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Inequalities for sine and cosine polynomials

Abstract. In this paper, we prove that, letting λ be a real number,

$$\text{(i)} \qquad \lambda \sum_{k=1}^{n} (-1)^k \sin(kx) \leq \sum_{k=1}^{n} \frac{\sin(kx)}{k}$$

is valid for all $n \geq 1$ and $x \in [0, \pi]$ if and only if $\lambda \in [0, 2]$. This extends the classical Fejér-Jackson inequality which states that (i) holds for $\lambda = 0$. An application of (i) reveals if $a > 0$ and b are real numbers, then

$$\text{(ii)} \qquad \frac{41}{96} + \sum_{k=1}^{n} \frac{\cos(kx)}{k+1} \geq a\big(\cos(x) + b\big)^2$$

holds for all $n \geq 2$ and $x \in [0, \pi]$ if and only if $a \leq 2/75$ and $b = 3/8$. This refines a result of Koumandos (2001) who proved that the expression on the left-hand side of (ii) is nonnegative for all $n \geq 2$ and $x \in [0, \pi]$. The cosine polynomial in (ii) was first studied by Rogosinski and Szegö in 1928.

Keywords. Sine polynomials, cosine polynomials, inequalities.

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1 - Introduction

The inequality of Fejér-Jackson is a classical result in the theory of trigonometric polynomials. It states that

$$(1.1) \qquad \sum_{k=1}^{n} \frac{\sin(kx)}{k} \geq 0 \quad (n \geq 1;\, 0 \leq x \leq \pi).$$

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The validity of (1.1) was conjectured by Fejér in 1910 and the first proof was given one year later by Jackson [**9**]. Since then, more than twenty proofs of (1.1) were discovered. A very short proof was published by Landau [**14**] in 1933. He used elementary properties of trigonometric functions to prove the inequality by induction on n. In 1988, Lupaş [**15**] presented the elegant integral representation

$$\sum_{k=1}^{n} \frac{\sin(k\arccos(t))}{k} = \frac{\sqrt{1-t}}{2}\int_{-1}^{t} \frac{1-P_n(y)}{1-y}\frac{dy}{\sqrt{t-y}} \quad (-1<t<1),$$

where P_n denotes the n-th Legendre polynomial. Since $|P_n(y)|<1$ for $y \in (-1,1)$, we obtain (1.1).

Many authors studied generalizations and numerous related results of (1.1). Moreover, it was shown that inequalities for trigonometric sums have applications in various fields, like, for instance, geometric function theory, approximation theory, combinatorics and number theory. For detailed information on this subject we refer to Askey [**2**], Askey and Gasper [**3**], Barnard et al. [**4**], Dimitrov and Merlo [**7**], Koumandos [**11**], Milovanović et al. [**17**, chap. 4] and Raigorodskii and Rassias [**18**].

The referee pointed out that "inequalities such as (1.1) and other relevant results on trigonometric sums have proved recently to be applicable to really essential problems in Mathematics, such as the Riemann Hypothesis"; see Derevyanko et al. [**6**], Maier et al. [**16**] and the references cited therein.

We note that the sine polynomial in (1.1) is closely connected to the partial sums of the Fourier series of the fractional part function,

$$\{x\} = \frac{1}{2} - \frac{1}{\pi}\sum_{k=1}^{\infty} \frac{\sin(2k\pi x)}{k} \quad (x \in \mathbb{R}\setminus\mathbb{Z}).$$

In 1913, Young [**22**] (see also Alzer and Kwong [**1**]) presented a companion to (1.1) for the cosine polynomial

$$F_n(x) = \sum_{k=1}^{n} \frac{\cos(kx)}{k},$$

namely,

$$1 + F_n(x) \ge 0 \quad (n \ge 1;\ 0 \le x \le \pi). \tag{1.2}$$

Brown and Koumandos [**5**] showed in 1997 that in (1.2) the additive constant 1 can be replaced by a smaller number if we assume that $n \ge 2$,

$$\frac{5}{6} + F_n(x) \ge 0 \quad (n \ge 2;\ 0 \le x \le \pi). \tag{1.3}$$

The constants 1 and 5/6, given in (1.2) and (1.3), respectively, are best possible. Recently, Fong et al. [**8**] discovered the following remarkable refinement of (1.3),

$$\frac{5}{6}+F_n(x)\geq\frac{1}{4}\big(\cos(x)+1\big)^2\quad(n\geq 2;\,0\leq x\leq\pi). \tag{1.4}$$

Equality holds in (1.4) if and only if $n=2$ and $x=\arccos(-1/3)$.

In 1928, Rogosinski and Szegö [**19**] published an interesting counterpart of (1.2) for the polynomial

$$C_n(x)=\sum_{k=1}^{n}\frac{\cos(kx)}{k+1}.$$

They proved that

$$\frac{1}{2}+C_n(x)\geq 0\quad(n\geq 1;\,0\leq x\leq\pi), \tag{1.5}$$

and in 2001 Koumandos [**10**] showed that (1.5) can be improved if $n\geq 2$,

$$\frac{41}{96}+C_n(x)\geq 0\quad(n\geq 2;\,0\leq x\leq\pi). \tag{1.6}$$

Both constants, 1/2 and 41/96, respectively, are sharp.

With regard to (1.4) it is natural to ask whether it is possible to replace in (1.6) the constant lower bound 0 by a simple nonnegative cosine polynomial. Here, we provide an affirmative answer to this question. In Section 2, we collect a few lemmas. They are needed to prove a new extension of the Fejér-Jackson inequality given in Section 3. This extension as well as some further lemmas, which we present in Section 4, play an important role in the proof of a refinement of Koumandos' inequality (1.6). Our refinement is given in Section 5.

In what follows, we maintain the notations introduced in this section. The numerical values have been calculated via the computer program MAPLE 13.

2 - Lemmas, part 1

First, we collect some inequalities for trigonometric functions. They are helpful for the proof of the theorem presented in Section 3.

Lemma 2.1. *For all real numbers $k>1$ and $x\in[0,\sigma/k]$ we have*

$$0\leq\frac{\sin((k-1)x)}{k-1}-\frac{\sin(kx)}{k},$$

where $\sigma=4.49\ldots$ is the smallest positive zero of $h(x)=\sin(x)-x\cos(x)$.

Proof. Let $\alpha \in (0,1]$, $y \in [0,\sigma]$ and

$$g(\alpha, y) = \frac{\sin(\alpha y)}{\alpha} - \sin(y).$$

Since $0 \le \alpha y \le \sigma$ and $h(t) \ge 0$ for $t \in [0,\sigma]$, we obtain

$$\alpha^2 \frac{\partial}{\partial \alpha} g(\alpha, y) = -h(\alpha y) \le 0.$$

Thus,

$$g(\alpha, y) \ge g(1, y) = 0.$$

Let $k > 1$ and $x \in [0, \sigma/k]$. We set $\alpha^* = 1 - 1/k$ and $y^* = kx$. Then, $\alpha^* \in (0,1]$ and $y^* \in [0,\sigma]$. It follows that

$$0 \le \frac{1}{k} g(\alpha^*, y^*) = \frac{\sin((k-1)x)}{k-1} - \frac{\sin(kx)}{k}.$$

□

Corollary 2.2. *For all even integers $k \ge 2$ and real numbers $t \in [\pi - 1.4\pi/k, \pi]$ we have*

$$0 \le \frac{\sin((k-1)t)}{k-1} + \frac{\sin(kt)}{k}.$$

Proof. We set $t = \pi - x$. Then, $x \in [0, \sigma/k]$. From Lemma 2.1 we conclude that

$$0 \le \frac{\sin((k-1)x)}{k-1} - \frac{\sin(kx)}{k} = \frac{\sin((k-1)t)}{k-1} + \frac{\sin(kt)}{k}.$$

□

Next, we present some properties of the function

$$f_m(y) = (-1)^{(m+1)/2} \int_0^y \frac{\cos(ms)}{\cos(s)} ds. \tag{2.1}$$

Lemma 2.3. *For all odd integers $m \ge 1$ and real numbers $y \in [0, \pi/2)$ we have*

$$f_m(y) \ge \frac{1}{m}\left(1 - \frac{2}{\cos(y)}\right). \tag{2.2}$$

P r o o f. Let $y \in [0, \pi/2)$. We distinguish two cases.

C a s e 1. $m \equiv 3 \pmod 4$.
Using integration by parts yields

$$\begin{aligned} f_m(y) &= \int_0^y \frac{\cos(ms)}{\cos(s)} ds \\ &= \frac{1}{m}\left(\frac{\sin(my)-1}{\cos(y)}+1\right) + \frac{1}{m}\int_0^y \bigl(1-\sin(ms)\bigr)\frac{\sin(s)}{\cos^2(s)} ds \\ &\geq \frac{1}{m}\left(\frac{\sin(my)-1}{\cos(y)}+1\right) \\ &\geq \frac{1}{m}\left(1-\frac{2}{\cos(y)}\right). \end{aligned}$$

C a s e 2. $m \equiv 1 \pmod 4$.
Again using integration by parts gives

$$\begin{aligned} f_m(y) &= -\int_0^y \frac{\cos(ms)}{\cos(s)} ds \\ &= \frac{1}{m}\left(\frac{-\sin(my)-1}{\cos(y)}+1\right) + \frac{1}{m}\int_0^y \bigl(1+\sin(ms)\bigr)\frac{\sin(s)}{\cos^2(s)} ds \\ &\geq \frac{1}{m}\left(\frac{-\sin(my)-1}{\cos(y)}+1\right) \\ &\geq \frac{1}{m}\left(1-\frac{2}{\cos(y)}\right). \end{aligned}$$

□

L e m m a 2.4. *For all odd integers $m \geq 11$ and real numbers $y \in \bigl[\pi/m, \pi/2 - 1.4\pi/m\bigr]$ we have*

$$f_m(y) + y - \tan(y/2) > 0. \tag{2.3}$$

P r o o f. Let $t \in [0, \pi/2]$ and

$$\eta(t) = 0.64t - \tan(t/2).$$

Since

$$\eta(0) = 0, \quad \eta(\pi/2) = 0.005\ldots, \quad \eta''(t) = -\frac{\sin(t)}{\bigl(1+\cos(t)\bigr)^2} \leq 0,$$

we conclude that η is nonnegative on $[0, \pi/2]$. Using this result and (2.2) gives for odd $m \ge 11$ and $y \in [\pi/m, \pi/2 - 1.4\pi/m]$,

$$f_m(y) + y - \tan(y/2) \ge \frac{1}{m}\left(1 - \frac{2}{\cos(y)}\right) + 0.36y = \phi_m(y), \quad \text{say.} \tag{2.4}$$

We obtain

$$\phi_m''(y) = \frac{-2}{m\cos^3(y)}\bigl(1 + \sin^2(y)\bigr) < 0,$$

$$m\phi_m(\pi/m) = 1 - \frac{2}{\cos(\pi/m)} + 0.36\pi \ge 2.13 - \frac{2}{\cos(\pi/11)} = 0.04\ldots,$$

$$\begin{aligned}\phi_m\bigl(\pi/2 - 1.4\pi/m\bigr) &= 0.18\pi + \frac{1}{m}(1 - 0.36 \cdot 1.4 \cdot \pi) - \frac{2}{m\sin(1.4\pi/m)}\\ &\ge \frac{1}{2} - \frac{2}{11 \cdot \sin(1.4\pi/11)}\\ &= 0.03\ldots.\end{aligned}$$

It follows that

$$\phi_m(y) \ge \min\bigl(\phi_m(\pi/m), \phi_m(\pi/2 - 1.4\pi/m)\bigr) > 0. \tag{2.5}$$

From (2.4) and (2.5) we conclude that (2.3) is valid. □

Lemma 2.5. *For all integers $n \ge 1$ and real numbers $x \in (0, \pi)$ we have*

$$\sum_{k=1}^{n}\left(2 - \frac{(-1)^k}{k}\right)\sin(kx) \ge f_{2n+1}(x/2) + \frac{x}{2} - \tan(x/4). \tag{2.6}$$

Proof. Let $x \in (0, \pi)$. We have

$$\begin{aligned}2\sum_{k=1}^{n}\sin(kx) &= \frac{\cos(x/2) - \cos((n+1/2)x)}{\sin(x/2)}\\ &\ge \frac{\cos(x/2) - 1}{\sin(x/2)}\\ &= -\tan(x/4)\end{aligned} \tag{2.7}$$

and

$$\sum_{k=1}^{n}\frac{(-1)^{k+1}}{k}\sin(kx) = \int_0^x \sum_{k=1}^{n}(-1)^{k+1}\cos(kt)dt \tag{2.8}$$
$$= \frac{x}{2} + \frac{(-1)^{n+1}}{2}\int_0^x \frac{\cos((n+1/2)t)}{\cos(t/2)}dt$$
$$= \frac{x}{2} + f_{2n+1}(x/2).$$

Using (2.7) and (2.8) reveals that (2.6) holds. □

The next statement is known as comparison principle; see Koumandos [**11**] and Kwong [**12**].

Lemma 2.6. *Let a_k, b_k and c_k $(k = 1, \dots, N)$ be real numbers such that*

$$a_k > 0 \quad (k = 1, \dots, N), \quad \frac{b_1}{a_1} \geq \frac{b_2}{a_2} \geq \dots \geq \frac{b_N}{a_N} \geq 0,$$
$$\sum_{k=1}^{m} a_k c_k \geq 0 \quad (m = 1, \dots, N).$$

Then, $\sum_{k=1}^{N} b_k c_k \geq 0$.

3 - An inequality for sine polynomials

The following extension of the Fejér-Jackson inequality (1.1) is valid.

Theorem 3.1. *Let λ be a real number. The inequality*

$$\lambda \sum_{k=1}^{n}(-1)^k \sin(kx) \leq \sum_{k=1}^{n}\frac{\sin(kx)}{k} \tag{3.1}$$

holds for all integers $n \geq 1$ and real numbers $x \in [0, \pi]$ if and only if $\lambda \in [0, 2]$.

Proof. We assume that (3.1) is valid for all $n \geq 1$ and $x \in [0, \pi]$. Let $x \in (0, \pi)$. From (3.1) with $n = 2$ we obtain

$$\lambda\big(-\sin(x) + \sin(2x)\big) \leq \sin(x) + \frac{\sin(2x)}{2}.$$

We multiply both sides by $1/\sin(x)$. This gives

$$\lambda\big(-1 + 2\cos(x)\big) \leq 1 + \cos(x).$$

If $x \to 0$, then $\lambda \le 2$, and if $x \to \pi$, then $-3\lambda \le 0$, that is, $\lambda \ge 0$.

Next, let $\lambda \in [0, 2]$. If $\lambda = 0$, then (3.1) reduces to (1.1). Hence, it remains to prove (3.1) with $\lambda = 2$. Let $x = \pi - t$ with $t \in [0, \pi]$. Then, (3.1) with $\lambda = 2$ is equivalent to

$$0 \le \sum_{k=1}^{n} \left(2 - \frac{(-1)^k}{k}\right) \sin(kt) = S_n(t), \quad \text{say.} \tag{3.2}$$

We have

$$S_1(t) = 3\sin(t), \quad S_2(t) = 3\sin(t)\big(\cos(t) + 1\big),$$

$$S_3(t) = \frac{28}{3}\sin(t)\left(\left(\cos(t) + \frac{9}{56}\right)^2 + \frac{143}{3136}\right),$$

$$S_4(t) = 14\sin(t)\big(\cos(t) + 1\big)\left(\left(\cos(t) - \frac{1}{6}\right)^2 + \frac{5}{252}\right).$$

These representations show that (3.2) is valid for $n = 1, 2, 3, 4$. Next, let $n \ge 5$. We consider five cases.

Case 1. $t \in [0, \pi/n]$.
Then, each term of $S_n(t)$ is nonnegative. Hence, $S_n(t) \ge 0$.

Case 2. n is odd and $t \in [\pi - \pi/n, \pi]$.
Let $n = 2N + 1$. Then,

$$S_n(t) = \sum_{k=1}^{N}(4k-1)\left(\frac{\sin((2k-1)t)}{2k-1} + \frac{\sin(2kt)}{2k}\right) + \frac{4N+3}{2N+1}\sin((2N+1)t).$$

Let $k \in \{1, \ldots, N\}$. Since $[\pi - \pi/(2N+1), \pi] \subset [\pi - 1.4\pi/(2k), \pi]$, we conclude from Corollary 2.2 that

$$\frac{\sin((2k-1)t)}{2k-1} + \frac{\sin(2kt)}{2k} \ge 0.$$

Moreover, we have $\sin((2N+1)t) \ge 0$. This implies that $S_n(t) \ge 0$.

Case 3. n is even and $t \in [\pi - 1.4\pi/n, \pi]$.
Let $n = 2N$. We have

$$S_n(t) = \sum_{k=1}^{N}(4k-1)\left(\frac{\sin((2k-1)t)}{2k-1} + \frac{\sin(2kt)}{2k}\right).$$

An application of Corollary 2.2 reveals that $S_n(t) \ge 0$.

Case 4. $t \in [\pi/n, \pi - 1.4\pi/n]$.
Let $y = t/2$, $m = 2n + 1 \geq 11$ and

$$L_m(y) = f_m(y) + y - \tan(y/2),$$

where f_m is defined in (2.1). From Lemma 2.5 we obtain

$$S_n(t) \geq L_m(y). \tag{3.3}$$

Since

$$y \in [\pi/(m-1), \pi/2 - 1.4\pi/(m-1)] \subset [\pi/m, \pi/2 - 1.4\pi/m],$$

we conclude from Lemma 2.4 and (3.3) that $S_n(t) \geq 0$.

Case 5. n is odd and $t \in [\pi - 1.4\pi/n, \pi - \pi/n]$.
Let $y = t/2$ and $m = 2n + 1 \geq 11$. Then,

$$S_n(t) \geq L_m(y). \tag{3.4}$$

We have

$$f'_m(y) = \frac{\cos(my)}{\cos(y)}.$$

Since

$$my \in [m\pi/2 - 1.4\pi m/(m-1), m\pi/2 - m\pi/(m-1)] \subset [2m_0\pi - \pi/2, 2m_0\pi + \pi/2]$$

$$\text{with} \quad m_0 = \frac{m-3}{4} \in \mathbb{N},$$

we conclude that $\cos(my) \geq 0$. It follows that f_m is increasing. Since $y \mapsto y - \tan(y/2)$ is increasing on $[0, \pi/2]$, we obtain

$$L_m(y) \geq L_m\big(\pi/2 - 1.4\pi/(m-1)\big). \tag{3.5}$$

We have $\pi/m \leq \pi/2 - 1.4\pi/(m-1) \leq \pi/2 - 1.4\pi/m$, so that Lemma 2.4 gives $L_m\big(\pi/2 - 1.4\pi/(m-1)\big) \geq 0$. Combining this result with (3.4) and (3.5) leads to $S_n(t) \geq 0$.

The Cases 1 - 5 reveal that $S_n(t) \geq 0$ for $n \geq 5$ and $t \in [0, \pi]$. $\square$

Remark 3.2. For each $n \in \mathbb{N}$ the sine polynomial on the left-hand side of (3.1) attains negative values on $(0, \pi)$. This means that (3.1) does not provide a refinement of the Fejér-Jackson inequality (1.1). Nevertheless, (3.1) with $\lambda = 2$ implies (1.1). To show this we set

$$a_k = 2 - \frac{(-1)^k}{k}, \quad b_k = \frac{1}{k}, \quad c_k = c_k(x) = \sin(kx) \quad (k = 1, \dots, n).$$

Then,

$$a_k > 0 \quad (k = 1, \ldots, n) \qquad \text{and}$$

$$b_k a_{k+1} - a_k b_{k+1} = \big(1 + (-1)^k\big)\frac{2}{k(k+1)} \geq 0 \quad (k = 1, \ldots, n-1).$$

Let $x \in [0, \pi]$. From (3.1) (with $\lambda = 2$ and $\pi - x$ instead of x) we obtain

$$\sum_{k=1}^{m} a_k c_k \geq 0 \quad (m \geq 1).$$

If follows from Lemma 2.6 that

$$0 \leq \sum_{k=1}^{n} b_k c_k = \sum_{k=1}^{n} \frac{\sin(kx)}{k}.$$

Remark 3.3. A remarkable extension of (1.1) and (1.2) was given by Vietoris [**20**] in 1958. He proved that the polynomials $\sum_{k=1}^{n} a_k \sin(kx)$ and $\sum_{k=0}^{n} a_k \cos(kx)$ are nonnegative for all $x \in [0, \pi]$ if

$$a_0 \geq a_1 \geq \cdots \geq a_n > 0 \quad \text{and} \quad a_{2k} \leq \frac{2k-1}{2k} a_{2k-1} \quad (1 \leq k \leq n/2). \tag{3.6}$$

By using inequality (3.2) and the comparison principle it can be shown that $\sum_{k=1}^{n} a_k \sin(kx) \geq 0$ $(0 \leq x \leq \pi)$ holds under weaker conditions than (3.6). For details we refer to Kwong [**12**]; see also Kwong [**13**].

4 - Lemmas, part 2

In this section, we provide two lemmas which play a role in the proof of Theorem 5.1 given in Section 5. The following lemma offers a property of

$$A_n(x) = \frac{41}{100} + \frac{12}{25}\cos(x) + \frac{8}{25}\cos(2x) + \sum_{k=3}^{n} \frac{\cos(kx)}{k+1}. \tag{4.1}$$

Lemma 4.1. *For all integers $n \in \{4, 5, \ldots, 12\}$ and real numbers $x \in [0, \pi]$ we have $A_n(x) > 0$.*

Proof. It suffices to show that if $n \in \{4, \ldots, 12\}$, then A_n has no zero on $[0, \pi]$. Let $x \in [0, \pi]$. We set $X = \cos(x) \in [-1, 1]$. Then, we obtain

$$A_4(x) = \frac{8}{5}X^4 + X^3 - \frac{24}{25}X^2 - \frac{27}{100}X + \frac{29}{100},$$

$$A_5(x) = \frac{8}{3}X^5 + \frac{8}{5}X^4 - \frac{7}{3}X^3 - \frac{24}{25}X^2 + \frac{169}{300}X + \frac{29}{100},$$

$$A_6(x) = \frac{32}{7}X^6 + \frac{8}{3}X^5 - \frac{184}{35}X^4 - \frac{7}{3}X^3 + \frac{282}{175}X^2 + \frac{169}{300}X + \frac{103}{700},$$

$$A_7(x) = 8X^7 + \frac{32}{7}X^6 - \frac{34}{3}X^5 - \frac{184}{35}X^4 + \frac{14}{3}X^3 + \frac{282}{175}X^2 - \frac{187}{600}X + \frac{103}{700},$$

$$A_8(x) = \frac{128}{9}X^8 + 8X^7 - \frac{1504}{63}X^6 - \frac{34}{3}X^5 + \frac{3944}{315}X^4 + \frac{14}{3}X^3 - \frac{3062}{1575}X^2 - \frac{187}{600}X + \frac{1627}{6300},$$

$$A_9(x) = \frac{128}{5}X^9 + \frac{128}{9}X^8 - \frac{248}{5}X^7 - \frac{1504}{63}X^6 + \frac{478}{15}X^5 + \frac{3944}{315}X^4 - \frac{22}{3}X^3 - \frac{3062}{1575}X^2 + \frac{353}{600}X + \frac{1627}{6300},$$

$$A_{10}(x) = \frac{512}{11}X^{10} + \frac{128}{5}X^9 - \frac{10112}{99}X^8 - \frac{248}{5}X^7 + \frac{54016}{693}X^6 + \frac{478}{15}X^5 - \frac{82616}{3465}X^4 - \frac{22}{3}X^3 + \frac{45068}{17325}X^2 + \frac{353}{600}X + \frac{11597}{69300},$$

$$A_{11}(x) = \frac{256}{3}X^{11} + \frac{512}{11}X^{10} - \frac{3136}{15}X^9 - \frac{10112}{99}X^8 + \frac{2776}{15}X^7 + \frac{54016}{693}X^6 - \frac{354}{5}X^5 - \frac{82616}{3465}X^4 + 11X^3 + \frac{45068}{17325}X^2 - \frac{197}{600}X + \frac{11597}{69300},$$

$$A_{12}(x) = \frac{2048}{13}X^{12} + \frac{256}{3}X^{11} - \frac{60928}{143}X^{10} - \frac{3136}{15}X^9 + \frac{552832}{1287}X^8 + \frac{2776}{15}X^7 - \frac{1781504}{9009}X^6 - \frac{354}{5}X^5 + \frac{1836592}{45045}X^4 + 11X^3 - \frac{661516}{225225}X^2 - \frac{197}{600}X + \frac{220061}{900900}.$$

Next, we apply Sturm's theorem to determine the number of distinct real roots of an algebraic polynomial located in an interval; see van der Waerden [**21**, sect. 79]. This gives that each of the nine polynomials in X has no zero in $[-1,1]$. We obtain the same result if we use $A_n(\pi) > 0$ for $n \in \{4,\dots,12\}$ and the MAPLE procedure "sturm" which provides the number of zeros in $(-1,1]$. It follows that each of the functions $A_4,\dots,A_{12}$ has no zero on $[0,\pi]$. $\square$

We define

$$u(x) = \frac{5}{14}\sin(x) + \frac{111}{700}\sin(2x) + \frac{2}{225}\sin(3x) \tag{4.2}$$

and

$$v_n(x) = \frac{\sin(nx) + \sin((n+1)x)}{n+2}. \tag{4.3}$$

Lemma 4.2. *For all integers $n \geq 13$ and real numbers $x \in (0,\pi)$ we have $u(x) + v_n(x) > 0$.*

Proof. Let $n \geq 13$. We consider three cases.

Case 1. $x \in (0, 0.2]$.

Let
$$p(x) = u(x) - 0.66x.$$
Then,
$$p''(x) = -\frac{8}{25}\sin(x)\bigl(\cos(x)+t_1\bigr)\bigl(\cos(x)+t_2\bigr) \tag{4.4}$$
with
$$t_1 = \frac{1}{56}\bigl(111-\sqrt{9605}\bigr) = 0.23\ldots, \quad t_2 = \frac{1}{56}\bigl(111+\sqrt{9605}\bigr) = 3.73\ldots.$$
It follows that $p''(x) < 0$ which implies that
$$p'(x) \geq p'(0.2) = 0.004\ldots \quad \text{and} \quad p(x) > p(0) = 0.$$
Since $\sin(t) > -t/4$ for $t > 0$, we obtain
$$\begin{aligned} u(x) + v_n(x) &> 0.66x + \frac{1}{n+2}\left(-\frac{nx}{4} - \frac{(n+1)x}{4}\right) \\ &= \left(0.66 - \frac{2n+1}{4(n+2)}\right)x \\ &> 0.16x > 0. \end{aligned}$$

Case 2. $x \in [0.2, 2.2]$.

Let
$$q(x) = u(x) - \frac{2}{15}.$$
Since $q''(x) = p''(x)$, we conclude from (4.4) that
$$q''(x) < 0 \quad \text{for} \quad x \in [0.2, t_0) \quad \text{and} \quad q''(x) > 0 \quad \text{for} \quad x \in (t_0, 2.2],$$
where $t_0 = \arccos(-t_1) = 1.80\ldots$. We have
$$q'(0.2) = 0.66\ldots, \quad q'(t_0) = -0.34\ldots, \quad q'(2.2) = -0.28\ldots.$$
This implies that there exists a number $x_0 \in (0.2, t_0)$ such that q' is positive on $[0.2, x_0)$ and negative on $(x_0, 2.2]$. Thus,
$$q(x) \geq \min\bigl(q(0.2), q(2.2)\bigr) = 0.004\ldots. \tag{4.5}$$

Using (4.5) yields

$$u(x)+v_n(x)>\frac{2}{15}-\frac{2}{n+2}\geq 0.$$

Case 3. $x\in[2.2,\pi)$.
Let

$$w(x)=u(x)-\frac{\pi-x}{15}.$$

We have $w''(x)=p''(x)$ and $\cos(x)\leq\cos(2.2)=-0.58\ldots$. From (4.4) we obtain $w''(x)>0$. It follows that

$$w'(x)<w'(\pi)=0 \quad\text{and}\quad w(x)>w(\pi)=0.$$

We have

$$|v_n(x)|=\frac{2}{n+2}\big|\sin((n+1/2)x)\cos(x/2)\big|\leq\frac{2}{15}\cos(x/2)\leq\frac{\pi-x}{15}.$$

Thus,

$$u(x)+v_n(x)>\frac{\pi-x}{15}-\frac{\pi-x}{15}=0.$$

This completes the proof of Lemma 4.2. $\square$

5 - An inequality for cosine polynomials

We are now in a position to present a refinement of Koumandos' inequality (1.6).

Theorem 5.1. *Let $a>0$ and b be real numbers. The inequality*

$$\frac{41}{96}+C_n(x)\geq a\big(\cos(x)+b\big)^2 \tag{5.1}$$

holds for all integers $n\geq 2$ and real numbers $x\in[0,\pi]$ if and only if $a\leq 2/75$ and $b=3/8$.

Proof. First, we assume that (5.1) is valid for all $n\geq 2$ and $x\in[0,\pi]$. Then, from (5.1) with $n=2$ and $x=\arccos(-3/8)$ we get

$$0=\frac{41}{96}+C_2(\arccos(-3/8))\geq a(-3/8+b)^2\geq 0.$$

This leads to $b=3/8$. It follows that we obtain from (5.1) with $n=3$ and $x=\pi$,

$$\frac{1}{96}=\frac{41}{96}+C_3(\pi)\geq a(-1+3/8)^2=\frac{25}{64}a.$$

Thus, $a \leq 2/75$.

Next, we assume that $0 < a \leq 2/75$ and $b = 3/8$. Then, we have to show that for $n \geq 2$ and $x \in [0, \pi]$,

$$\frac{41}{96} + C_n(x) \geq \frac{2}{75}\big(\cos(x) + 3/8\big)^2. \tag{5.2}$$

We have

$$\frac{41}{96} + C_2(x) - \frac{2}{75}\big(\cos(x) + 3/8\big)^2 = \frac{16}{25}\big(\cos(x) + 3/8\big)^2$$

and

$$\frac{41}{96} + C_3(x) - \frac{2}{75}\big(\cos(x) + 3/8\big)^2 = \frac{1}{100}\big(\cos(x) + 1\big)\big(9\big(2\cos(x) - 1\big)^2 + 64\cos^2(x)\big).$$

This implies that (5.2) is valid for $n = 2, 3$ and $x \in [0, \pi]$. Moreover, if $n = 2$, then equality holds in (5.2) if and only if $x = \arccos(-3/8)$; and if $n = 3$, then equality is valid if and only if $x = \pi$.

We have

$$\frac{41}{96} + C_n(x) - \frac{2}{75}\big(\cos(x) + 3/8\big)^2 = A_n(x),$$

where A_n is defined in (4.1).

Next, we show that $A_n(x) > 0$ for $n \geq 4$ and $x \in [0, \pi]$. From Lemma 4.1 we conclude that this is true if $n \in \{4, 5, \ldots, 12\}$.

Let $n \geq 13$. Since

$$A_n(0) \geq A_{13}(0) = 2.62\ldots \quad \text{and} \quad A_n(\pi) \geq A_{13}(\pi) = 0.07\ldots,$$

it remains to prove that $A_n(x) > 0$ for $x \in (0, \pi)$. Let

$$B_n(x) = 2\sin(x)A_n(x). \tag{5.3}$$

Then, we obtain the sine polynomial

$$B_n(x) = \frac{1}{2}\sin(x) + \frac{23}{100}\sin(2x) + \frac{3}{25}\sin(3x) + \sum_{k=4}^{n-1}\frac{2\sin(kx)}{k(k+2)} + \frac{\sin(nx)}{n} + \frac{\sin((n+1)x)}{n+1}.$$

We have the representation

$$B_n(x) = u(x) + v_n(x) + H_n(x), \tag{5.4}$$

where u and v_n are defined in (4.2) and (4.3), respectively, and

$$H_n(x) = \frac{1}{7}\sin(x) + \frac{1}{14}\sin(2x) + \frac{1}{9}\sin(3x) + \sum_{k=4}^{n} \frac{2\sin(kx)}{k(k+2)} + \frac{\sin((n+1)x)}{(n+1)(n+2)}.$$

Let

$$a_k = 2 - \frac{(-1)^k}{k} \quad (k = 1, \ldots, n+1),$$

$$b_1 = \frac{1}{7}, \quad b_2 = \frac{1}{14}, \quad b_3 = \frac{1}{9}, \quad b_k = \frac{2}{k(k+2)} \quad (k = 4, \ldots, n),$$

$$b_{n+1} = \frac{1}{(n+1)(n+2)},$$

$$c_k = c_k(x) = \sin(kx) \quad (k = 1, \ldots, n+1).$$

By direct computation we obtain

$$a_k > 0 \quad (k = 1, \ldots, n+1), \qquad \frac{b_1}{a_1} \geq \frac{b_2}{a_2} \geq \cdots \geq \frac{b_{n+1}}{a_{n+1}} > 0,$$

and from Theorem 3.1 (with $\lambda = 2$ and $\pi - x$ instead of x) we conclude that for $x \in (0, \pi)$ we have

$$0 \leq \sum_{k=1}^{m} a_k c_k \quad (m = 1, \ldots, n+1).$$

Applying Lemma 2.6 reveals that

$$0 \leq \sum_{k=1}^{n+1} b_k c_k = H_n(x). \tag{5.5}$$

Next, we use Lemma 4.2, (5.4) and (5.5). This leads to $B_n(x) > 0$, and from (5.3) we conclude that $A_n(x) > 0$. □

The proof of Theorem 5.1 shows that the following result is valid.

Corollary 5.2. *For all integers $n \geq 2$ and real numbers $x \in [0, \pi]$ we have*

$$\frac{41}{96} + C_n(x) \geq \frac{2}{75}\bigl(\cos(x) + 3/8\bigr)^2.$$

The sign of equality holds if and only if $n = 2$, $x = \arccos(-3/8)$ or $n = 3$, $x = \pi$.

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