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Analytic properties of the standard twist of L -functions

Abstract. We study some consequences of the functional equation satisfied by the standard twist $F(s, \alpha)$ of the L -functions $F(s)$ from the extended Selberg class. The shape of such a functional equation differs significantly from the classical one of Riemann-type satisfied by $F(s)$; for example, it contains an error term which can identically vanish only in some special but well described cases. In this paper we show that this unusual functional equation can nevertheless be used to investigate convexity bounds, asymptotic formulae for the average of the coefficients and distribution of zeros of $F(s, \alpha)$.

Keywords. L -functions, standard twist, convexity bounds, distribution of zeros, nonlinear exponential sums, Selberg class.

Mathematics Subject Classification: 11M41.

1 - Introduction

1.1 - Notation and definitions

Throughout the paper we write $s = \sigma + it$, an empty sum equals 0 and an empty product equals 1, logarithms have the principal value unless otherwise specified and $f(s) \equiv 0$ means that $f(s)$ vanishes identically.

The extended Selberg class $\mathcal{S}^\sharp$ consists of the non identically vanishing Dirichlet series

$$F(s) = \sum_{n=1}^{\infty} \frac{a(n)}{n^s},$$

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absolutely convergent for $\sigma > 1$, such that $(s-1)^m F(s)$ is entire of finite order for some integer $m \geq 0$ and satisfying a functional equation of type

$$F(s)\gamma(s) = \omega\overline{\gamma}(1-s)\overline{F}(1-s).$$

Here $|\omega| = 1$, $\overline{f}(s)$ means $\overline{f(\overline{s})}$ and the γ -factor $\gamma(s)$ is defined by

$$\gamma(s) = Q^s \prod_{j=1}^r \Gamma(\lambda_j s + \mu_j)$$

with $Q > 0$, $r \geq 0$, $\lambda_j > 0$ and $\Re \mu_j \geq 0$. We refer to our survey papers [5], [3], [11], [12], [13], [14] for definitions, examples and the basic theory of the Selberg class. We recall that degree d , conductor q , ξ -invariant ξ_F and root number ω_F of $F(s)$ are the invariants defined by

$$\begin{aligned} d = d_F &:= 2 \sum_{j=1}^r \lambda_j, & q = q_F &:= (2\pi)^d Q^2 \prod_{j=1}^r \lambda_j^{2\lambda_j}, \\ \xi_F &= 2 \sum_{j=1}^r \left(\mu_j - \frac{1}{2}\right) = \eta_F + id\theta_F, & \omega_F &= \omega \prod_{j=1}^r \lambda_j^{-2i\Im(\mu_j)} \end{aligned}$$

with $\eta_F, \theta_F \in \mathbb{R}$. The invariant θ_F is called the internal shift of $F(s)$, and the classical L -functions usually have $\theta_F = 0$. Moreover, m_F is the order of pole of $F(s)$ at $s = 1$ and

$$(1.1) \quad \sum_{m=1}^{m_F} \frac{\gamma_m}{(s-1)^m}$$

is its polar part. In this paper we always assume that the degree d is positive, and hence $d \geq 1$ thanks to the results in Conrey-Ghosh [1] and in [4].

Let $F \in \mathcal{S}^\sharp$ with degree $d \geq 1$ and $\alpha > 0$. The standard twist of $F(s)$ is defined for $\sigma > 1$ as

$$F(s, \alpha) = \sum_{n=1}^{\infty} \frac{a(n)}{n^s} e(-\alpha n^{1/d}),$$

where $e(x) = e^{2\pi i x}$. The spectrum of $F(s)$ is

$$\text{Spec}(F) = \{\alpha > 0 : a(n_\alpha) \neq 0\} = \left\{ d \left(\frac{m}{q} \right)^{1/d} : m \in \mathbb{N} \text{ with } a(m) \neq 0 \right\},$$

where

$$n_\alpha = qd^{-d}\alpha^d \quad \text{and} \quad a(n_\alpha) = 0 \text{ if } n_\alpha \notin \mathbb{N}.$$

Moreover, we write

$$(1.2) \quad s_\ell = \frac{d+1}{2d} - \frac{\ell}{d} \quad \ell = 0, 1, 2, \dots$$

and

$$s_\ell^* = s_\ell - i\theta_F.$$

1.2 - Properties of the standard twist $F(s, \alpha)$

We first recall some properties of the standard twist. We have that $F(s, \alpha)$ is entire if $\alpha \notin \text{Spec}(F)$, while for $\alpha \in \text{Spec}(F)$ it is meromorphic on $\mathbb{C}$ with at most simple poles at the points s_ℓ^* . Denoting by $\rho_\ell(\alpha)$ the residue of $F(s, \alpha)$ at $s = s_\ell^*$, it is known that $\rho_0(\alpha) \neq 0$ when $\alpha \in \text{Spec}(F)$; hence $F(s, \alpha)$ has always a simple pole at $s = s_0^*$ if $\alpha \in \text{Spec}(F)$. Moreover, $F(s, \alpha)$ has polynomial growth on every vertical strip, although the known bounds are weak in general. We refer to [6] and [8] for these and other results on $F(s, \alpha)$.

In a recent paper [10] we showed that $F(s, \alpha)$ satisfies a functional equation whose shape differs significantly from the classical functional equation of Riemann-type satisfied by $F(s)$. Actually, its shape is of general Hurwitz-Lerch type and contains a new error term function $H_k(s, \alpha)$; see (1.10) below; we need further definitions, taken from [10], to state it. With the notation in Section 1.1 we let

$$h_F(s) = \frac{\omega}{(2\pi)^r} Q^{1-2s} \prod_{j=1}^r (\Gamma(\lambda_j(1-s) + \bar{\mu}_j) \Gamma(1 - \lambda_j s - \mu_j)).$$

Then, the structural coefficients d_ℓ , a new set of invariants introduced in [10], are defined by the following asymptotic expansion of the function $h_F(s)$

$$(1.3) \quad h_F(s) \approx \frac{\omega_F}{\sqrt{2\pi}} \left(\frac{q^{1/d}}{2\pi d} \right)^{d(\frac{1}{2}-s)} \sum_{\ell=0}^{\infty} d_\ell \Gamma(d(s_\ell^* - s)), \quad (d_0 \neq 0)$$

valid for large $|s|$ outside an arbitrarily small angular region containing the positive real axis. Here $\approx$ means that cutting the sum at $\ell = M$ one gets a meromorphic remainder which is $\ll$ than the modulus of the M -th term times $1/|s|$; see Section 1.3 of [10]. Moreover, with γ_m as in (1.1) we write

$$(1.4) \quad R(s, \alpha) = (2\pi i \alpha)^{-ds} \sum_{m=1}^{m_F} d^m \gamma_m \sum_{h=0}^{m-1} \frac{(-1)^h \log^h(2\pi i \alpha) \Gamma^{(m-1-h)}(ds)}{h!(m-1-h)!},$$

hence $R(s, \alpha) \equiv 0$ if $F(s)$ is entire. Further, let

$$S_F(s) = 2^r \prod_{j=1}^r \sin(\pi(\lambda_j s + \mu_j)) = \sum_{j=-N}^N a_j e^{i\pi d \omega_j s}$$

with a certain $N \in \mathbb{N}$, $a_j \in \mathbb{C}$ and

$$(1.5) \quad -\frac{1}{2} = \omega_{-N} < \cdots < \omega_N = \frac{1}{2}$$

satisfying $\omega_{-j} = -\omega_j$. Note that $a_j \neq 0$ for $j \neq 0$, and $N \geq 1$ since $r \geq 1$. For $\ell = 0, 1, \dots$ we also define

$$(1.6) \quad \overline{F}_\ell(s, \alpha) = \sum_{j=-N}^N a_j e^{i\pi d \omega_j (1-s)} \sum_{n \geq 1}^b \frac{\overline{a(n)}}{n^s} \left(1 + e^{i\pi(\frac{1}{2} - \omega_j)} \left(\frac{n_\alpha}{n} \right)^{1/d} \right)^{d(1-s-s_\ell^*)},$$

where the symbol b in the inner sum indicates that if $j = -N$ then the term $n = n_\alpha$ is omitted. Hence $\overline{F}_\ell(s, \alpha)$ is well defined since $1 + e^{i\pi(\frac{1}{2} - \omega_j)} \left(\frac{n_\alpha}{n} \right)^{1/d} \neq 0$ always. Note that if $\alpha \notin \text{Spec}(F)$ we may omit b , since $a(n_\alpha) = 0$ in this case. Note also that the inner sum in (1.6) is a general Dirichlet series with complex frequencies, absolutely convergent for $\sigma > 1$, and

$$(1.7) \quad \left(1 + e^{i\pi(\frac{1}{2} - \omega_j)} \left(\frac{n_\alpha}{n} \right)^{1/d} \right)^{d(1-s-s_\ell^*)} = e^{d(1-s-s_\ell^*) \text{Log} \left(1 + e^{i\pi(\frac{1}{2} - \omega_j)} \left(\frac{n_\alpha}{n} \right)^{1/d} \right)}$$

where the branch of Log on $\overline{\mathbb{H}} \setminus \{0\}$ has argument in $[0, \pi]$, $\mathbb{H}$ being the upper half-plane.

We refer to [10] for further information on the above defined quantities. Here we recall in particular that, given $F \in \mathcal{S}^\sharp$ with degree $d \geq 1$ and $\alpha > 0$, for $\ell = 0, 1, 2, \dots$ the function $\overline{F}_\ell(s, \alpha)$ is entire and not identically vanishing. Moreover, uniformly for σ in any bounded interval, as $|t| \rightarrow \infty$ we have

$$(1.8) \quad \overline{F}_\ell(s, \alpha) \ll e^{\frac{\pi}{2}d|t|} |t|^{c(\sigma)}$$

with a certain $c(\sigma) \geq 0$ independent of ℓ and α , satisfying $c(\sigma) = 0$ for $\sigma > 1$; see Theorem 1 of [10].

Now we are ready to state the functional equation of $F(s, \alpha)$ for any $F \in \mathcal{S}^\sharp$ with degree $d \geq 1$. Let $\alpha > 0$ and

$$(1.9) \quad e_\ell = \frac{d-1}{2} + \ell + i\theta_F.$$

Then for any integer $k \geq 0$ and s in the strip $s_{k+1} < \sigma < s_k$ we have

$$(1.10) \quad F(s, \alpha) = \frac{\omega_F}{\sqrt{2\pi}} \left(\frac{q^{1/d}}{2\pi d} \right)^{d(\frac{1}{2}-s)} \sum_{\ell=0}^k d_\ell \Gamma(d(1-s) - e_\ell) \overline{F}_\ell(1-s, \alpha) \\ + R(1-s, \alpha) + H_k(s, \alpha),$$

where the structural coefficients d_ℓ are defined by (1.3) and the functions $H_k(s, \alpha)$ are holomorphic in the above strips and meromorphic over $\mathbb{C}$ with all poles in a horizontal strip of bounded height. Moreover, there exists $\theta = \theta(d) > 0$ such that for any $\sigma \in [s_{k+1}, s_k] \cap (-\infty, 0)$ we have

$$(1.11) \quad H_k(s, \alpha) \ll |t|^{-\theta} \quad \text{as } |t| \rightarrow \infty.$$

We refer to [10] for a discussion of the functional equation (1.10) and for further results on it; see in particular Theorem 2 in [10]. Here we only recall that the constant in the $\ll$ -symbol may depend on σ , $F(s)$ and α , that the value of θ is usually close to $1/2$ but may become smaller when σ is near 0, and that by (1.2) and (1.9) the Γ -factors in (1.10) may also be rewritten as $\Gamma(d(s_\ell^* - s))$. We finally recall that the term $R(1-s, \alpha)$ is present only when $F(s)$ has a pole at $s = 1$, and that there are examples of functions $F(s)$ such that $H_k(s, \alpha) \equiv 0$ for k larger than a certain value h ; see [9] and [10]. Actually, Theorem 4 of [10] describes exactly the cases where $H_k(s, \alpha) \equiv 0$.

1.3 - New results

The principal goal of this paper is to show that in spite of the unusual form of functional equation (1.10), suitable versions of the basic consequences of the Riemann-type functional equation can still be derived. In particular, we deal with convexity bounds, the distribution of zeros and the asymptotics for the average of the coefficients of $F(s, \alpha)$. Note that, although here we restrict ourselves mainly to qualitative results (in particular concerning the distribution of trivial zeros), it will be clear from the proofs that more precise statements can be obtained in any concrete case.

We first consider the Lindelöf μ -function of $F(s, \alpha)$

$$\mu_F(\sigma, \alpha) = \inf\{\xi : F(\sigma + it, \alpha) \ll_{\sigma, \alpha} |t|^\xi \text{ as } |t| \rightarrow \infty\}.$$

Theorem 1. *Let $F \in \mathcal{S}^\#$ with $d \geq 1$ and $\alpha > 0$. Then $\mu_F(\sigma, \alpha) = 0$ for $\sigma \geq 1$ and*

$$\mu_F(\sigma, \alpha) \leq \begin{cases} \frac{d}{2}(1-\sigma) & \text{if } 0 \leq \sigma \leq 1, \\ \frac{d}{2}(1-2\sigma) & \text{if } \sigma \leq 0. \end{cases}$$

Note that the upper bound for $\mu_F(\sigma, \alpha)$ is exactly the same as for the μ -function of $F(s)$, namely it coincides with the typical convexity bound of degree d . We do not enter here the issue of lower bounds for $\mu_F(\sigma, \alpha)$ when $\sigma < 0$, although it can be approached by refining the arguments in the proof of Theorem 1.

Next we deal with the distribution of zeros of $F(s, \alpha)$. As already remarked in [9], where we deal with the special case of the L -functions associated with half-integral cusp forms, distinguishing between trivial and nontrivial zeros is more delicate and unconventional in the case of the standard twist $F(s, \alpha)$. Indeed, due to the different shape of the functional equation (1.10), the trivial zeros of $F(s, \alpha)$ come essentially from the interferences between the terms in the sum over j on the right hand side of (1.6), rather than from the poles of the Γ -factors as in the classical case of $F(s)$. Moreover, as in [9] we consider only a rough definition of trivial zeros, which in principle does not characterize them uniquely. Actually, giving a really satisfactory definition of the trivial zeros appears to be a nontrivial problem.

For simplicity in the next theorems we assume that $F(s)$ satisfies the Ramanujan conjecture, namely $a(n) \ll n^\varepsilon$ for every $\varepsilon > 0$, although similar results can be proved unconditionally. Given $\varepsilon > 0$ let

$$(1.12) \quad \mathcal{L}_\varepsilon = \bigcup_{j=1}^J \{s \in \mathbb{C} : |s - L_j| < \varepsilon\}$$

denote a finite union of tubular regions of width 2ε around certain lines L_j in the complex plane. Our description of the trivial zeros is based on the following theorem.

Theorem 2. *Let $F(s)$ and α be as in Theorem 1, and assume the Ramanujan conjecture. For every $\varepsilon > 0$ there exist $\sigma^- = \sigma^-(\varepsilon) > 0$ and $t^+ = t^+(\varepsilon) > 0$ such that for $\sigma < -\sigma^-$ and $|t| > t^+$ we have $F(s, \alpha) \neq 0$ unless $s \in \mathcal{L}_\varepsilon(F, \alpha)$, where $\mathcal{L}_\varepsilon(F, \alpha)$ is a certain region of type (1.12).*

Clearly, σ^- and t^+ may depend on $F(s)$ and α as well, and from the proof of Theorem 2 it is clear that σ^-, t^+ increase to $+\infty$ as $\varepsilon \rightarrow 0$. The notation $\mathcal{L}_\varepsilon(F, \alpha)$ means that the lines L_j involved in (1.12) are determined only by $F(s)$ and α . We do not compute the equations of such lines in the general case, although this is certainly possible for any given $F(s)$ and α . For example, these equations are explicitly computed in the special case treated in [9]. Moreover, here we do not give an asymptotic formula, similar to (1.23) of [9], for the number of zeros in the tubular regions of $\mathcal{L}_\varepsilon(F, \alpha)$ as $\sigma \rightarrow -\infty$. Again, this is certainly possible for any given $F(s, \alpha)$.

The meaning of Theorem 2 is that the zeros of $F(s, \alpha)$ in the region defined by $\sigma < \sigma^-$ and $|t| > t^+$ lie close to a finite number of lines. Hence, similarly as in [9], we define the trivial zeros of $F(s, \alpha)$ as those zeros lying in the region $\sigma < -\sigma^-$, with some small $\varepsilon > 0$. More precisely, let $\varepsilon_0 > 0$ be so small that the various tubular regions of $\mathcal{L}_{\varepsilon_0}(F, \alpha)$ do not intersect for $\sigma < -\sigma^-$ and, moreover, the vertical and horizontal parts of $\mathcal{L}_{\varepsilon_0}(F, \alpha)$ lie inside the regions $\sigma \geq \sigma^-$ and $|t| \leq t^+$, respectively. We thus define the trivial zeros as above, choosing some $\varepsilon \leq \varepsilon_0$. Clearly, the trivial zeros lie either in the oblique parts of $\mathcal{L}_\varepsilon(F, \alpha)$ or in the horizontal half strip $\sigma < \sigma^-$, $|t| \leq t^+$, but in principle are not defined in a canonical way since they depend on the choice of ε . However, changing $\varepsilon \leq \varepsilon_0$ the trivial zeros change by a finite set.

Turning to the nontrivial zeros, let σ^+ be the upper bound of the real parts of the zeros of $F(s, \alpha)$ and σ^- be as above. The nontrivial zeros of $F(s, \alpha)$ are those in the vertical strip $-\sigma^- \leq \sigma \leq \sigma^+$, and their counting function is defined by

$$N_F(T, \alpha) = \{\rho = \beta + i\gamma : F(\rho, \alpha) = 0, -\sigma^- \leq \beta \leq \sigma^+, |\gamma| \leq T\}.$$

We denote by $\bar{n}$ the smallest $n \geq 1$ with $a(n) \neq 0$ and define m_+ and m_- as follows:

$$m_+ = n_+^{1/d} - n_\alpha^{1/d}, \quad \text{where } n_+ = \min\{n > n_\alpha : a(n) \neq 0\},$$

and

$$m_- = \begin{cases} n_\alpha^{1/d} - n_-^{1/d} & \text{if } n_- < n_\alpha \\ n_\alpha^{1/d} + n_-^{1/d} & \text{if } n_- \geq n_\alpha, \end{cases}$$

where

$$n_- = \begin{cases} \max\{n < n_\alpha : a(n) \neq 0\} & \text{if there exists } n < n_\alpha \text{ with } a(n) \neq 0, \\ \min\{n \geq n_\alpha : a(n) \neq 0\} & \text{if } a(n) = 0 \text{ for } 1 \leq n < n_\alpha. \end{cases}$$

Then the analog of the classical Riemann-von Mangoldt formula for $F(s, \alpha)$ is given by the following theorem.

Theorem 3. *Let $F(s)$ and α be as in Theorem 2. Then as $T \rightarrow \infty$ we have*

$$N_F(T, \alpha) = \frac{d}{\pi} T \log T + \frac{T}{\pi} \log \left(\frac{q}{\bar{n}(2\pi e \sqrt{m_+ m_-})^d} \right) + O(\log T).$$

The constant in the O -symbol may depend on $F(s)$, α and the choice of σ^- . Note that letting $\alpha = 0$ in Theorem 3 we formally obtain the well known asymptotic formula for the counting function of the nontrivial zeros of $F(s)$. Indeed, when $\alpha = 0$ we have $m_+ = m_- = \bar{n}^{1/d}$, and hence the mid term of the above formula becomes

$$\frac{T}{\pi} \log \left(\frac{q}{\bar{n}^2 (2\pi e)^d} \right).$$

Finally, we consider the nonlinear exponential sums

$$S_F(x, \alpha) = \sum_{n \leq x} a(n) e(-\alpha n^{1/d}),$$

where $a(n)$ are the coefficients of a function $F(s)$ as in Theorem 2. We may assume that $d \geq 2$, since the behavior of $S_F(x, \alpha)$ is well known when $d = 1$ thanks to the characterisation of the degree 1 functions in [4], and [7] proves that there are no functions in $\mathcal{S}^\sharp$ with $1 < d < 2$.

Theorem 4. *Let $F(s)$ and α be as in Theorem 2, and suppose that $d \geq 2$. Then for every $\varepsilon > 0$ we have*

$$S_F(x, \alpha) = \frac{2d}{d+1} \rho_0(\alpha) x^{s_0^*} + O(x^{\frac{d-1}{d+1} + \varepsilon}),$$

where $s_0^* = (d+1)/2d - i\theta_F$ and $\rho_0(\alpha)$ is the residue of $F(s, \alpha)$ at $s = s_0^*$.

The constant in the O -symbol may depend on $F(s)$, α and ε . Note that the error term in Theorem 4 is the typical convexity bound in the asymptotic formulae for the coefficient sums of L -functions of degree d , although here the main term is different since it reflects the polar structure of $F(s, \alpha)$, instead of $F(s)$. Note also that Theorem 4 provides a genuine asymptotic formula when $2 \leq d < 2 + \sqrt{5}$, hence in particular for degrees $d = 2, 3, 4$. If instead of $S_F(x, \alpha)$ one considers the smoothed version

$$S_F(x, \alpha, \phi) = \sum_{n=1}^{\infty} a(n) e(-\alpha n^{1/d}) \phi\left(\frac{n}{x}\right)$$

with a smooth function $\phi(u)$ on $(0, \infty)$ with compact support, then we refer to Theorem 3 of [6] for a full asymptotic expansion in the general case of $F \in \mathcal{S}^\sharp$.

2 - Proofs

2.1 - Proof of Theorem 1

First, $\mu_F(\sigma, \alpha) = 0$ for $\sigma \geq 1$ and every $\alpha > 0$ follows from general principles. Let now $\sigma < 0$ be given, thus by (1.8)

$$\overline{F}_\ell(1-s, \alpha) \ll e^{\frac{\pi}{2}d|t|}.$$

Hence from functional equation (1.10), the definition (1.4) of $R(1-s, \alpha)$, the bound (1.11) for $H_k(s, \alpha)$ and Stirling's formula, as $|t| \rightarrow \infty$ we get that

$$(2.1) \quad F(s, \alpha) \ll |t|^{\frac{d}{2}(1-2\sigma)}$$

for every $\alpha > 0$, since $R(1-s, \alpha) + H_k(s, \alpha) = o(1)$ for any given $\sigma < 0$. Thus, from (2.1) and the continuity of the Lindelöf μ -function, for $\sigma \leq 0$ we have

$$\mu_F(\sigma, \alpha) \leq \frac{d}{2}(1-2\sigma).$$

The bound for $\mu_F(\sigma, \alpha)$ when $0 < \sigma < 1$ follows by convexity, and the proof of Theorem 1 is complete.

2.2 - Proof of Theorem 2

We only give a sketch of the argument, pointing out the main changes with respect to Section 3.1 of [9], where a similar result is proved in a special case; further details can be found there.

Following the argument in Section 3.1 of [9], in order to detect the zeros of $F(s, \alpha)$ for σ large and negative, and $|t|$ sufficiently large, we have to detect the zeros of $\overline{F}_0(1-s, \alpha)$ in the same region. Indeed, by Stirling's formula the term with $\ell = 0$ dominates the sum over ℓ in (1.10). Moreover, the functions $R(1-s, \alpha)$ and $H_k(s, \alpha)$ in (1.10) are small with respect to such a sum in suitable neighborhoods of the zeros, provided $|t|$ is sufficiently large. This can be checked directly for $R(1-s, \alpha)$ using Stirling's formula, and follows from a weak uniform version in σ of (1.11) in the case of $H_k(s, \alpha)$. The required uniform bound is obtained analysing the dependence on k in Lemma 3.2 of [10]; note that a rather weak bound is sufficient here in view of (1.10) and (1.8), since σ is negative and large. We do not provide details of such routine estimates here. Our claim follows then by an application of Rouché's theorem, as in Section 3.1 of [9].

Let $\sigma < 0$ and $|\sigma|, |t|$ be sufficiently large. As remarked in Section 1.2, the series

$$(2.2) \quad \bar{F}_0(1-s, \alpha) = \sum_{j=-N}^N a_j e^{i\pi d\omega_j s} \sum_{n \geq 1}^b \frac{\overline{a(n)}}{n^{1-s}} \left(1 + e^{i\pi(\frac{1}{2}-\omega_j)} \left(\frac{n_\alpha}{n} \right)^{1/d} \right)^{d(s-s_0^*)}$$

is absolutely convergent; see (1.6) for the meaning of the symbol b . An inspection of the above quantities shows that (2.2) can be rewritten as

$$(2.3) \quad \begin{aligned} \bar{F}_0(1-s, \alpha) &= n_\alpha^s \sum_{j=-N}^N \sum_{n \geq 1}^b c_{j,n} (i + e^{i\pi\omega_j} \xi_n)^{ds} \\ &= n_\alpha^s \sum_{j=-N}^N \sum_{n \geq 1}^b c_{j,n} \theta_{j,n}(\sigma, t) \rho_{j,n}^{-d|\sigma|} e^{-dt\phi_{j,n}}, \end{aligned}$$

where

$$\begin{aligned} \rho_{j,n} &= |i + e^{i\pi\omega_j} \xi_n|, \quad \phi_{j,n} = \arg(i + e^{i\pi\omega_j} \xi_n), \quad \xi_n = (n/n_\alpha)^{1/d}, \\ c_{j,n} &= a_j \frac{\overline{a(n)}}{n} \left(1 + e^{i\pi(\frac{1}{2}-\omega_j)} \left(\frac{n_\alpha}{n} \right)^{1/d} \right)^{-ds_0^*} \quad \text{and} \quad \theta_{j,n}(\sigma, t) = e^{id(t \log(\rho_{j,n}) + \sigma \phi_{j,n})}. \end{aligned}$$

Thanks to the Ramanujan conjecture we have that $|c_{j,n}| \leq C$ for some $C > 0$ depending at most on $F(s)$ and α , and clearly $|\theta_{j,n}(\sigma, t)| = 1$. Moreover,

$$\rho_{-N,n} = |1 - \xi_n| < \rho_{j,n} \text{ for every } j \neq -N \text{ and } n \neq n_\alpha$$

and, as n varies over the values such that $a(n) \neq 0$,

the minimum of $\rho_{-N,n}$ is attained at most at two points $n_1 < n_\alpha < n_2$, say;

see Section 2.1 of [10]. If $\rho_{-N,n}$ has only one minimum, then either n_1 or n_2 does not exist and we replace its definition by

$$n_1 = \max\{n < n_\alpha : a(n) \neq 0\} \quad \text{or} \quad n_2 = \min\{n > n_\alpha : a(n) \neq 0\},$$

respectively. Note that, in principle, n_1 may not exist also with this second definition, since it may happen that $a(n) = 0$ for every $n < n_\alpha$. However, we always have that

$$\rho_{-N,n_1} < \rho_{j,n} \text{ for every } (j, n) \neq (-N, n_1) \text{ with } n < n_\alpha \text{ and } a(n) \neq 0,$$

$$\rho_{-N,n_2} < \rho_{j,n} \text{ for every } (j, n) \neq (-N, n_2) \text{ with } n > n_\alpha \text{ and } a(n) \neq 0;$$

of course, the first line is meaningful only if n_1 really exists.

Now we consider separately two cases. Suppose first that $t > 0$. Clearly we have

$$(2.4) \quad e^{-dt\phi_{-N,n}} = \begin{cases} e^{-dt\pi/2} & \text{if } n < n_\alpha \\ e^{dt\pi/2} & \text{if } n > n_\alpha, \end{cases}$$

hence the term in (2.3) with $(j, n) = (-N, n_2)$ dominates over all other terms with $n > n_\alpha$ for $\sigma < 0$ with $|\sigma|$ sufficiently large, since always $|\phi_{j,n}| \leq \pi/2$. As a consequence, writing

$$c_{j,n}(\sigma, t) = c_{j,n,\ell} \theta_{j,n}(\sigma, t),$$

in this case (2.3) becomes

$$(2.5) \quad \begin{aligned} \bar{F}_0(1-s, \alpha) = n_\alpha^s \Big(\sum_{j=-N}^N \sum_{n \leq n_\alpha}^b c_{j,n}(\sigma, t) \rho_{j,n}^{-d|\sigma|} e^{-dt\phi_{j,n}} + c_{-N,n_2}(\sigma, t) \rho_{-N,n_2}^{-d|\sigma|} e^{dt\pi/2} \Big) \\ \times (1 + o(1)). \end{aligned}$$

Suppose now that $t < 0$. In view of (2.4), the term in (2.3) with $(j, n) = (-N, n_1)$ dominates over all the other terms with $n < n_\alpha$, provided n_1 exists. Moreover, for every $j = -N, \dots, N$ we have that

$$\rho_{j,n} > \rho_{j,n_2} \text{ and } \phi_{j,n} \leq \phi_{j,n_2} \text{ for } n > n_2.$$

Therefore, in this case (2.3) becomes

$$(2.6) \quad \begin{aligned} \bar{F}_0(1-s, \alpha) = n_\alpha^s \Big(c_{-N,n_1}(\sigma, t) \rho_{-N,n_1}^{-d|\sigma|} e^{-dt\pi/2} \\ + \sum_{j=-N}^N \sum_{n_\alpha \leq n \leq n_2}^b c_{j,n}(\sigma, t) \rho_{j,n}^{-d|\sigma|} e^{-dt\phi_{j,n}} \Big) \times (1 + o(1)). \end{aligned}$$

As we shall see in a moment, what really matters in (2.5) and (2.6) is that the double sum contains finitely many terms. Hence, in the present general case, we do not investigate the possibility of excluding further non-dominating terms from the right hand side of (2.5) and (2.6), although this is certainly possible and easy to do in any concrete case.

Finally we observe that for $\sigma < 0$ and $|\sigma|, |t|$ sufficiently large both (2.5) and (2.6) have the form

$$(2.7) \quad \bar{F}_0(1-s, \alpha) = n_\alpha^s \left(\sum_{k=1}^K c_k \theta_k(\sigma, t) A_k^\sigma B_k^t \right) (1 + o(1))$$

with some $K \geq 1$, $A_k, B_k > 0$, $|c_k| \leq C$ and $|\theta_k(\sigma, t)| = 1$. Hence, a necessary condition for the vanishing of the main term in the right hand side of (2.7) is that at least two of the terms in the sum have the same modulus. Since this implies a linear relation between σ and t , thanks to Rouché's theorem for $\sigma < 0$ and $|\sigma|, |t|$ sufficiently large the zeros of $\bar{F}_0(1-s, \alpha)$ lie in an arbitrarily narrow tubular region around a finite number of lines. In view of the argument at the beginning of this section, the sketch of proof of Theorem 2 is now complete.

2.3 - Proof of Theorem 3

Again we give only a sketch of the proof of Theorem 3 since it goes along fairly standard lines, modulo some technical points. Actually, we follow the argument and the notation in Section 3.2 of [9]; in particular, let $a, b, \bar{n}, N^\pm(T), T_0, \mathcal{R}^\pm$ and $\mathcal{L}_j$ be as in that section. We recall that a and b are positive and sufficiently large; moreover, in the present case $T_0 > 0$ is sufficiently large in terms of a . Then, exactly as in Section 3.2 of [9], we have

$$N_F(T, \alpha) = N^+(T) + N^-(T) + O(1),$$

$$N^+(T) = \frac{1}{2\pi} \sum_{j=1}^4 \Delta_{\mathcal{L}_j} \arg(F(s, \alpha)) = \frac{1}{2\pi} (\Delta_1^+ + \cdots + \Delta_4^+),$$

say, and

$$\Delta_1^+ = O(1), \quad \Delta_2^+ = -T \log \bar{n} + O(1), \quad \Delta_3^+ = O(\log T).$$

On $\mathcal{L}_4$ we use the functional equation of $F(s, \alpha)$; recall that $\sigma = -a$ and $t \geq T_0 > 0$ for $s \in \mathcal{L}_4$. Since the shape of the functional equation in the present general case is more complicated than in [9], here we give some details. Recalling (1.4), from the properties of the functions $\bar{F}_\ell(s, \alpha)$ and $H_k(s, \alpha)$ (see in particular (1.8), (1.10) and (1.11)) and Stirling's formula we have that for $s \in \mathcal{L}_4$

$$(2.8) \quad F(s, \alpha) = \frac{\omega_F}{\sqrt{2\pi}} \left(\frac{q^{1/d}}{2\pi d} \right)^{d(\frac{1}{2}-s)} d_0 \Gamma(d(1-s) - e_0) \bar{F}_0(1-s, \alpha) \\ \times (1 + O(t^{-1})) + O(t^{-\theta})$$

for some $\theta > 0$. As in Section 2.2, we have to detect the term in $\bar{F}_0(1-s, \alpha)$ which dominates over the whole sum on $\mathcal{L}_4$. Taking into account that $\sigma = -a$, $t \geq T_0$ and now T_0 is sufficiently large in terms of a , a similar analysis shows

that for $s \in \mathcal{L}_4$ the dominating term in (2.3) must have $\phi_{j,n} = -\pi/2$. Thus, in view of (2.4), it is the one with $(j, n) = (-N, n_+)$, where

$$n_+ = \min\{n > n_\alpha : a(n) \neq 0\}.$$

Hence, writing

$$m_+ = n_+^{1/d} - n_\alpha^{1/d}$$

and observing that the real part of the argument in the Γ factor in (2.8) is $> 1/2$, from (2.8) and (2.3) we deduce that for $s \in \mathcal{L}_4$

$$F(s, \alpha) = c \left(\frac{q^{1/d}}{2\pi d} \right)^{-idt} \Gamma(d(1-s) - e_0) m_+^{idt} e^{\frac{\pi}{2}dt} (1 + o(1))$$

with a certain $c = c(a)$. Stirling's formula gives then

$$\Delta_4^+ = dT \log T + T \log \left(\frac{q}{(2\pi e m_+)^d} \right) + O(\log T),$$

hence

$$N^+(T) = \frac{d}{2\pi} T \log T + \frac{T}{2\pi} \log \left(\frac{q}{\bar{n}(2\pi e m_+)^d} \right) + O(\log T).$$

The treatment of $N^-(T)$ is analogous, this time computing the variation of the argument of $F(s, \alpha)$ along the rectangle $\mathcal{R}^-$. Using an obvious notation, again we give some details only for the computation of Δ_4^- . For $t \leq -T_0$ the dominating term in (2.3) must have $\phi_{j,n} = \pi/2$, which is attained if $j = -N$ and $n < n_\alpha$ or if $j = N$ and $n \geq 1$ arbitrary. Since $\rho_{\pm N, n} = |1 \pm \xi_n|$, again an analysis as in the previous section shows that such dominating term corresponds to $(j, n) = (-N, n_-)$ with

$$n_- = \max\{n < n_\alpha : a(n) \neq 0\} \quad \text{if there exists } n < n_\alpha \text{ with } a(n) \neq 0,$$

or $(j, n) = (N, n_-)$ with

$$n_- = \min\{n \geq n_\alpha : a(n) \neq 0\} \quad \text{otherwise.}$$

Hence we write

$$m_- = \begin{cases} n_\alpha^{1/d} - n_-^{1/d} & \text{if } n_- < n_\alpha \\ n_\alpha^{1/d} + n_-^{1/d} & \text{if } n_- \geq n_\alpha \end{cases}$$

and arguing as before we finally obtain that

$$N_-(T) = d \frac{T}{2\pi} \log T + \frac{T}{2\pi} \log \left(\frac{q}{\bar{n}(2\pi e m_-)^d} \right) + O(\log T),$$

thus completing the sketch of the proof of Theorem 3.

2.4 - Proof of Theorem 4

Let $\varepsilon > 0$ be arbitrarily small, $\delta > 0$ be sufficiently small in terms of ε and let $10 \leq T \leq x$. We start with the standard truncated Perron's formula, where the integration is over the segment $[1 + 1/\log x - iT, 1 + 1/\log x + iT]$ and the error is $O(x^{1+\varepsilon}/T)$ thanks to the Ramanujan conjecture, and then shift the vertical segment to $\sigma = -\delta$. A standard computation based on Theorem 1 shows that

$$(2.9) \quad S_F(x, \alpha) = r(x, \alpha) + \frac{1}{2\pi i} \int_{-\delta-iT}^{-\delta+iT} F(s, \alpha) \frac{x^s}{s} ds + O\left(\frac{T^{d/2} + x}{T} x^\varepsilon\right),$$

where $r(x, \alpha)$ is the sum of the residues at the possible simple poles of the integrand at $s = s_k^*$ with $s_k > 0$ and at the pole in the point $s = 0$.

Now we apply (1.10) with $s = -\delta + it$. By (1.11), (1.4) and Stirling's formula we have that the contribution to the integral in (2.9) of the terms $R(1-s, \alpha)$ and $H_k(s, \alpha)$ (with the relevant value of k) is absorbed into the error term in (2.9). Moreover, thanks to (1.8) and Stirling's formula, the same holds for the terms with $\ell \geq 1$, if any, in (1.10). Hence we have to deal only with the term $\ell = 0$ in (1.10); let it be denoted by $G_0(s, \alpha)$.

In order to treat $G_0(s, \alpha)$ we use once more the absolutely convergent expansion (1.6) of $\bar{F}_0(1-s, \alpha)$, and apply Stirling's formula to the Γ factor with $\ell = 0$ in (1.10). Writing

$$e^{as+b} = \frac{\omega_F}{\sqrt{2\pi}} \left(\frac{q^{1/d}}{2\pi d} \right)^{\frac{d}{2}-ds},$$

a computation taking into account (2.2) shows that for $s = -\delta + it$ with large $|t|$ we have, using a different notation with respect to (2.3), that

$$(2.10) \quad G_0(s, \alpha) \frac{x^s}{s} = cx^{-\delta} \sum_{j=-N}^N \sum_{n \geq 1}^b b_{j,n} g_{j,n}(t) e^{if_{j,n}(t,x)} (1 + O(\frac{1}{|t|})),$$

where $c \neq 0$ is a certain constant,

$$b_{j,n} = e^{-a\delta} a_j e^{-i\pi d \omega_j \delta} \frac{\overline{a(n)}}{n^{1+\delta}} \rho_j(n, \alpha)^{-d(\delta+s_0^*)}, \quad \rho_j(n, \alpha) = 1 + e^{i\pi(\frac{1}{2}-\omega_j)} \left(\frac{n_\alpha}{n} \right)^{1/d},$$

$$g_{j,n}(t) = |dt|^{d(\delta+\frac{1}{2})} t^{-1} e^{-\frac{\pi}{2}d|t|} e^{-dt(\pi\omega_j + \arg(\rho_j(n, \alpha)))}$$

and

$$f_{j,n}(t, x) = -dt \log |dt| + (a + d)t + t \log (nx |\rho_j(n, \alpha)|^d) + f(t),$$

with $f(t) = -\frac{\pi}{2}(d\delta + d/2)\text{sgn}(t)$. Note, in view of (1.5) and (1.7), that by elementary geometry we have

$$|\pi\omega_j + \arg(\rho_j(n, \alpha))| \leq \frac{\pi}{2}$$

for every $j = -N, \dots, N$ and $n \geq 1$ appearing in (2.10). Hence, as before, the contribution of the error term in (2.10) to the integral in (2.9) is absorbed into the error term present in (2.9).

The contribution of the main term in (2.10) is now estimated by the second derivative test for oscillating integrals, see Lemma 5.1.3 of Huxley [2]. To this end we first proceed to a dyadic dissection of the range of integration in (2.9) and, writing $T_\nu = T2^{-\nu}$, we denote by I_ν the integral of $g_{j,n}(t)e^{if_{j,n}(t,x)}$ over $[-T_\nu, -T_{\nu+1}] \cup [T_{\nu+1}, T_\nu]$. Since the functions $g_{j,n}(t)$ are eventually monotonic we have that

$$I_\nu \ll T_\nu^{(d-1)/2+d\delta}$$

uniformly in j, n and x . Hence the contribution of the main term in (2.10) is

$$\ll x^{-\delta} \sum_{0 \leq \nu \ll \log T} T_\nu^{(d-1)/2+d\delta} \ll T^{(d-1)/2} x^\varepsilon.$$

Gathering all previous computations we finally deduce that

$$\begin{aligned} (2.11) \quad S_F(x, \alpha) &= r(x, \alpha) + O\left(\frac{T^{(d+1)/2} + x}{T} x^\varepsilon\right) \\ &= r(s, \alpha) + O\left(x^{\frac{d-1}{d+1} + \varepsilon}\right) \end{aligned}$$

by the choice $T = x^{2/(d+1)}$. Theorem 4 follows now from (2.11), recalling (1.2) and observing that

$$r(x, \alpha) = \frac{2d}{d+1} \rho_0(\alpha) x^{\frac{d+1}{2d} - i\theta_F} + O\left(x^{\frac{d-1}{2d}}\right),$$

where $\rho_0(\alpha)$ is the residue of $F(s, \alpha)$ at the point $s_0^* = \frac{d+1}{2d} - i\theta_F$.

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