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Roth's ascending powers in the primes

Abstract. The number of variables in Roth's ascending powers problem with prime variables is reduced by three.

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1 - The problem

In 1951 Roth [12] considered the diophantine equation

$$(1) \quad n = \sum_{k=1}^s x_k^{k+1}$$

and showed that when $s = 50$ and the natural number n is sufficiently large, then (1) admits solutions in natural numbers x_j . There is therefore a smallest s with the latter property, say s_0 , and Roth's result is then that $s_0 \leq 50$. This problem attracted many researchers in additive number theory. The upper bound for s_0 has been improved by Thanigasalam [13], Vaughan [16, 17], Thanigasalam [14, 15], this writer [1, 2] and Ford [5]. The current record, due to Ford [6], is $s_0 \leq 14$. It is immediate that $s_0 \geq 3$, and we expect that $s_0 = 3$. Indeed Roth [11] showed that when $s = 3$ then almost all natural numbers n can be written in the form (1) with $x_j \in \mathbb{N}$.

Vaughan [17] studied the equation (1) with the variables x_j constrained to sequences that are not too thin. In particular, it follows from his work that when $s \geq 30$ and $n \equiv s \pmod{2}$ is large, then (1) has solutions in primes x_j . Thanigasalam [14] obtained the same result when $s \geq 23$. The work

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in [14] on Roth's original question extends to prime variables in a natural way, but later contributions to Roth's ascending powers depend on a complex interplay of major and minor arc technology [1, 2, 15] or on mean values for smooth Weyl sums [5, 6]. Both techniques do not immediately imply results with prime variables, and so, Thanigasalam's work [14] involving 23 primes remained unsurmounted for fourty years. Here we make redundant three of these primes.

Theorem. *Let $s = 20$. Then all sufficiently large even numbers n have a representation in the form (1), with all x_j prime.*

Our result has the trivial corollary that *whenever $s \geq 21$ then for all large n the equation (1) has solutions with all x_j prime.*

We establish the theorem by an iterated use of the Hardy-Littlewood method. This strategy is familiar from the work of Vaughan [18], but its implementation for the problem at hand turns out to be rather delicate. We will have recourse to recent work of Zhao [20] on the exponential sum over differenced cubes on which we comment in due course. Beyond that, our work is a blend of existing techniques so that we go in medias res immediately.

Throughout we apply the convention that whenever the letter ε occurs in a statement then it is asserted that the statement is true for any positive real value assigned to ε . Constants hidden in Landau's or Vinogradov's well known symbols may depend on the value of ε .

2 - Greetings from the past

We recall an almost forgotten concept in additive number theory. Given natural numbers $k_1, \dots, k_r$ and real numbers $\lambda_1, \dots, \lambda_r$ with $0 < \lambda_j \leq 1$, let $U(N) = U(N; \mathbf{k}, \boldsymbol{\lambda})$ denote the number of solutions of the diophantine equation

$$(2) \quad \sum_{j=1}^r (x_j^{k_j} - y_j^{k_j}) = 0$$

with the variables constrained to

$$N^{\lambda_j/k_j} < x_j \leq 2N^{\lambda_j/k_j}, \quad N^{\lambda_j/k_j} < y_j \leq 2N^{\lambda_j/k_j} \quad (1 \leq j \leq r).$$

We say that the sequence (k_j, λ_j) with $1 \leq j \leq r$ form *admissible exponents with weight*

$$w = w(\mathbf{k}, \boldsymbol{\lambda}) = \sum_{j=1}^r \frac{\lambda_j}{k_j}$$

if the estimate $U(N) \ll N^{w+\varepsilon}$ holds. This upper bound is nearly best possible; solutions of (2) with $x_j = y_j$ show that $U(N) \gg N^w$. The idea goes back to Hardy and Littlewood [7], the terminology to Davenport and Erdős [4].

We illustrate this with an example. Take $k_1 = 18, k_2 = 17, k_3 = 12, k_4 = 11$. We wish to construct admissible exponents. Put

$$\theta_1 = \theta_2 = 1, \quad \theta_3 = \frac{16}{17}, \quad \theta_4 = \frac{16}{17} \cdot \frac{11}{12} \quad \text{and} \quad \delta_3 = \frac{7}{264}.$$

Then, with $\delta_1 = \delta_2 = 0$ and $\delta_4 = \delta_3$, the numbers $\lambda_j = (1 + \delta_j)\theta_j$ make $(18, 1), (17, 1), (12, \lambda_3), (11, \lambda_4)$ admissible exponents with weight

$$w_0 = \frac{1}{18} + \frac{1}{17} + (1 + \delta_3) \left(\frac{\theta_3}{12} + \frac{\theta_4}{11} \right) > 0.275401.$$

This follows from [14, Theorem 3]. We now apply Davenport's iterative method [3] to add in an eighth power. The augmented version of this method due to Vaughan [18, Lemma 4] allows us to perform the iteration within the framework of admissible exponents. In Vaughan's Lemma 4 we take $k = 8, j = 2$ and

$$\lambda = 3/(3 + w_0),$$

and we let $R(n)$ denote the number of solutions of the equation

$$n = x_1^{18} + x_2^{17} + x_3^{12} + x_4^{11}$$

with $N^{\lambda\lambda_j/k_j} < x_j \leq 2N^{\lambda\lambda_j/k_j}$. Then, as one readily confirms, Vaughan's Lemma 4 shows that $(18, \lambda), (17, \lambda), (12, \lambda\lambda_3), (11, \lambda\lambda_4), (8, 1)$ are admissible exponents with weight

$$w_1 = \frac{1}{8} + \lambda w_0 > 0.37724.$$

Repetition of this argument with $k = 6, j = 2$ and $\lambda' = 3/(3 + w_1)$ (for λ) in Lemma 4 of Vaughan [18] confirms that $(18, \lambda\lambda'), (17, \lambda\lambda'), (12, \lambda\lambda'\lambda_3), (11, \lambda\lambda'\lambda_4), (8, \lambda'), (6, 1)$ form admissible exponents with weight

$$(3) \quad w_2 = \frac{1}{6} + \lambda' w_1 > \frac{1}{2}.$$

Now define the exponents

$$l_1 = 18, l_2 = 17, l_3 = 12, l_4 = 11, l_5 = 8, l_6 = 6, l_7 = 2$$

and the numbers

$$\mu_1 = \mu_2 = \lambda\lambda', \quad \mu_3 = \lambda\lambda'\lambda_3, \quad \mu_4 = \lambda\lambda'\lambda_4, \quad \mu_5 = \lambda', \quad \mu_6 = \mu_7 = 1.$$

The lower bound for w_2 shows that

$$v = \sum_{j=1}^7 \frac{\mu_j}{l_j}$$

exceeds 1. The following result is our first basic estimate.

Lemma 1. *One has $U(N; \mathbf{l}, \boldsymbol{\mu}) \ll N^{2v-1+\varepsilon}$.*

Proof. Recall the equation (2) where now the k_j are the l_j . The solutions with $x_7 = y_7$ contribute

$$\ll N^{1/2} U(N; (l_1, \dots, l_6), (\mu_1, \dots, \mu_6)) \ll N^{1/2+w_2+\varepsilon}$$

which is satisfactory in view of (3). For the remaining solutions, we choose x_j, y_j ($1 \leq j \leq 6$) with

$$(4) \quad \sum_{j=1}^6 (x_j^{l_j} - y_j^{l_j}) \neq 0.$$

There are $O(N^{2v-1})$ such choices of which we fix one now. The number on the left hand side of (4) is bounded by $O(N)$, and equals $y_7^2 - x_7^2 = (y_7 - x_7)(y_7 + x_7)$. A divisor function estimate shows that there are no more than $O(N^\varepsilon)$ choices for x_7, y_7 . The lemma follows immediately. $\square$

3 - Stay tuned

We require an estimate of the same precision as in Lemma 1 for the remaining powers. The smallest available power now the cube, this forces us to apply the circle method in the final iteration. We use the exponents

$$k_1 = 20, k_2 = 19, k_3 = 16, k_4 = 15, k_5 = 14, k_6 = 13, k_7 = 10, k_8 = 9, k_9 = 7$$

and apply [14, Theorem 3]. Then, with $\theta_1 = \theta_2 = 1$ and $\theta_{j+1} = \theta_j(1 - k_j^{-1})$ for $j \geq 2$ one finds that the numbers δ_j defined by (11) and (12) of Thanigasalam [14] are non-negative, and therefore there are numbers λ_j with $\theta_j \leq \lambda_j \leq 1$ such that the sequence (k_j, λ_j) are admissible exponents. Their weight w'_0 satisfies

$$w'_0 \geq \sum_{j=1}^9 \frac{\theta_j}{k_j} > 0.56278.$$

The numerical value for w'_0 may be increased by computing the δ_j in Theorem 3 of [14]. If one uses Davenport's iterative method for the seventh power as in the deduction of our Lemma 1, then again one obtains a slightly larger value for w'_0 . However, these marginal improvements leave no trace in the final result.

We follow the tactics applied in the previous section and add in the remaining powers step by step. We begin by choosing $k = 5$, $j = 3$ and $\lambda = 32/(35 + 5w'_0)$ in [18, Lemma 4] to confirm that the sequence $(k_j, \lambda\lambda_j)$ with $1 \leq j \leq 9$ together with $(5, 1)$ form admissible exponents with weight

$$w'_1 = \frac{1}{5} + \lambda w'_0 > 0.67625.$$

Then we add in the fourth power by taking $k = 4$, $j = 2$ and $\lambda' = 3/(3 + w'_1)$ (for λ) in [18, Lemma 4]. One then finds that $(4, 1)$, $(5, \lambda')$ and $(k_j, \lambda\lambda'\lambda_j)$ ($1 \leq j \leq 9$) form admissible exponents with weight w'_2 where

$$w'_2 = \frac{1}{4} + \lambda' w'_1 > 0.8015.$$

For the success of our method it is vital that $w'_2 \geq 4/5$, so there is not that much to spare. We will now add in a cube by the Hardy-Littlewood method, and this will allow us to conclude as follows. We define

$$m_1 = 20, m_2 = 19, m_3 = 16, m_4 = 15, m_5 = 14, m_6 = 13, m_7 = 10,$$

$$m_8 = 9, m_9 = 7, m_{10} = 5, m_{11} = 4, m_{12} = 3$$

and

$$\nu_j = \frac{5}{6} \lambda \lambda' \lambda_j \quad (1 \leq j \leq 9), \quad \nu_{10} = \frac{5}{6} \lambda', \quad \nu_{11} = \frac{5}{6}, \quad \nu_{12} = 1$$

as well as

$$u = \sum_{j=1}^{12} \frac{\nu_j}{k_j}.$$

Note that

$$u = \frac{1}{3} + \frac{5}{6} w'_2 > 1.$$

Lemma 2. *One has $U(N, \mathbf{m}, \boldsymbol{\nu}) \ll N^{2u-1+\varepsilon}$.*

4 - Zhao's device

In this section we establish Lemma 2. It is here where the recent work of Zhao [20] is applied. Define the function $\omega : \mathbb{R} \rightarrow \mathbb{R}$ by

$$\omega(t) = \exp(1/((t - 3/2)^2 - 1)) \quad \text{when } \frac{1}{2} < t < \frac{5}{2},$$

and $\omega(t) = 0$ for all other t . Note that ω is smooth. We remark that Zhao [20] works with a slightly differently scaled weight, but it is clear from his argument that any compactly supported smooth function can be substituted for ω , so his estimates apply with our choice of ω . Let $P = N^{1/3}$ and put

$$f(\alpha) = \sum_{x=1}^{\infty} \omega(x/P) e(\alpha x^3).$$

Further, for $1 \leq j \leq 11$ let $M_j = N^{\nu_j/m_j}$ and

$$f_j(\alpha) = \sum_{M_j < y \leq 2M_j} e(\alpha y^{m_j}).$$

We claim that

$$(5) \quad U(N, \mathbf{m}, \nu) \ll \int_0^1 |f(\alpha) f_1(\alpha) f_2(\alpha) \dots f_{11}(\alpha)|^2 d\alpha.$$

Indeed, by orthogonality, the integral in (5) counts the number of solutions of

$$(6) \quad y^3 - x^3 = \sum_{j=1}^{11} (x_j^{m_j} - y_j^{m_j})$$

in integers constrained by $M_j < x_j, y_j \leq 2M_j$, each solution counted with weight $\omega(x/P)\omega(y/P)$. But $\omega(t) \geq \exp(-\frac{4}{3})$ for $1 \leq t \leq 2$, so (5) follows.

Now consider solutions of (6) where $x = y$. There are no more than $O(P)$ such choices where $\omega(x/P) \neq 0$, and we can use that the numbers $(4, 1)$, $(5, \lambda')$ and $(m_j, \lambda' \lambda_j)$ are admissible exponents to conclude that the solutions of (6) with $x = y$ contribute to (5) no more than

$$(7) \quad \ll P^{1+\varepsilon} M_1 M_2 \dots M_{11} \ll N^{u+\varepsilon} \ll N^{2u-1}$$

which is acceptable.

Next observe that solutions of (6) with $x > y$ will make the same contribution as the solutions with $y > x$, even when counted with weight $\omega(x/P)\omega(y/P)$. Hence, we concentrate on solutions with $y > x$ and put $y = x + h$. Then $h \geq 1$, and (6) transforms into

$$(8) \quad h(3x^2 + 3xh + h^2) = \sum_{j=1}^{11} (x_j^{m_j} - y_j^{m_j}),$$

solutions now being counted with weight $\omega(x/P)\omega((x+h)/P)$. For any choice of natural numbers x_j, y_j with $x_j \leq 2M_j$, $y_j \leq 2M_j$, the right hand side in (8)

is bounded above by $(2N^{5/24})^4 + o(N^{5/6}) \leq 18N^{5/6}$, at least when N is large. Further $\omega(x/P) > 0$ implies $x > \frac{1}{2}P$ whence, by (8), we have $h \leq H$ where

$$H = 24N^{1/2}.$$

Again, this is not the same scaling as in Zhao's work [20], he has 24 replaced by a smaller number, but as before, his estimates are not affected. We put

$$F(\alpha) = \sum_{1 \leq h \leq H} \sum_{x=1}^{\infty} \omega(x/P) \omega((x+h)/P) e(\alpha h(3x^2 + 3xh + h^2)).$$

By orthogonality and the preceding discussions, it follows from (5) that

$$(9) \quad U(N, \mathbf{m}, \nu) \ll N^{2u-1} + \int_0^1 F(\alpha) |f_1(\alpha) \cdots f_{11}(\alpha)|^2 d\alpha.$$

We apply the circle method to the integral on the right hand side of (9). When $0 \leq a \leq q$, $1 \leq q \leq P^{3/4}$ and $(a, q) = 1$ let $\mathfrak{M}(q, a)$ denote the subinterval of $[0, 1]$ defined by $|\alpha q - a| \leq P^{-7/4}$, and let $\mathfrak{M}$ denote the union of the $\mathfrak{M}(q, a)$. Define the function $\Upsilon : [0, 1] \rightarrow \mathbb{R}$ by

$$\Upsilon(\alpha) = q^{-2/3} H P (1 + H P^2 |\alpha - (a/q)|)^{-1} \quad \text{when } \alpha \in \mathfrak{M}(q, a),$$

and by $\Upsilon(\alpha) = 0$ otherwise. The next lemma is the core of Zhao's work [20].

Lemma 3. *Uniformly for $\alpha \in [0, 1]$ one has $F(\alpha) \ll P^{1+\varepsilon} + P^\varepsilon \Upsilon(\alpha)$.*

Proof. We have mentioned before that H and ω in [20] differ slightly by irrelevant constants from ours, but with this observation in mind the function $\mathcal{F}^+$ in [20] is our $F(\alpha)$. Hence, Lemma 4.1 and 4.2 in [20] show that $F(\alpha) \ll P^{1+\varepsilon}$ holds for $\alpha \notin \mathfrak{M}$. For $\alpha \in \mathfrak{M}$ the desired estimate for $F(\alpha)$ is contained in the first display on p. 774 of Zhao [20]. $\square$

We apply Lemma 3 within (9) and obtain

$$U(N, \mathbf{m}, \nu) \ll N^{2u-1} + P^{1+\varepsilon} I_1 + P^\varepsilon I_2$$

where

$$I_1 = \int_0^1 |f_1(\alpha) \cdots f_{11}(\alpha)|^2 d\alpha, \quad I_2 = \int_{\mathfrak{M}} \Upsilon(\alpha) |f_1(\alpha) \cdots f_{11}(\alpha)|^2 d\alpha.$$

By orthogonality, the estimation of I_1 is the same as the count of the diagonal solutions (with $x = y$) in (6), and so, by (7), we have $P I_1 \ll N^{2u-1+\varepsilon}$. Hence, in order to establish Lemma 2, it suffices to show that $I_2 \ll N^{2u-1+\varepsilon}$.

To initiate the treatment of I_2 , we apply [19, Theorem 4.1 and Theorem 4.2] with $k = 4$ to see that whenever $\alpha \in \mathfrak{M}(q, a)$ then

$$f_{11}(\alpha) \ll q^{1/2+\varepsilon} + q^{-1/4}N^{5/24} \ll q^{-1/4}N^{5/24}.$$

Consequently we have

$$\Upsilon(\alpha)|f_{11}(\alpha)|^2 \ll HPN^{5/12}(1 + HP^2|\alpha - (a/q)|)^{-1}q^{-7/6}$$

and may now follow the proof of [2, Lemma 1], with $\Psi(\alpha) = |f_1 f_2 \cdots f_{10}|^2$ in the notation of this source. The argument given there then yields

$$I_2 = \int_{\mathfrak{M}} \Upsilon(\alpha)|f_{11}(\alpha)|^2 \Psi(\alpha) d\alpha \ll J_1 + J_2$$

where $J_1 \ll P^{\varepsilon-1}N^{5/12}\Psi(0) \ll N^{2u-1+\varepsilon}$ and

$$J_2 = P^{\varepsilon-1}N^{5/12} \sum_{q \leq P^{3/4}} q^{-1/6} \int_0^1 |f_1 f_2 \cdots f_{10}|^2 d\alpha.$$

Routinely we find that

$$J_2 \ll P^{\varepsilon-1}N^{5/12}P^{5/8}M_1 \cdots M_{10} \ll N^{2u-1},$$

and the lemma follows.

5 - The theorem

We are ready to establish the theorem. Let $L = \{l_j : 1 \leq j \leq 7\}$ be the exponents occurring in Lemma 1, and let $M = \{m_j : 1 \leq j \leq 12\}$ be the numbers in Lemma 2. Then $[2, 20] \cap \mathbb{N}$ is the disjoint union of L and M . If $k \in L$, then $k = l_j$ for some j , and we put $\psi_k = \mu_j$. If $k \in M$, then $k = m_j$ for some j , and we put $\psi_k = \nu_j$. Also, put $\psi_{21} = 1$.

Now let n be a large natural number and put $N = n/8$. For $2 \leq k \leq 21$ let

$$g_k(\alpha) = \sum_{N^{\psi_k/k} < p \leq 2N^{\psi_k/k}} e(\alpha p^k) \log p$$

where the sum is over primes p . By orthogonality, the integral

$$\varrho(n) = \int_0^1 g_2(\alpha)g_3(\alpha) \cdots g_{21}(\alpha)e(-\alpha n) d\alpha$$

counts certain solutions of (1) in primes x_j with weight no larger than $(\log n)^{20}$. Thus, showing that $\rho(n) > 0$ holds for all large even n establishes the Theorem.

To define major arcs, let $\Delta = 2^{-40}$ and $\delta = \Delta^3$. For $0 \leq a \leq q$, $1 \leq q \leq N^\Delta$ and $(a, q) = 1$ let $\mathfrak{N}(q, a)$ be the subinterval of $[0, 1]$ defined by $|q\alpha - a| \leq N^{\Delta-1}$, and let $\mathfrak{N}$ denote the union of these intervals. The minor arcs $\mathfrak{n} = [0, 1] \setminus \mathfrak{N}$ are swiftly dealt with. A routine application of Harman [8, Theorem 1], for example, shows that $g_{21}(\alpha) \ll N^{1/21-2\delta}$ holds uniformly for $\alpha \in \mathfrak{n}$. Moreover, on considering the underlying diophantine equations, Lemmata 1 and 2 yield

$$\int_0^1 \prod_{l \in L} |g_l(\alpha)|^2 d\alpha \ll N^{2v-1+\varepsilon}, \quad \int_0^1 \prod_{m \in M} |g_m(\alpha)|^2 d\alpha \ll N^{2u-1+\varepsilon}.$$

Hence, by Schwarz's inequality,

$$(10) \quad \int_{\mathfrak{n}} |g_2(\alpha)g_3(\alpha) \cdots g_{21}(\alpha)| d\alpha \ll N^{u+v-20/21-\delta}.$$

A thoroughly standard argument also deals with the core of the major arcs. Let $Q = (\log N)^9$, and when $0 \leq a \leq q$, $1 \leq q \leq Q$, $(a, q) = 1$ let $\mathfrak{K}(q, a)$ denote the subset of $[0, 1]$ defined by $|\alpha - (a/q)| \leq Q/N$. Write $\mathfrak{K}$ for the union of all $\mathfrak{K}(q, a)$. Note that $\mathfrak{K} \subset \mathfrak{N}$.

Next, define

$$S_k(q, a) = \sum_{\substack{x=1 \\ (x, q)=1}}^q e\left(\frac{ax^k}{q}\right), \quad V_k(\beta) = \frac{1}{k} \sum_{N^{\psi_k} < m \leq 2^k N^{\psi_k}} m^{(1-k)/k} e(\beta m).$$

Then, by the Siegel-Walfisz theorem and partial summation (Hua [9, Lemma 6], for example), one finds that whenever $(a, q) = 1$ and β is a real number,

$$(11) \quad g_k\left(\frac{a}{q} + \beta\right) = \frac{S_k(q, a)}{\phi(q)} V_k(\beta) + O\left(\frac{N^{\psi_k} q}{(\log N)^{99}} (1 + N^{\psi_k} |\beta|)\right).$$

Now consider k with $\psi_k < 1$. Then, since the number λ' defined in paragraph preceding (3) is no larger than $9/10$, we see that $\psi_k < 9/10$. For $\alpha \in \mathfrak{K}$ let a, q be the uniquely determined pair with $0 \leq a \leq q$, $1 \leq q \leq Q$, $(a, q) = 1$ and $\alpha \in \mathfrak{K}(q, a)$. Let $\beta = \alpha - (a/q)$ and note that $|\beta| \leq Q/N$. Thus $e(m\beta) = 1 + O(N^{\psi_k-1+\varepsilon})$ for $m \ll N^{\psi_k}$, and so, $V_k(\beta) = N^{\psi_k/k} + O(N^{\psi_k/k-\delta})$. Hence, for all $\alpha \in \mathfrak{K}$ and all k where $\psi_k < 1$, the evaluation (11) simplifies to

$$(12) \quad g_k(\alpha) = \phi(k)^{-1} S_k(q, a) N^{\psi_k/k} + O(N^{\psi_k/k} (\log N)^{-90}).$$

Note that we have $\psi_k = 1$ for $k = 2, 3, 4, 21$ only. Hence, we define

$$S(q, a) = \prod_{k=2}^{21} S_k(q, a), \quad X = \prod_{k=5}^{20} N^{\psi_k/k},$$

and when $\alpha \in \mathfrak{K}$ then we put

$$G^*(\alpha) = \phi(q)^{-20} S(q, a) X V_2(\beta) V_3(\beta) V_4(\beta) V_{21}(\beta)$$

where $\beta = \alpha - (a/q)$ as before. By (11) and (12), one has

$$g_2(\alpha) \cdots g_{21}(\alpha) = G^*(\alpha) + O(N^{u+v+1/21}(\log N)^{-80})$$

while the measure of $\mathfrak{K}$ is $O(N^{-1}Q^3)$. It follows that

$$\int_{\mathfrak{K}} g_2(\alpha) \cdots g_{21}(\alpha) e(-\alpha n) d\alpha = \int_{\mathfrak{K}} G^*(\alpha) d\alpha + O(N^{u+v-20/21}(\log N)^{-50}).$$

Here we expand the integral on the right, yielding

$$\int_{\mathfrak{K}} g_2(\alpha) \cdots g_{21}(\alpha) e(-\alpha n) d\alpha = \mathfrak{S}(n; Q) \mathfrak{J}(n; Q) + O(N^{u+v-20/21}(\log N)^{-50})$$

where

$$\mathfrak{S}(n; Q) = \sum_{q \leq Q} \sum_{\substack{a=1 \\ (a,q)=1}}^q S(q, a) e(-an/q)$$

and

$$\mathfrak{J}(n; Q) = \int_{-Q/N}^{Q/N} V_2(\beta) V_3(\beta) V_4(\beta) V_{21}(\beta) e(-\beta n) d\beta.$$

With the aid of Lemma 6.2 of Vaughan [19] it is readily seen that $J(n; Q) = J(n; \frac{1}{2}N) + O(N^{11/84}Q^{-3})$, and by orthogonality,

$$J(n; \tfrac{1}{2}N) = \frac{1}{504} \sum_{n_2+n_3+n_4+n_{21}=n} n_2^{-1/2} n_3^{-2/3} n_4^{-3/4} n_{21}^{-20/21}$$

where the sum on the right is over all n_2, n_3, n_4, n_{21} with $N < n_2 \leq 4N$ and $n_k > N$ for all other k (recall here that $N = \frac{1}{8}n$). In particular, we infer that $J(n; Q) \gg N^{11/84}$.

By Lemma 5 of Hua [9], whenever $(q, a) = 1$ one has $S(q, a) \ll q^{\varepsilon-10}$. Consequently, the limit $\mathfrak{S}(n) = \lim_{R \rightarrow \infty} \mathfrak{S}(n; R)$ exists, the resulting series $\mathfrak{S}(n)$ is absolutely convergent and $\mathfrak{S}(n) - \mathfrak{S}(n; Q) \ll Q^{-5}$. Standard endgame technique for the singular series readily shows that $\mathfrak{S}(n) \gg 1$ uniformly for even n . On collecting together we have now confirmed that

$$(13) \quad \int_{\mathfrak{K}} g_2(\alpha) \cdots g_{21}(\alpha) e(-\alpha n) d\alpha \gg N^{u+v-20/21}.$$

This leaves the range $\mathfrak{N} \setminus \mathfrak{K}$. Here we apply upper bounds for Weyl sums over primes that are of major arc type. A convenient reference is Kumchev [10, Theorem 1]. This implies that when $0 \leq a \leq q$, $1 \leq q \leq N^\Delta$ with $(a, q) = 1$, then for $\alpha \in \mathfrak{N}(q, a)$ and $2 \leq k \leq 4$ one has

$$g_k(\alpha) \ll N^{1/k} q^{-1/2k} (1 + N|\alpha - (a/q)|)^{-1/2}$$

while for $5 \leq k \leq 21$ the weaker $g_k(\alpha) \ll N^{\psi_k/k} q^{-1/2k}$ holds. It suffices to note that

$$\Sigma = \sum_{k=2}^{21} \frac{1}{k} > 2$$

to conclude that

$$\int_{\mathfrak{N} \setminus \mathfrak{K}} |g_2 \cdots g_{21}| d\alpha \ll N^{u+v+1/21} (A_1 + A_2)$$

where

$$A_1 = \sum_{q>Q} q^{1-\Sigma} \int_{-\infty}^{\infty} (1 + N|\beta|)^{-3/2} d\beta \ll Q^{2-\Sigma} N^{-1}$$

and

$$A_2 = \sum_{q=1}^{\infty} q^{1-\Sigma} \int_{Q/N}^{\infty} (1 + N|\beta|)^{-3/2} d\beta \ll Q^{-1/2} N^{-1}.$$

Combining these last estimates with (13) and (10) we see that whenever n is even and large, then $\rho(n) \gg n^{u+v-20/21}$. This proves the theorem.

References

- [1] J. BRÜDERN, *Sums of squares and higher powers, II*, J. London Math. Soc. (2) **35** (1987), 244–250.
- [2] J. BRÜDERN, *A problem in additive number theory*, Math. Proc. Cambridge Philos. Soc. **103** (1988), 27–33.
- [3] H. DAVENPORT, *On sums of positive integral k -th powers*, Amer. J. Math. **64** (1942), 189–198.
- [4] H. DAVENPORT and P. ERDŐS, *On sums of positive integral k^{th} powers*, Ann. of Math. **40** (1939), 533–536.
- [5] K. B. FORD, *The representation of numbers as sums of unlike powers*, J. London Math. Soc. (2) **51** (1995), 14–26.
- [6] K. B. FORD, *The representation of numbers as sums of unlike powers, II*, J. Amer. Math. Soc. **9** (1996), 919–940.

- [7] G. H. HARDY and J. E. LITTLEWOOD, *Some problems of 'Partitio Numerorum' (VI): Further researches in Waring's Problem*, Math. Z. **23** (1925), 1–37.
- [8] G. HARMAN, *Trigonometric sums over primes, I*, Mathematika **28** (1981), 249–254.
- [9] L.-K. HUA, *Some results in the additive prime-number theory*, Quart. J. Math. Oxford Ser. (2) **9** (1938), 68–80.
- [10] A. V. KUMCHEV, *On Weyl sums over primes and almost primes*, Michigan Math. J. **54** (2006), 243–268.
- [11] K. F. ROTH, *Proof that almost all positive integers are sums of a square, a positive cube and a fourth power*, J. London Math. Soc. **24** (1949), 4–13.
- [12] K. F. ROTH, *A problem in additive number theory*, Proc. London Math. Soc. (2) **53** (1951), 381–395.
- [13] K. THANIGASALAM, *On additive number theory*, Acta Arith. **13** (1967/1968), 237–258.
- [14] K. THANIGASALAM, *On sums of powers and a related problem*, Acta Arith. **36** (1980), 125–141.
- [15] K. THANIGASALAM, *On certain additive representations of integers*, Portugal. Math. **42** (1983/84), 447–465.
- [16] R. C. VAUGHAN, *On the representation of numbers as sums of powers of natural numbers*, Proc. London Math. Soc. (3) **21** (1970), 160–80.
- [17] R. C. VAUGHAN, *On sums of mixed powers*, J. London Math. Soc. (2) **3** (1971), 677–688.
- [18] R. C. VAUGHAN, *On Waring's problem for smaller exponents*, Proc. London Math. Soc. (3) **52** (1986), 445–463.
- [19] R.C. VAUGHAN *The Hardy-Littlewood method*, 2nd ed., University Press, Cambridge, 1997.
- [20] L. ZHAO, *The additive problem with one cube and three cubes of primes*, Michigan Math. J. **63** (2014), 763–779.

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