbb{P}$, and $V_0 1 = p$, where $V_0$ is the potential kernel associated with $\mathbb{V}$ . Moreover $\mathbb{E}_\mathbb{V} = \mathcal{W}$, resp. $\mathbb{E}_\mathbb{P} = \mathcal{W}$, if and only if $p$ is strict [**4**, Chap. II, Theor. 7.8 and Theor. 8.6].

Let $\mathbb{V} = (V_\lambda)$ be a resolvent family of Borel kernels on $X$. We say that a function $f \in \mathcal{B}(X)$ is finite $\mathbb{V}$-a.e. if the function $V_0(1_{\{f=\pm\infty\}})$ is identically zero. In the following, we denote by $\mathcal{E}_\mathbb{V}$ the cone of all nonnegative excessive functions of $\mathbb{V}$ which are finite $\mathbb{V}$-a.e.

Examples 2.3. 1. Let $\mathbb{P} = (P_t)_{t>0}$ be the brownian semigroup on $\mathbb{R}^n$, $n \geq 3$, defined by:

$$\mathbb{P}_t f(x) = \int g_t(x-y) f(y) \lambda^n(dy), \ \ \forall f \in \mathcal{B}^+(\mathbb{R}^n),$$

where $g_t(x) := (\frac{1}{2\pi t})^{1/2} \exp(-||x||^2/2t)$, $x \in \mathbb{R}^n$, and where $\lambda^n$ is the Lebesgue measure on $\mathbb{R}^n$. Let $\mathcal{W} = \mathbb{E}_\mathbb{P}$, then $(\mathbb{R}^n, \mathcal{W})$ is a balayage space. Moreover, every hyperharmonic function $p \in \mathcal{C}_0^+(\mathbb{R}^n)$ is a potential.

2. Consider the semigroup $\mathbb{T} = (T_t)_{t>0}$ on $\mathbb{R}$ defined by: $\mathbb{T}_t(x, \centerdot) := \varepsilon_{x-t}$, $t > 0$, $x \in \mathbb{R}$, ($\mathbb{T}$ is called a *translation* on $\mathbb{R}$). Then $(\mathbb{R}, \mathcal{W})$ is a balayage space

with $\mathcal{W} = \mathbb{E}_{\mathbb{T}}$. Furthermore, any increasing function $p \in \mathcal{C}^+(\mathbb{R})$ is a potential if and only if $\lim_{x\to-\infty} p(x) = 0$, (i.e. $\mathcal{P}$ is the set of all functions $x \mapsto \mu(]-\infty, x])$ where $\mu \in \mathcal{M}^+(\mathbb{R})$ is non atomic such that $\mu(]-\infty, 0]) < +\infty$).

**2.3 -** *Balayage spaces and harmonic kernels.*

Let $(X, \mathcal{W})$ be a balayage space and $\mathcal{U}$ a base of relatively compact open subsets of $X$.

Let $(H_U)_{U\in\mathcal{U}}$ be a family of sweeping kernels on $X$, that is, for any $U \in \mathcal{U}$, we have $H_U(x, U) = 0$ for every $x \in U$ and $H_U(x, \cdot) = \varepsilon_x$ for every $x \in X \setminus U$.

For any open subset $U$ of $X$, we define the set of all nonnegative Borel measurable functions which are hyperharmonic, resp. superharmonic, resp. harmonic, on $U$ by

$$\begin{aligned}
\mathcal{H}^*_+(U) &:= \{u \in \mathcal{B}^+(X) : u_{|U} \text{ is l.s.c, } H_V u \le u\ \forall V \in \mathcal{U},\ \overline{V} \subset U\},\\
\mathcal{S}^+(U) &:= \{s \in \mathcal{H}^*_+(U) : H_V s \text{ is continuous on every open set } V \in \mathcal{U},\ \overline{V} \subset U\},\\
\mathcal{H}^+(U) &:= \{h \in \mathcal{S}^+(U) : H_V h = h \text{ for every open set } V \in \mathcal{U},\ \overline{V} \subset U\}\\
&= \{h \in \mathcal{B}^+(X) : h_{|U} \in \mathcal{C}(U),\ H_V h = h,\ \forall V \in \mathcal{U},\quad \overline{V} \subset U\}.
\end{aligned}$$

The family of kernels $(H_U)_{U\in\mathcal{U}}$ will be called *harmonic* if:

(a) $\forall x \in X$, $\lim_{U\in\mathcal{U}, x\in U} H_U(x, \cdot) = \varepsilon_x$, or $R_1^{\{x\}}$ is l.s.c. at $x$.

(b) $H_V H_U = H_U$, for every open subset $V \subset \overline{V} \subset U$.

(c) If $f \in \mathcal{B}_b(X)$ is zero outside a compact set, then $H_U f$ is continuous and bounded on $U$.

(d) $\forall x \in U$, there exists $w \in \mathcal{H}^*_+(X)$ such that $w(x) < \infty$ and $w(x_n) \to +\infty$ for every purely irregular sequence $(x_n)$ in $U$ (that is, $(x_n)$ has a limit $z \in \partial U$ and there is no subsequence $(x_{\varphi(n)})$ of $(x_n)$ satisfying: $\lim_{n\to+\infty} H_U(x_{\varphi(n)}, .) = \varepsilon_z$).

(e) $\mathcal{H}^*_+(X)$ is linearly separating and there exists a function $s \in \mathcal{H}^*_+(X) \cap \mathcal{C}(X)$, $s > 0$, such that for every open subset $V \subset \overline{V} \subset U$, $V \in \mathcal{U}$, the function $H_V s$ is continuous on $V$.

Let $U$ be an open subset of $X$ and $H_U$ the kernel defined by $H_U(x, \cdot) := \overset{\circ}{\varepsilon}{}_x^{X\setminus U}$ (cf. [**4**, Chap. II, Cor. 5.4, p. 67]) which coincides with $\varepsilon_x^{X\setminus U}$ if $x \in U$, and $H_U(x, \cdot) := \varepsilon_x$, if $x \in X \setminus U$. Then the family of kernels $(H_U)_{U\in\mathcal{U}}$ is harmonic

([**4**, Chap. III, Theor. 2.8, p. 101]), and we have $\mathcal{W} = \mathcal{H}^*_+(X)$. Furthermore, a function $p \in \mathcal{W} \cap C(X)$ is a potential if and only if $\inf\{R_p^{X\setminus K} : K \text{ compact} \subset X\} = 0$, if and only if, $(\forall h \in \mathcal{H}^+(X),\ 0 \le h \le p \Rightarrow h = 0)$.

Conversely, if $(X, \mathcal{H}^*_+(X))$ is a balayage space, then $H_U(x, \centerdot) := \overset{\circ}{\varepsilon}{}_x^{X\setminus U} = \varepsilon_x^{X\setminus U}$ for any $U \in \mathcal{U}$ and for any $x \in U$ ([**4**, Chap. II, Theor. 7.8 et Theor. 8.6]).

Remark 2.4. A harmonic space (in the sense of Constintinescu and Cornea [**11**]) is a balayage space. However, a balayage space $(X, \mathcal{W})$ is not in general a harmonic space, a counter-example is given in [**14**, p. 78]. If $\mathcal{W} = \mathcal{H}^*_+(X)$, then $(X, \mathcal{W})$ is a harmonic space if and only if $X$ has no finely isolated points and if $\mathcal{W}$ has the local truncation property (that is, for any $u, v \in \mathcal{W}$ such that $u \ge v$ on $\partial U$, the function $w$ defined by: $w = u \wedge v$ on $U$ and $w = u$ on $X \smallsetminus U$ belongs to $\mathcal{W}$ [**4**, Chap. III, p. 130]), if and only if, for any $U \in \mathcal{U}$, and for any $x \in U$, the measure $H_U(x, \centerdot)$ is supported by the boundary $\partial U$ of $U$ (i.e., $H_U(x, X \smallsetminus \partial U) = 0$ for any $U \in \mathcal{U}$ and any $x \in U$) and, for any $x \in X$, there exists $U \in \mathcal{U}$ such that $x \in U$ and $H_U(x, \partial U) > 0$ ([**4**, Chap. III, Theor. 8.5, p. 130]).

## 3 - Integral representation in the cone $\overline{\mathcal{S}}(X)$

In the following $(X, \mathcal{W})$ is a balayage space. According to [**4**, Remark 7.9, p. 81] we may assume that $1 \in \mathcal{W}$. Let $p$ be a bounded and strict potential on $X$, $\mathbb{V} = (V_\lambda)$ the resolvent associated with $p$ (cf. 2.2) and $\mathcal{E}_{\mathbb{V}}$ the cone of all excessive functions of $\mathbb{V}$ which are finite $\mathbb{V}$-a.e., that is outside of a set of potential zero (the potential of $A \in \mathcal{B}(X)$ is the function $V_0(1_A)$ where $V_0$ is the potential kernel of $\mathbb{V}$, a subset $A$ of $X$ is of potential zero if, for any $x \in X$, $A$ is contained in a set $B \in \mathcal{B}(X)$ such that $V_0(x, B) = 0$). We also recall that the resolvent $\mathbb{V}$ is said to be basic if it is absolutely continuous with respect to a $\sigma$-finite measure $\tau$ on $(X, \mathcal{B}(X))$ (we will then say that $\mathbb{V}$ is of base $\tau$). We shall say that a property $P(x), x \in X$, holds $\mathbb{V}$-almost everywhere ($\mathbb{V}$-a.e. in abbreviated form) if it holds for all $x$ in the complement of a set of potential zero.

Let $f$ be a $\mathbb{V}$-supermedian function. We denote $\widetilde{f}$ the excessive regularized function of $f$, that is, the excessive function of $\mathbb{V}$ defined by $\widetilde{f} = \sup_{\lambda>0} \lambda V_\lambda f$. It is the greatest excessive minorant of $f$. In [**5**] and [**12**], this function is denoted $\widehat{f}$ which means in our notations the l.s.c. regularized of $f$, but one can show that it is the same. Indeed, the function $\widetilde{f}$ is l.s.c. (as any $\mathbb{V}$-excessive function) and we have $\widetilde{f} \le f$, then $\widetilde{f} \le \widehat{f} \le f$. The set $A = \{\widetilde{f} < \widehat{f}\}$ is of potential zero according to [**12**, Théor. 12, p. 8]. Since $A$ is a finely open subset of $X$, we deduce that it is empty because, since $p$ is strict, the nonempty finely open subsets of $x$ are of nonzero potentials. Then $\widetilde{f} = \widehat{f}$.

The excessive functions of the resolvent $\mathbb{V}$ are l.s.c., then the resolvent $\mathbb{V}$ is absolutely continuous with respect to a nonnegative Borel measure $\sigma$-finite $\tau$ on $X$ according to [**12**, no 41, p. 25]. Replacing $\tau$ by an equivalent measure, we can suppose that it is finite (and then $\tau \in \mathcal{M}^+(X)$). We deduce from [**5**, Example 3, p. 113], that the cone $\mathcal{E}_{\mathbb{V}}$ of excessive functions which are finite $\mathbb{V}$-a.e. is a standard $H$-cone of functions. We then endow $\mathcal{E}_{\mathbb{V}}$ with the natural topology in the sense of Boboc, Bucur and Cornea ([**5**, Chap. IV, p. 141]). This topology is induced on $\mathcal{E}_{\mathbb{V}}$ by that one of a locally convex topological vector space (l.c.s in abbreviated form) $E$ in which $\mathcal{E}_{\mathbb{V}}$ is a convex cone of vertex 0. Let us recall here that this topology is metrizable and if a filter $\mathcal{F}$ on $\mathcal{E}_{\mathbb{V}}$ converges to $u$, then $u = \sup_{M\in\mathcal{F}} \widehat{\inf}_{v\in M} v$ ([**5**, Theor. 4.5.2, p. 143]). In particular, if a sequence $(s_n) \subset \mathcal{E}_{\mathbb{V}}$ is a convergent in $\mathcal{E}_{\mathbb{V}}$ with respect to the natural topology (we shall say naturally convergent), then we have $\lim s_n = \lim \widehat{\inf} s_n$ with $\sup \widehat{\inf} s_n := \sup_{n\in\mathbb{N}} \widehat{\inf}_{p\geq n} s_p$.

We denote $\mathcal{S}^+(X)$ the subcone of $\mathcal{E}_{\mathbb{V}}$ formed by the nonnegative superharmonic functions on $X$. It is clear that $\mathcal{S}^+(X)$ is an $H$-cone of functions in the sense of [**5**].

Proposition 3.1. *Let $\mathbb{V} = (V_\lambda)$ be a basic resolvent whose cone of excessive functions is equal to $\mathcal{W}$. Then the cone $\mathcal{E}_{\mathbb{V}}$ of excessive functions which are finite $\mathbb{V}$-a.e. does not depend on the resolvent $\mathbb{V}$.*

Proof. Let $\mathbb{V}_1$ and $\mathbb{V}_2$ are two basic resolvents whose cones of excessive functions are equal to $\mathcal{W}$. Let $s$ be an element of $\mathcal{E}_{\mathbb{V}_1}$ and let $A = \{s = +\infty\}$. Then $\widehat{R}_s^A = 0$ and $R_s^A = +\infty$ on $A$, hence $A \subset \{\widehat{R}_s^A < R_s^A\}$, so $A$ is of $\mathbb{V}_2$-potential zero according to [**12**, Théor. 12, p. 8] and therefore $s \in \mathcal{E}_{\mathbb{V}_2}$. We deduce that $\mathcal{E}_{\mathbb{V}_1} \subset \mathcal{E}_{\mathbb{V}_2}$. By exchanging $\mathbb{V}_1$ and $\mathbb{V}_2$, we obtain the inverse inclusion. $\square$

The cone $\mathcal{E}_{\mathbb{V}}$, which does not depend on the resolvent $\mathbb{V}$, will be denoted simply by $\overline{\mathcal{S}}(X)$.

Remark 3.2. If $(X, \mathcal{W})$ is a harmonic space, then an excessive function of the resolvent $\mathbb{V}$ is finite $\mathbb{V}$-a.e. if and only if it is superharmonic. Hence we have $\overline{\mathcal{S}}(X) = \mathcal{S}^+(X)$.

Remark 3.3. The (natural) topology of $\overline{\mathcal{S}}(X)$ is independent of the resolvent $\mathbb{V}$.

Proposition 3.4. *For any point $x \in X$ and any subset $A$ of $X$, the functions $s \mapsto s(x)$ and $s \mapsto \widehat{R}_s^A(x)$, with values in $\overline{\mathbb{R}}_+$, are l.s.c. affine functions on $\overline{\mathcal{S}}(X)$.*

P r o o f. For any $x \in X$ and for any $A \subset X$, the function $s \mapsto \widehat{R}_s^A(x)$ is affine (cf. subsection 2.1). Let $(s_n)$ be a sequence in $\overline{\mathcal{S}}(X)$ which converges (in the natural topology) to $s \in \overline{\mathcal{S}}(X)$. We have $s = \lim_n \widehat{\inf} s_n = \sup_n \widehat{\inf}_{k \geq n} s_k$ and then $\widehat{R}_s^A(x) = \widehat{R}^A_{\sup_n \widehat{\inf}_{k\geq n} s_k}(x) = \sup_n \widehat{R}^A_{\widehat{\inf}_{k\geq n} s_k}(x) \leq \sup_n \widehat{\inf}_{k\geq n} \widehat{R}^A_{s_k}(x) \leq \sup_n \inf_{k\geq n} \widehat{R}^A_{s_k}(x) = \liminf_n \widehat{R}^A_{s_n}(x)$ (the first equality follows from properties of the reduite of functions and the second inequality follows from the fact that for any $n$ one has $\widehat{R}^A_{\widehat{\inf}_{k\geq n} s_k}(x) \leq \widehat{R}^A_{s_n}$). We deduce that the function $s \mapsto \widehat{R}_s^A(x)$ is l.s.c. on $\overline{\mathcal{S}}(X)$. For the function $s \mapsto s(x)$, it suffices to apply the above result for $A = X$. $\square$

P r o p o s i t i o n 3.5. *Let $U$ be a relatively compact open subset of $X$ and $x \in U$. Then the function $s \mapsto H_U(s)(x)$ from $\overline{\mathcal{S}}(X)$ into $\overline{\mathbb{R}}_+$ is l.s.c.*

P r o o f. Let $(s_n) \subset \overline{\mathcal{S}}(X)$ be a convergent sequence to $s \in \overline{\mathcal{S}}(X)$, we have $s = \lim \widehat{\inf} s_n = \sup_m \widehat{\inf}_{n\geq m} s_n$. Then by Fatou's lemma we have $H_U(s)(x) = H_U(\lim \widehat{\inf} s_n)(x) \leq H_U(\liminf s_n)(x) \leq \liminf H_U(s_n)(x)$, which clearly proves that the function $s \mapsto H_U(s)(x)$ is l.s.c. on $\overline{\mathcal{S}}(X)$. $\square$

Let $S$ be a convex cone of a locally convex topological vector space. We will say by abuse of language that an element of $S$ is extreme if it belongs to an extremal ray of $S$. We denote $\mathrm{Ext}(S)$ the union of extremal rays of $S$. We say that a subset $C$ of $S$ is a *cap* of $S$ if $C$ is a convex compact containing 0 whose complementary (in $S$) is convex. The cone $S$ is said to be *well capped* if $S$ is the union of its caps. The set of the extreme points of $C$ is denoted by $\mathrm{Ext}(C)$.

According to [**5**, Cor. 4.2.5, p. 107 and Theor. 4.5.8, p. 147], the cone $\overline{\mathcal{S}}(X)$ is well capped.

The natural topology of $\overline{\mathcal{S}}(X)$ is metrizable by [**5**, Sect. 4.5, p. 141], then if $C$ is a cap of $\overline{\mathcal{S}}(X)$, the set $\mathrm{Ext}(C)$ of extreme points of $C$ is by [**8**, Cor. 27.3] a $G_\delta$ subset of $C$. Furthermore, the cone $\overline{\mathcal{S}}(X) = \mathcal{E}_{\mathbb{V}}$ is a lattice in its own order (the specific order) according to [**5**, Example 5, p. 38, and Theor. 2.1.5, p. 41].

Let $C$ be a cap of $\overline{\mathcal{S}}(X)$ and $\mu$ be a nonnegative Radon measure on $C$. For any point $x \in X$ and for any subset $A$ of $X$, it follows by Proposition 3.9 that the integrals $\int p(x)d\mu(p)$ and $\int \widehat{R}_p^A(x)d\mu(p)$ are well defined and the functions $x \mapsto \int p(x)d\mu(p)$ and $x \mapsto \int \widehat{R}_p^A(x)d\mu(p)$ on $X$ belong to $\overline{\mathcal{S}}(X)$. We denote them simply by $\int p d\mu(p)$ and $\int \widehat{R}_p^A d\mu(p)$ respectively.

T h e o r e m 3.6. *Let $C$ be a cap of $\overline{\mathcal{S}}(X)$. Then for any function $s \in C$,* *there exists a unique Radon measure* $\mu \geq 0$ *on* $C$ *supported by* $\mathrm{Ext}(C)$ *such that*

$$s = \int u d\mu(u).$$

Proof. According to Choquet's integral representation theorem [**8**, Theor. 30.20]), $s$ is the barycenter of a unique measure $\mu$ on $C$ supported by $\mathrm{Ext}(C)$. Let $x \in X$, the function $f : u \mapsto u(x)$ is affine and l.s.c. (taking values in $[0, +\infty]$) on $C$, then by [**1**, Cor. I.1.4] we can find an increasing sequence $(l_n)$ of continuous affine forms on $C$ such that $f = \sup l_n$, and hence, according to the monotone convergence theorem, we have $f(s) = \sup_n l_n(s) = \sup_n \int_C l_n(u) d\mu(u) = \int_C f(u) d\mu(u)$, and then $s(x) = \int_C u(x) d\mu(u)$. □

The measure $\mu$ associated with $s$ in the precedent theorem will be called the *maximal measure* on $C$ representing $s$.

Proposition 3.7. *Let* $C$ *be a cap of* $\overline{\mathcal{S}}(X)$, $\mu$ *a positive (Radon) measure on* $C$ *and* $u = \int_C q d\mu(q)$. *Then, for any subset* $A$ *of* $X$, *we have* $\widehat{R}_u^A = \int \widehat{R}_q^A(x) d\mu(q)$.

Proof. The case where $\mu = 0$ being trivial, let us suppose that $\mu \neq 0$. The measure $\mu$ is necessarily finite, we can then suppose that it is a probability measure on $C$. For any $x \in X$, the function $g : q \mapsto \widehat{R}_q^A(x)$ (with values in $[0, +\infty]$) is affine and l.s.c. on $C$, we can find as in the proof of the precedent theorem an increasing sequence $(l_n)$ of continuous affine forms on $C$ such that $g = \sup_n l_n$. Then we have $\widehat{R}_u^A(x) = \sup_n l_n(u) = \sup_n \int l_n(q) d\mu(q) = \int \widehat{R}_q^A(x) d\mu(q)$, and, since $x$ is an arbitrary point in $X$, it follows that $\widehat{R}_u^A = \int \widehat{R}_q^A d\mu(q)$. □

Proposition 3.8. *Let* $s \in \mathrm{Ext}(\overline{\mathcal{S}}(X))$. *For any subset* $A$ *of* $X$ *we have* $\widehat{R}_s^A = s$ *or* $\widehat{R}_s^{X \setminus A} = s$.

Proof. Suppose that $s \neq \widehat{R}_s^A$ (then in particular $s > 0$ on a set of nonempty fine interior), and denote $u$ the function defined on $X$ by $u(x) = s(x) - \widehat{R}_s^A(x)$ if $\widehat{R}_s^A(x) < +\infty$ and $u(x) = +\infty$ otherwise. Then, according to [**5**, Prop. 1.1.6, p. 14]), we have $\widehat{R}_u \prec s$ ($\prec$ designed the specific order in $\overline{\mathcal{S}}(X)$). Since $s \in \mathrm{Ext}(\overline{\mathcal{S}}(X))$ and $\widehat{R}_u \neq 0$, there exists $\alpha > 0$ such that, $s = \alpha \widehat{R}_u$, and, since $u = 0$ on a finely dense subset in $A$, we have $\widehat{R}_u = \widehat{R}_u^{X \setminus A}$, and consequently $\widehat{R}_u = \widehat{R}_{\widehat{R}_u}^{X \setminus A}$, so that $s = \widehat{R}_s^{X \setminus A}$. □

Proposition 3.9. *Let* $C$ *be a cap of the cone* $\overline{\mathcal{S}}(X)$ *and* $A \subset X$. *Then the set* $\mathrm{Ext}_A(C) = \{s \in \mathrm{Ext}(C) : \widehat{R}_s^A = s\}$ *is Borel measurable subset of* $C$.

P r o o f. Let $\tau$ be a $\sigma$-finite measure on $(X, \mathcal{B}(X))$ such that the resolvent $\mathbb{V}$ is of base $\tau$ (see the beginning of the present section). Replacing the measure $\tau$ by an equivalent measure, we may suppose that the constant functions are $\tau$-integrable. As two superharmonic functions which are equal $\tau$-a.e. are necessarily equal everywhere, we see that $\mathrm{Ext}_A(C) = \bigcap_n C_n$, where for every integer $n$, $C_n = \{s \in \mathrm{Ext}(C) : \int s \wedge n d\tau = \int \widehat{R}_s^A \wedge n d\tau\}$. For any integer $n$, the set $C_n$ is Borel measurable subset of $C$ because the functions $s \mapsto \int s \wedge n d\mu$ and $s \mapsto \widehat{R}_s^A \wedge n d\mu$ are l.s.c. on $C$ (which easily follows from the Fatou lemma). On the other hand, $\mathrm{Ext}(C)$ is a Borel subset (a $G_\delta$) of $C$, then $\mathrm{Ext}_A(C)$ is a Borel measurable subset of $C$. $\square$

As an application of the integral representation in $\overline{\mathcal{S}}(X)$ we shall prove a Brelot decomposition type (see [**7**]) in $\overline{\mathcal{S}}(X)$, namely: for any subset $A$ of $X$, every element $s$ of $\overline{\mathcal{S}}(X)$ has a decomposition $s = s_1 + s_2$, where $\widehat{R}_{s_1}^A = s_1$ and $\widehat{R}_{s_2}^{X \setminus A} = s_2$ (with uniqueness of the decomposition if we take $u_2$ the greatest specific minorant $v$ of $u$ such that $R_v^{X \smallsetminus A} = v$). Any element $s$ of $\mathcal{S}^+(X)$ has this property, with $s_1, s_2 \in \mathcal{S}^+(X)$.

A function $s \in \overline{\mathcal{S}}(X)$ will be called *autoreduite* on a subset $A$ of $X$ if $\widehat{R}_s^A = s$. It is clear that $s \in \overline{\mathcal{S}}(X)$ is autoreduite on $A$ if and only if $R_s^A = s$.

T h e o r e m 3.10. *Let $s \in \overline{\mathcal{S}}(X)$ and $A \subset X$. Then there exists a decomposition $s = s_1 + s_2$ of $s$ in $\overline{\mathcal{S}}(X)$ such that $s_1$ is autoreduite on $A$ and $s_2$ is autoreduite on $X \smallsetminus A$.*

P r o o f. Let $s \in \overline{\mathcal{S}}(X)$, $C$ a cap of $\overline{\mathcal{S}}(X)$ containing $s$ and $\mu$ be the maximal measure on $C$ representing $s$. Putting $s_1 = \int_{\mathrm{Ext}_A(C)} u d\mu(u)$ and $s_2 = \int_{\mathrm{Ext}_{X \smallsetminus A}(C)} u d\mu(u)$, we have $s = \int_{\mathrm{Ext}(C)} u d\mu(u) = \int_{\mathrm{Ext}_A(C)} u d\mu(u) + \int_{\mathrm{Ext}(C) \smallsetminus \mathrm{Ext}_A(C)} u d\mu(u) = s_1 + s_2$ and, according to Proposition 4.5,

$$\widehat{R}_{s_1}^A = \int\limits_{\mathrm{Ext}_A(C)} \widehat{R}_u^A d\mu(u) = \int\limits_{\mathrm{Ext}_A(C)} u d\mu(u) = s_1, \text{ and}$$

$$\widehat{R}_{s_2}^{X \smallsetminus A} = \int\limits_{\mathrm{Ext}(C) \smallsetminus \mathrm{Ext}_A(C)} \widehat{R}_u^{X \smallsetminus A} d\mu(u) = \int\limits_{\mathrm{Ext}(C) \smallsetminus \mathrm{Ext}_A(C)} u d\mu(u) = s_2.$$

$\square$

R e m a r k 3.11. We have the uniqueness of the decomposition of $s \in \overline{\mathcal{S}}(X)$ in the previous theorem if we impose to $s_2$ (respectively $s_1$) to be the specific greatest minorant of $s$ which is autoreduite on $X \smallsetminus A$ (respectively on $A$).

According to [**4**, Cor. 4.5.1, p. 108], the union of all open subsets $U$ of $X$ where a nonnegative superharmonic function ($u \in \mathcal{S}^+(X)$) is harmonic is the greatest open subset of $X$ in which $u$ is harmonic; its complementary $S(u)$ is called the *harmonic support* of $u$. It is also the smallest closed set in $X$ in the complement of which $u$ is harmonic.

As an application of the above decomposition, we have the following result:

Theorem 3.12. *Every extreme potential $p$ of $\mathcal{S}^+(X)$, $p \not\equiv 0$, is of harmonic support reduced to a single point.*

Proof. Let $p$ be an extreme potential not identically zero in $X$. Suppose that $S(p)$ contains two points $y_1$ and $y_2$ such that $y_1 \neq y_2$. Then according to Theorem 3.10, applied to $p$ and to a relatively compact open subset $V \subset X$, such that $y_1 \in V$ and $y_2 \in X \smallsetminus \overline{V}$, we have $p = p_1 + p_2$ in $X$, where $p_1, p_2 \in \mathcal{S}^+(X)$ are such that $p_1 = \widehat{R}_{p_1}^V$ and $p_2 = \widehat{R}_{p_2}^{X \smallsetminus V}$. It follows from [**4**, Prop. 2.3, p. 345 and Cor. 2.8, p. 347] that $p_1$, resp. $p_2$, is harmonic on an open neighborhood of $y_2$, resp. $y_1$, and then $p_1$ and $p_2$ are two nonproportional potentials. This contradicts the fact that $p$ is an extreme potential. Hence the support of $p$ is necessarily reduced to a single point. □

Let $C$ be a cap of $\overline{\mathcal{S}}(X)$. We denote $\mathrm{Ext}_p(C)$ and $\mathrm{Ext}_h(C)$ respectively the set of extreme potentials and that of extreme harmonic functions of $C$. Then we have $\mathrm{Ext}_h(C) = \mathrm{Ext}(C) \cap \mathcal{H}^+(X)$ and $\mathrm{Ext}_p(C) = (\mathrm{Ext}(C) \cap \mathcal{S}^+(X) \smallsetminus \mathrm{Ext}_h(C)) \cup \{0\}$. We shall show that $\mathrm{Ext}_h(C)$ and $\mathrm{Ext}_p(C)$ are Borel measurable subsets of $C$.

Proposition 3.13. *The sets $\mathcal{S}^+(X)$ and $\mathcal{H}^+(X)$ are Borel measurable subsets of $\overline{\mathcal{S}}(X)$.*

Proof. Let $(U_n)$ be an increasing sequence of relatively compact open subsets of $X$ such that $\overline{U}_n \subset U_{n+1}$ for any $n$ and $\bigcup_n U_n = X$. For any $n \in \mathbb{N}$, consider a sequence $(x_n^j)$ of points of $U_n$ which is dense in $U_n$. Then, a function $s \in \overline{\mathcal{S}}(X)$ belongs to $\mathcal{S}^+(X)$ if and only if for any integer $n$ there exists a constant $k \in \mathbb{N}$ such that $H_{U_{n+1}}(x_n^j) \leq k$ for any $j$. We deduce that

$$\mathcal{S}^+(X) = \bigcap_n \bigcup_k \bigcap_j \{s \in \overline{\mathcal{S}}(X) : H_{U_{n+1}}(s)(x_n^j) \leq k\}.$$

As well, a function $s \in \overline{\mathcal{S}}(X)$ belongs to $\mathcal{H}^+(X)$ if and only if $s \in \mathcal{S}^+(X)$ and if for any integer $n$ we have $H_{U_n}(s)(x_n^j) = s(x_n^j)$ for any $j$, and then

$$\mathcal{H}^+(X) = \bigcap_n \bigcap_j \{s \in \overline{\mathcal{S}}(X) : H_{U_n}(s)(x_n^j) = s(x_n^j)\} \cap \mathcal{S}^+(X).$$

The sets $\{s \in \overline{\mathcal{S}}(X) : H_{U_{n+1}}(s)(x_n^j) \le k\}$, $\{s \in \mathcal{S}^+(X) : H_n(s)(x_n^j) = s(x_n^j)\}$, $j,k,n \in \mathbb{N}$, are Borel measurable subsets of $\overline{\mathcal{S}}(X)$ according to Proposition 3.5. It follows that $\mathcal{S}^+(X)$ and $\mathcal{H}^+(X)$ are Borel measurable subsets of $\overline{\mathcal{S}}(X)$. $\square$

Corollary 3.14. *Let $C$ be a cap of $\overline{\mathcal{S}}(X)$. Then $\mathrm{Ext}_h(C)$ and $\mathrm{Ext}_p(C)$ are Borel measurable subsets of $C$.*

Proof. Indeed, we have $\mathrm{Ext}_h(C) = \mathrm{Ext}(C) \cap \mathcal{H}^+(X)$ and $\mathrm{Ext}(C)$ is a $G_\delta$ of $C$ ($C$ being a metrizable compact convex set), then $\mathrm{Ext}_h(C)$ is a Borel subset of $C$. We deduce that $\mathrm{Ext}_p(C) = (\mathrm{Ext}(C) \cap \mathcal{S}^+(X) \smallsetminus \mathrm{Ext}_h(C)) \cup \{0\}$ is a Borel subset of $C$. $\square$

Theorem 3.15. *Suppose that $\overline{\mathcal{S}}(X) = \mathcal{S}^+(X)$ (which is the case if $(X, \mathcal{W})$ is a harmonic space) and let $C$ be a cap of $\overline{\mathcal{S}}(X)$ and $s \in C$ admitting the integral representation $s = \int u d\mu(u)$, where $\mu$ is the maximal measure on $C$ representing $s$. If $s$ is an harmonic function (resp. a potential), then the measure $\mu$ is supported by $\mathrm{Ext}_h(C)$ (resp. $\mathrm{Ext}_p(C)$).*

Proof. The measure $\mu$ is supported by $\mathrm{Ext}(C) = \mathrm{Ext}_h(C) \cup \mathrm{Ext}_p(C)$, and the sets $\mathrm{Ext}_h(C)$ and $\mathrm{Ext}_p(C)$ are Borel measurable subsets of $C$ according to the precedent corollary, hence $s = \int u d\mu(u) = \int_{\mathrm{Ext}_p(C)} u d\mu(u) + \int_{\mathrm{Ext}_h(C)} u d\mu(u)$. Suppose that $s$ is harmonic and let $(U_n)$ be an increasing sequence of relatively compact open subsets of $X$ such that $\bigcup_n U_n = X$. Then, for any $n \in \mathbb{N}$, we have

$$
\begin{aligned}
s &= \widehat{R}_s^{X \smallsetminus U_n} = \int \widehat{R}_u^{X \smallsetminus U_n} d\mu(u) \\
&= \int_{\mathrm{Ext}_p(C)} \widehat{R}_u^{X \smallsetminus U_n} d\mu(u) + \int_{\mathrm{Ext}_h(C)} \widehat{R}_u^{X \smallsetminus U_n} d\mu(u) \\
&= \int_{\mathrm{Ext}_p(C)} \widehat{R}_u^{X \smallsetminus U_n} d\mu(u) + \int_{\mathrm{Ext}_h(C)} u d\mu(u).
\end{aligned}
$$

Passing to the infimum over $n$, we obtain according to the dominated convergence theorem that $s = \int_{\mathrm{Ext}_h(C)} u d\mu(u)$ on the set $\{s < +\infty\}$ (since $\inf_n \widehat{R}_u^{X \smallsetminus U_n} = 0$ for any potential $u$), then everywhere, and by the uniqueness of the integral representation we have $\mu = 1_{\mathrm{Ext}_h(C)} \mu$, so that $\mu$ is supported by $\mathrm{Ext}_h(C)$. $\square$

Corollary 3.16. *Suppose that $\overline{\mathcal{S}}(X) = \mathcal{S}^+(X)$ and let $h \in \mathcal{H}^+(X)$ (resp. $p$ be a potential on $X$) and $C$ a cap of $\mathcal{S}^+(X)$ containing $h$ (resp. $p$), then* *there exists a unique Radon measure $\mu \geq 0$ on $C$ supported by* $\mathrm{Ext}_h(C)$ *(resp.* $\mathrm{Ext}_p(C)$*) such that $p = \int q d\mu(q)$.*

### 4 - Potentials representable by a Green kernel of $X$

A Green kernel on the balayage space $(X, \mathcal{W})$ (when it exists) is a Borel measurable function $G : X \times X \to \overline{\mathbb{R}}^+ : (x, y) \mapsto G(x, y)$ with the following properties:

($a$) $G$ is locally bounded outside of the diagonal $X \times X$.

($b$) For any $y \in X$, the function $G_y : x \mapsto G(x, y)$ is a potential of harmonic support $S(G_y) = \{y\}$.

($c$) For any $x \in X$, the function $G_x : y \mapsto G(x, y)$ is l.s.c. on $X$, and continuous on $X \setminus \{x\}$. If $x$ is a finely isolated (that is, isolated with respect to the fine topology) and not isolated, then $G(x, \cdot)$ is continuous on $X$.

If $\mu \in \mathcal{M}^+(X)$, we denote by $G\mu$ the function defined on $X$ by $G\mu := \int G(\cdot, y) d\mu(y)$. We say that a function $s \in \mathcal{S}^+(X)$ is representable by $G$ if we can find a measure $\mu \in \mathcal{M}^+(X)$ such that $s = G\mu$.

Proposition 4.1 ([**15**, Lemma 2.1]). *Let $\mu$ be a nonnegative Radon measure on $X$ such that $p = G\mu$ is a potential, then $S(G\mu) = Supp(\mu)$.*

There exists a $\sigma$-finite measure $\tau$ on $X$ such that the resolvent $\mathbb{V}$ (see the beginning of Section 3) is of base $\tau$ and that any subset of $X$ is of potential zero if and only if it is $\tau$-negligible. In fact, let $(x_n)$ be a sequence dense in $X$ and let $(\alpha_n)$ be a sequence of real numbers $> 0$ such that $\sum_n \alpha_n V_0(1)(x_n) < +\infty$, the measure $\tau = \sum_n \alpha_n V_0(x_n, \cdot)$ satisfies the required condition, $V_0$ being the potential kernel of $\mathbb{V}$.

Proposition 4.2. *Let $(h_n)$ be a sequence of functions in $\mathcal{S}^+(X)$, harmonic on an open subset $U$ of $X$ and converging $\tau$-a.e. to a function $h$ l.s.c. on $X$. Suppose that there exists $s \in \mathcal{S}^+(X)$ such that $h_n \leq s$ for any $n$, then $h$ is harmonic on $U$.*

Proof. We have $h = \sup_n \widehat{\inf}_{j \geq n} h_j \in \mathcal{W}$ and $h$ is majorized by an element of $\mathcal{S}^+(X)$, then $h \in \mathcal{S}^+(X)$. Let $V$ be an open subset of $X$ such that $V \subset \overline{V} \subset U$ and $W, W'$ two open subsets of $X$ such that $\overline{V} \subset W \subset \overline{W} \subset W' \subset \overline{W'} \subset U$. By Dini's Lemma the sequence $((h - \widehat{\inf}_{j \geq n} h_j)_{|\overline{V}})$ of u.s.c. functions on $\overline{V}$ converges uniformly to 0 on $\overline{V}$. Hence, for any $\epsilon > 0$, there exists an integer $n_1$ such that

$n \geq n_1 \Rightarrow h - h_n < \epsilon$ on $\overline{V}$. On the other hand, the function $H_{W'}(s)$ is finite and continuous on $\overline{V}$, hence there exists a nonnegative constant $c$ such that for any $n$ we have $h_n \leq H_{W'}(s) \leq c$ on $\overline{W}$. By applying the above proceeding to the sequence of the restrictions of the functions $c - h_n$ to $W$ we can find an integer $n_2$ such that $n \geq n_2 \Rightarrow h_n - h < \epsilon$ on $\overline{V}$. For any $n \geq \max(n_1, n_2)$ we have $|h - h_n| < \epsilon$ on $\overline{V}$. Hence the sequence $(h_n)$ converges to $h$ uniformly on $\overline{V}$ and therefore $H_V(h) = h$ on $V$. Since $V$ is arbitrary it follows that $h$ is harmonic on $U$. □

Corollary 4.3. *Let $(h_n)$ be a sequence of harmonic functions $\geq 0$ in $X$ which converges in $\overline{\mathcal{S}}(X)$ (in the natural topology on $\overline{\mathcal{S}}(X)$) to a function $h$. If the functions $h_n$, $n \in \mathbb{N}$, are bounded from above by a function $s \in \mathcal{S}^+(X)$, then $h$ is harmonic on $X$.*

Proof. In fact, by [**12**, Lemme 94, p. 81] we can find a subsequence $(h_{n_k})$ of $(h_n)$ which converges $\tau$-a.e. to $h$. The corollary follows from the previous proposition applied to the sequence $(h_{n_k})$. □

Lemma 4.4. *Let $K$ be a compact of $X$. Then the maps $\varphi_K : y \mapsto G(\centerdot, y)$ is an homeomorphism from $K$ on its image $A = \varphi_K(K)$ in $\overline{\mathcal{S}}(X)$.*

Proof. The map $\varphi_K$ is injective because for any $y \in X$ the function $G(\centerdot, y)$ is a potential harmonic on $X \setminus \{y\}$. It remains to prove that $\varphi$ is continuous. Let $(y_n)$ be a sequence of points in $K$ converging to $y \in K$. Let $\mathcal{U}$ an ultrafilter finer than the filter of sections of $\mathbb{N}$. Since for any $x \in X \setminus \{y\}$ the function $G(x, \centerdot)$ is continuous at $y$, we have $\lim \widehat{\inf}_{\mathcal{U}} G(\centerdot, y_n) = G(\centerdot, y)$ on $X \setminus \{y\}$. If $y$ is not isolated point of $X$ we necessarily have $\lim \widehat{\inf}_{\mathcal{U}} G(\centerdot, y_j) = G(\centerdot, y)$ on $X$. If $y$ is an isolated point of $X$, then $y_n = y$ for any $n$ large enough, and hence $\lim \widehat{\inf}_{\mathcal{U}} G(\centerdot, y_n)(y) = G(y, y)$. It follows that in any case $\lim_{\mathcal{U}} G(\centerdot, y_n) = G(\centerdot, y)$. Hence $\varphi_K$ is continuous. □

For any Borel subset $E$ of $X$ and any Borel measure $\beta$ on $E$, we also denote by $\beta$ the measure image of $\beta$ by the inclusion map from $E$ into $X$.

Proposition 4.5. *Let $K$ be a compact subset of $X$ and $(\mu_n)$ a sequence of measures in $\mathcal{M}^+(X)$ supported by $K$ and converging (weakly) to a measure $\mu \in \mathcal{M}^+(X)$ (necessarily supported $K$). Then $\lim G\mu_n = G\mu$ in $\overline{\mathcal{S}}(X)$.*

Proof. Let $\beta \in \mathcal{M}^+(X)$ such that $\beta(K) > 0$ and $G\beta \in \mathcal{S}^+(X)$ and $\epsilon$ a real $> 0$. Let us denote again by $\mu_n$, $\mu$ the restrictions of $\mu_n$, $\mu$ to $K$, $n \in \mathbb{N}$, and by $\nu_n$, $\nu$ the images of $\mu_n + \epsilon\beta$ and $\mu + \epsilon\beta$ by $\varphi_K$. Since

$\varphi_K$ is continuous, it follows that the sequence $(\nu_n)$ converges (weakly) to the (finite) measure $\nu$ on the compact subset $A = \varphi_K(K)$ of $\overline{\mathcal{S}}(X)$. The sequence of reals $|\nu_n| > 0$ converges to $|\nu| > 0$, and the sequence of barycenters $b((1/|\nu_n|)\nu_n)$ (in the convex hull $conv(A)$ of $A$ in $\overline{\mathcal{S}}(X)$) of the probability measures $(1/|\nu_n|)\nu_n$ converges to $b((1/|\nu|))\nu)$. It follows that the sequence of the functions $G(\mu_n + \epsilon\beta) = \int_A q d\nu_n(q)$ converges in $\overline{\mathcal{S}}(X)$ and that $\lim(G\mu_n + \epsilon G\beta) = \lim \int_A q d\nu_n(q) = \int_A q d\nu(q) = G\mu + \epsilon G\beta$. By letting $\epsilon \to 0$, we obtain $\lim G\mu_n = G\mu$. □

Lemma 4.6. *Let $s$ be a nonnegative superharmonic function on $X$. Then the set $\{u \in \overline{\mathcal{S}}(X) : u \leq s\}$ is compact in $\overline{\mathcal{S}}(X)$.*

Proof. The lemma follows easily from [**5**, Theor. 4.5.8]. □

Theorem 4.7. *Let $(\mu_n)$ be a sequence of measures in $\mathcal{M}^+(X)$ converging weakly to a measure $\mu \in \mathcal{M}^+(X)$ and suppose that there exists a potential $q$ such that $G\mu_n \leq q$ for any integer $n$. Then the sequence of potentials $G\mu_n$ converges to $G\mu$ in $\overline{\mathcal{S}}(X)$.*

Proof. By Fubini's theorem and the hypothesis on $G$, we have $\liminf G\mu_n \geq G\mu$ and therefore $\lim \widehat{\inf} G\mu_n \geq G\mu$. Let $(U_j)$ be an increasing sequence of relatively compact open subsets of $X$ such that $K_j = \overline{U_j} \subset U_{j+1}$ for any $j$ and $\bigcup_j U_j = X$. For each $j \in \mathbb{N}$ we have $G\mu_n = G(1_{K_j\mu_n}) + G(1_{X \setminus K_j}\mu_n)$. By using the fact that the total masses of the measures $\mu_n$ are uniformly bounded on each compact of $X$, the above lemma and the diagonal proceeding, we can find a strictly increasing sequence $(n_k)$ of integers such that $(1_{K_j}\mu_{n_k})$ converges to a measure $\lambda_j$, the sequence $(G(1_{K_j}\mu_{n_k}))$ converges in $\mathcal{S}^+(X)$ and $(G(1_{X \setminus K_j}\mu_{n_k}))$ converges in $\mathcal{S}^+(X)$ to a function $h_j$ (and hence a potential since $h_j \leq q$). By Proposition 4.5 we have $\lim \widehat{\inf} G\mu_{n_k} = G\lambda_j + h_j \leq G\mu + h_j$. Since $1_{K_j}\mu_{n_k} \leq 1_{K_{j+1}}\mu_{n_k} \leq \mu_{n_k}$ we have $\lambda_j \leq \lambda_{j+1} \leq \mu$ and hence $G\lambda_j \leq G\mu$. On the other hand the sequence $(h_j)$ is decreasing and each $h_j$ is harmonic on $U_j$ (by Corollary 4.3) and $h_j \leq G\mu$ for each $j$. It follows that the sequence $(h_j)$ converges in $\overline{\mathcal{S}}(X)$ to $h = \widehat{\inf} h_j$. According to Corollary 4.3, the function $h$ is harmonic on $X$ and majorized there by a potential, hence $h = 0$ and then $\lim \widehat{\inf} G\mu_{n_k} \leq G\mu$. It follows that $\lim \widehat{\inf} G\mu_n \leq G\mu$ and therefore $\lim \widehat{\inf} G\mu_n = G\mu$. By applying this to any convergent subsequence of $(G\mu_n)$, we deduce that the sequence $(G\mu_n)$ is convergent in $\overline{\mathcal{S}}(X)$ and its limit is equal to $G\mu$. □

The following result of representability of potentials by $G$ is bit stronger than the result of Hansen-Netuka [**15**, Theor. 1.1].

Theorem 4.8. *Let $(p_n)$ be a sequence of potentials $p_n = G\mu_n$ representable by $G$ and converging in $\mathcal{S}^+(X)$ (naturally) to $p$, where for every $n \in \mathbb{N}$, $\mu_n \in \mathcal{M}^+(X)$. Suppose that there is a potential $q$ such that $p_n \leq q$ on $X$, then $p$ is a potential representable by $G$. More precisely, the sequence $(\mu_n)$ converges weakly on $X$ to a measure $\mu \in \mathcal{M}^+(X)$ and one has $p = G\mu$.*

Proof. We have $p = \widehat{\liminf} p_n \leq q$ and hence $p$ is a potential. Now let $y \in X$. If $y$ is an isolated point of $X$, then $\mu_n(\{y\})G(y,y) \leq G\mu_n(y) \leq q(y) < +\infty$ for any $n \in \mathbb{N}$. If $y$ is not isolated in $X$ there exists a point $x_y \in X$ such that $G(x_y, y) > 0$ and $q(x_y) < +\infty$. Since the function $G(x_y, \cdot)$ is l.s.c. on $X$, there exists a compact neighborhood $V$ of $x_y$ and a real $\alpha > 0$ such that $G(x_y, z) \geq \alpha$ on $V$. Hence $\mu_n(V) \leq 1/\alpha G\mu_n(x_y) \leq 1/\alpha q(x_y)$ for any $n \in \mathbb{N}$. Since $X$ is separable, it follows that the sequence $(\mu_n)$ converges weakly to a measure $\mu \in \mathcal{M}^+(X)$. By Theorem 4.7 we have $p = G\mu$. □

Proposition 4.9. *Let $(s_n)$ be a sequence of functions in $\overline{\mathcal{S}}(X)$ converging pointwise to $s \in \overline{\mathcal{S}}(X)$. Then $(s_n)$ converges to $s$ in the natural topology on $\overline{\mathcal{S}}(X)$.*

Proof. Let $\mathcal{U}$ be an ultrafilter on $\mathbb{N}$ which is finer than the filter of sections of $\mathbb{N}$. Then we have $\widehat{\liminf}_{\mathcal{U}} \leq s$. According to Choquet's lemma [**4**, Lemma 1.8, p. 19], we have $\widehat{\liminf}_{\mathcal{U}} \in \mathcal{S}^+(X)$. On the other hand, we have $\widehat{\liminf} s_n \leq \widehat{\liminf}_{\mathcal{U}} \leq s = \liminf s_n$. Since $\widehat{\liminf} s_n = \liminf s_n$ $\mathbb{V}$-a.e., then $\widehat{\liminf}_{\mathcal{U}} = s$ $\mathbb{V}$-a.e., hence everywhere because two $\mathbb{V}$-excessive functions which are equal $\mathbb{V}$-a.e. are equal everywhere. It follows that $\mathcal{U}$ converge to $s$ according to [**5**, Theor. 4.5.8, p. 147]. Since $\mathcal{U}$ is an arbitrary ultrafilter, we deduce that the sequence $(s_n)$ converges to $s$ in the natural topology on $\overline{\mathcal{S}}(X)$. □

As a corollary of the previous theorem, we have the following result of Hansen-Netuka:

Corollary 4.10. *Let $(p_n)$ be a sequence of potentials which representable by $G$, majorized by a potential $q$ and converging pointwise to a potential $p$ outside a semi-polar set, then $p$ is representable by $G$.*

## 5 - Integral representation of the potentials and the hypothesis of uniqueness

In a $\mathcal{P}$-harmonic space $X$, the integral representation of potentials and its relationship with the hypothesis of uniqueness, also called axiom of proportionality, (that is, the potentials with the same harmonic support reduced to one point are proportional) has been studied by many authors. In [**17**], Janssen showed that if the space $X$ satisfies the hypothesis of uniqueness and a condition $(A)$ (see [**17**]), then it has a Green function and any potential is the Green potential of a positive Radon measure on $X$. When the harmonic space $X$ has a Green potential $G$, Boukricha [**6**] showed that if any continuous finite potential of compact support can be represented by the function $G$, then the hypothesis of uniqueness is satisfied. Schirmeier in [**22**] established that, if at least one strict bounded continuous potential can be represented by the function $G$, then the hypothesis of uniqueness is satisfied. El Kadiri [**13**] showed that if the space $X$ admits a Green function $G$, and if the hypothesis of uniqueness is satisfied, then any potential is representable by $G$.

In the present section we shall study the integral representation and its relationship with the hypothesis of uniqueness in a balayage space. Throughout this section, we assume that the constant function 1 is superharmonic in the space $(X, \mathcal{W})$, that the potentials on $X$ of superharmonic support reduced to a single point are proportional (the hypothesis of uniqueness or proportionality), and that there is a Green kernel $G$ on $X$. We also assume that $\mathcal{S}^+(X) = \overline{\mathcal{S}}(X)$, so that the cone $\mathcal{S}^+(X)$ is well capped (which is the case if $(X, \mathcal{W})$ is a harmonic space).

Proposition 5.1. *For any $y \in X$, the function $G(\centerdot, y)$ is an extreme potential (in the cone $\mathcal{S}^+(X)$).*

Proof. Let $p_1, p_2 \in \mathcal{S}^+(X)$ such that $G(\centerdot, y) = p_1 + p_2$, then the functions $p_1$ and $p_2$ are two potentials, harmonic on $X \setminus \{y\}$ and proportional to $G(\centerdot, y)$ according to the hypothesis of uniqueness. $\square$

Proposition 5.2. *Every extreme potential $p \not\equiv 0$ of $\mathcal{S}^+(X)$ is of the form $\alpha . G(\centerdot, y)$ with $\alpha \geq 0$.*

Proof. Let $p \not\equiv 0$ be an extreme potential of $\mathcal{S}^+(X)$. According to Theorem 3.12, $p$ is of harmonic support reduced to a single point, then proportional to $G(\centerdot, y)$, where $\{y\}$ is the harmonic support of $p$. $\square$

Let $C$ be a cap of $\mathcal{S}^+(X)$, then for any extreme potential $p \not\equiv 0$ of $C$ there exists a unique real $\alpha > 0$, such that $p = \alpha G(\centerdot, y)$, where $\{y\}$ is the harmonic support of $p$. Let us recall that the gauge $l$ of $C$ is the function on $\mathcal{S}^+(X)$ with values $\overline{\mathbb{R}}^+$ defined by $l(u) := \inf\{\lambda > 0,\ u/\lambda \in C\}$ for every $u \in \mathcal{S}^+(X)$. It follows from [**8**, p. 202] that $l$ is a l.s.c. affine form and we have $C = \{u \in \mathcal{S}^+(X) :\ l(u) \leq 1\}$. Moreover, since $C$ is compact we have $l(u) > 0$ for any $u \in C$ except for $u = 0$.

To establish the theorem of the integral representation of potentials, we still need some lemmas:

Lemma 5.3. *The function $y \mapsto G(\centerdot, y)$ from $X$ into $\mathcal{S}^+(X)$ is continuous on $X$.*

Proof. Let $\mathcal{U}$ be an ultrafilter on $X$ finer than the filter of neighborhoods of $y$, then $\lim_{z,\mathcal{U}} G(x,z) = G(x,y)$ for any $x \in X \smallsetminus \{y\}$ according to the assumptions made on $G$. Hence $\lim_{z,\mathcal{U}} G(\centerdot, z) \in \mathcal{S}^+(X)$ (the last limit is taken in the sense of natural topology). Let $h \in \mathcal{H}^+(X)$ such that $h \le \lim_{z,\mathcal{U}} G(\centerdot, z)$ on $X$ and $U$ a relatively compact open subset of $X$ containing $y$, then we have $h \le G(\centerdot, y)$ on $X \smallsetminus \{y\}$, and $h(y) = H_U(h)(y) = \int h d\epsilon_y^{X\smallsetminus U} \le \int G(\centerdot, y) d\epsilon_y^{X\smallsetminus U} \le G(y,y)$, the second inequality follows from the fact that the measure $\epsilon_y^{X\smallsetminus U}$ is supported by $X \smallsetminus U$ according to [**4**, Cor. 5.4, pp. 64-68]. We deduce that $h \le G(\centerdot, y)$ on $X$ and then $h \equiv 0$. Hence $\lim_{z,\mathcal{U}} G(\centerdot, z)$ is a potential, harmonic on $X \smallsetminus \{y\}$ following Corollary 4.3, and then $\lim_{z,\mathcal{U}} G(\centerdot, z) = \alpha G(\centerdot, y)$ for some $\alpha > 0$. It follows from the beginning of the proof that $\alpha = 1$, and hence $\lim_{z,\mathcal{U}} G(\centerdot, z) = G(\centerdot, y)$. We deduce that $G(\centerdot, z)$ converges naturally (in the sense of natural topology) to $G(\centerdot, y)$ as $z \to y$, and this completes the proof. $\square$

Lemma 5.4. *The set $A = \{y \in X : l(G(\centerdot, y)) < +\infty\}$ is a Borel measurable subset of $X$ and the mapping $\varphi$ from $\mathrm{Ext}_p(C) \smallsetminus \{0\}$ into $X$ which associates to each point $p \in \mathrm{Ext}_p(C) \smallsetminus \{0\}$ the unique point of its harmonic support, is a bijective Borel measurable function from $\mathrm{Ext}_p(C) \smallsetminus \{0\}$ into $A$, and so is its inverse.*

Proof. The mapping $\varphi$ is obviously a bijection from $\mathrm{Ext}_p(C) \smallsetminus \{0\}$ into $A$. The first part of the lemma follows from the preceding lemma and the fact that the function $l$ is l.s.c. on $\mathcal{S}^+(X)$. Let us prove the second part. Let $(V_n)$ be a sequence of relatively compact open subsets of $X$ such that $\bigcup_n V_n = X$. Note that $\mathrm{Ext}_p(C) = \bigcup_n E_n \cup \{0\}$, where $E_n = \{\alpha G(\centerdot, y) \in C : y \in \overline{V}_n \text{ and } 1/n \le \alpha \le n\} \cap \mathrm{Ext}(C)$. Let $(\alpha_j G(\centerdot, y_j))$ be a sequence of points of $E_n$ which converges to a function $\alpha G(\centerdot, y) \in E_n$. Consider a subsequence $(y_{j_k})$ of $(y_j)$ converging to a point $z \in \overline{V}_n$. By replacing the sequence $(\alpha_j)$ by a convergent subsequence, we may suppose that it is convergent to a real $\beta \in [1/n, n]$ and then the sequence $(G(\centerdot, y_{j_k}))$ is convergent to $\frac{\alpha}{\beta} G(\centerdot, y)$. Then $\frac{\alpha}{\beta} G(\centerdot, y) = \widehat{\liminf} G(\centerdot, y_{j_k}) = G(\centerdot, z)$ by the above lemma. It follows from the hypothesis of uniqueness that $y = z$ and therefore $\alpha = \beta$. Thus, we proved that any convergent subsequence of $(y_j)$ has limit $y$ and hence it follows that $(y_j)$ converges to $y$ and, consequently, $\widehat{\liminf} G(\centerdot, y_j) = G(\centerdot, y)$ according to the above lemma. Then the restriction of $\varphi$ to $E_n$ is continuous. The lemma is proved. $\square$

Lemma 5.5. *For any $y \in X$, there exists at least one point $x \in X$ such that $G(x, y) > 0$..*

Proof. The lemma follows easily from the fact that $G(\cdot, y)$ is of harmonic support the set reduced to the point $y$. $\square$

We will also need the following lemma of uniqueness:

Proposition 5.6 ([**15**, Prop. 5.2]). *If $\sigma$ and $\nu$ are two nonnegative Radon measures on $X$ such that $G_\sigma = G_\nu$, then $\sigma = \nu$.*

Now we can now prove the following result:

Theorem 5.7. *For any potential $p$ on $X$, there exists a unique Radon measure $\mu \geq 0$ on $X$ such that $p = G\mu$.*

Proof. The uniqueness of the measure $\mu$ follows from Proposition 5.6. Let us prove the existence of $\mu$. Let $C$ be a cap of $\mathcal{S}^+(X)$ containing $p$, $l$ its gauge and $\nu$ the nonnegative Radon measure on $C$ associated with $p$ in Corollary 3.16. Let $\mu$ be the (Borel) measure on $X$ with density the function $y \mapsto 1/l(G(\cdot, y))$ with respect to the image measure by the mapping $\varphi$ of Lemma 5.4 of the restriction of the measure $\nu$ to $\mathrm{Ext}_p(C) \setminus \{0\}$. Then we have $p = G\mu$. It remains to show that $\mu$ is a Radon measure on $X$, and as $X$ is with countable base, it suffices to show that it is finite on the compact subsets of $X$. Let $y \in X$, then, according to Lemma 5.5, there exists at least a point $x \in X$ such that $G(x, y) > 0$ so that the set $\{x \in X : G(x, y) > 0\}$ is a nonempty open subset of $X$, and therefore there exists a point $x' \in X$ such that $p(x') < +\infty$ and $G(x', y) > 0$ because the set $\{x \in X : p(x) < +\infty\}$ is dense in $X$. The function $z \mapsto G(x', z)$ is l.s.c. on $X$, then the set $U_y = \{z \in X : G(x', z) > 0\}$ is an open subset of $X$ containing $y$. For any $y \in X$, let $O_y$ be a relatively compact open subset of $X$ containing $y$ such that $\overline{O}_y \subset U_y$; then $X = \bigcup_{y \in X} O_y$ and for any $y \in X$, we have $\inf_{z \in O_y} G(x', z) = c(y) > 0$ and, consequently,

$$\mu(O_y) \leq \frac{1}{c(y)} \int G(x', z) d\mu(z) = \frac{p(x')}{c(y)} < +\infty.$$

It follows that any compact subset of $X$ is of finite $\mu$-measure, and hence $\mu$ is a Radon measure on $X$. $\square$

Now we prove the converse of the previous theorem:

Theorem 5.8. *If any potential on $X$ is representable by the Green function $G$, then the potentials of harmonic support reduced to a single point are proportional.*

Proof. Let $p$ be a potential on $X$ of harmonic support reduced to the point $y \in X$. We have $p = G\mu$, where $\mu$ is a nonnegative Radon measure. According to Proposition 4.1 we have $S(G\mu) = Supp(\mu) = \{y\}$ and then the measure $\mu$ and the Dirac measure at the point $y$ are proportional, so that there exists a real $\alpha \geq 0$ such that $\mu = \alpha.\varepsilon_y$ and consequently $p = \alpha.G(\textbf{.}, y)$. $\square$

Here is an application of the integral representation theorem (Theorem 5.7):

Corollary 5.9. *Let $(X, \mathcal{W})$ be a balayage space admitting a Green kernel $G$. Then the following two assertions are equivalent:*

1. *Every potential $p$ is representable by means of $G$.*
2. *For any $y \in X$, every extreme potential $p$ of support $\{y\}$ is of the form $p = \alpha G(., y)$, $\alpha \geq 0$.*

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