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On the Frölicher spectral sequence of the Iwasawa manifold and its small deformations

Abstract. We determine the successive pages of the Frölicher spectral sequence of the Iwasawa manifold and some of its small deformations, providing new examples and counterexamples on its properties, including the behaviour under small deformations.

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Introduction

Thanks to a result by J. Stelzig ([11, Theorem 3]) we obtain the explicit algebraic structures of the complexes of differential forms of the Iwasawa manifold and some of its small deformations. We introduce these examples with the aim of studying some properties of the Frölicher spectral sequence; in particular, for each one of the analysed deformations, we compute the explicit dimensions of the components of the successive pages. We observe that:

- (i) the Frölicher spectral sequence of the Iwasawa manifold degenerates at the second page, but there are some deformations for which it degenerates at the first page;
- (ii) for some deformations the dimensions of the components of the second page of the Frölicher spectral sequence can be in general higher or lower than those of the corresponding components for the Iwasawa manifold;

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- (iii) for each page of the Frölicher spectral sequence and for every $p, q \in \mathbb{Z}$ the component of type (p, q) has the same dimension as the component of type $(3 - p, 3 - q)$.

Fact (i) was already known, by considering the Hodge numbers of the Iwasawa manifold and its small deformations determined by I. Nakamura in [7], in which it is shown that the degeneration at the second page is not in general a stable property under small deformations.

Fact (ii) shows instead that the dimensions of the various components do not vary in general either upper or lower semi-continuously under small deformations.

These two considerations were already provided by M. Ceballos, A. Otal, L. Ugarte and R. Villacampa in [3], where they considered a particular example of nilmanifold and a family of its small deformations. In that case the behaviour of the Frölicher spectral sequence is different; indeed, it degenerates at the second page for some specific deformations, while it degenerates at the third page for each of the others.

Finally, fact (iii) is a direct consequence of a generalization of the Serre duality for the successive pages of the Frölicher spectral sequence, provided for the first time in [8] by D. Popovici. More recently other proofs are given in [6], in [11] and in [9].

1 - Preliminaries

A double complex over a field K is a triad $(A, \partial_1, \partial_2)$, where $A = \bigoplus_{p,q \in \mathbb{Z}} A^{p,q}$ is a bigraded K -vector space and ∂_1, ∂_2 are two endomorphisms of bidegree $(1, 0)$ and $(0, 1)$, i.e. such that for each $p, q \in \mathbb{Z}$

$$\partial_1(A^{p,q}) \subseteq A^{p+1,q}, \quad \partial_2(A^{p,q}) \subseteq A^{p,q+1},$$

satisfying the *boundary conditions* $\partial_1^2 = 0, \partial_2^2 = 0$ and the anti-commutativity property given by $\partial_1 \partial_2 + \partial_2 \partial_1 = 0$. Throughout the paper the term *complex* refers to a double complex over a field K .

A result by J. Stelzig ([11, Theorem 3]) guarantees that every bounded complex can be decomposed in an essentially unique way as direct sum of *indecomposable* complexes, i.e. those complexes that cannot be seen as direct sum of two other nontrivial complexes. This class is composed by *squares*, defined

as those complexes whose representation is of the type

$$\begin{array}{ccc} A^{p,q+1} & \xrightarrow{\partial_1} & A^{p+1,q+1} \\ \partial_2 \uparrow & & \uparrow \partial_2 \\ A^{p,q} & \xrightarrow{\partial_1} & A^{p+1,q} \end{array}$$

and *zigzags*, given by

$$\begin{array}{ccccccc} A^{p,q} & , & A^{p,q} \xrightarrow{\partial_1} A^{p+1,q} & , & A^{p,q+1} & , & A^{p,q+1} \\ & & & & \uparrow \partial_2 & & \uparrow \partial_2 \\ & & & & A^{p,q} & & A^{p,q} \xrightarrow{\partial_1} A^{p+1,q} \\ \\ A^{p,q+1} \xrightarrow{\partial_1} A^{p+1,q+1} & , & A^{p,q+1} \xrightarrow{\partial_1} A^{p+1,q+1} & , & \dots & \\ \uparrow \partial_2 & & \uparrow \partial_2 & & & & \\ A^{p+1,q} & & A^{p+1,q} \xrightarrow{\partial_1} A^{p+2,q} & & & & \end{array}$$

In both cases all the drawn components represent one-dimensional vector spaces and all drawn arrows represent isomorphisms. The number of non-zero components is said *length* of the zigzag.

Let us consider now the complex of complex-valued differential forms on a complex manifold X , denoted by $(\Omega_{\mathbb{C}}(X), \partial, \bar{\partial})$. If $X = \Gamma \backslash G$ is a *nilmanifold*, i.e. a compact quotient of a connected and simply connected nilpotent Lie group G by a co-compact discrete subgroup Γ , endowed with a left-invariant complex structure, then, under some appropriate hypotheses, all cohomological information can be obtained from the subcomplex of left-invariant forms, namely differential forms whose pullback to G is invariant by left-translations, denoted by $(\Omega_{\mathbb{C}}^{\text{inv}}(X), \partial, \bar{\partial})$ (see [2, Chapter 3] and the references therein), which is isomorphic to the complex $(\bigwedge^{\bullet, \bullet} \mathfrak{g}_{\mathbb{C}}^*, \partial, \bar{\partial})$, where $\mathfrak{g}_{\mathbb{C}} := \mathfrak{g} \otimes \mathbb{C}$ represents the complexification of the Lie algebra $\mathfrak{g}$ naturally associated to G . Using the results obtained by L. A. Cordero, M. Fernández, L. Ugarte and A. Gray in [4, Theorem 1, Theorem 3] this last complex also permits us to compute easily the dimensions of the successive pages of the Frölicher spectral sequence $\{(E_r^{\bullet, \bullet}, d_r)\}_{r \in \mathbb{N}}$.

A classical example of nilmanifold endowed with a left-invariant complex structure is the *Iwasawa manifold*, defined as the quotient

$$\mathbb{I} := \mathbb{H}(3; \mathbb{Z}[i]) \backslash \mathbb{H}(3; \mathbb{C}),$$

where $\mathbb{H}(3; \mathbb{C})$ is the 3-dimensional *Heisenberg group* over $\mathbb{C}$, defined by

$$\mathbb{H}(3; \mathbb{C}) := \left\{ \begin{pmatrix} 1 & z_1 & z_3 \\ 0 & 1 & z_2 \\ 0 & 0 & 1 \end{pmatrix} \in \text{GL}(n, \mathbb{C}) \mid z_1, z_2, z_3 \in \mathbb{C} \right\}$$

and $\mathbb{H}(3, \mathbb{Z}[i])$ corresponds to the lattice

$$\mathbb{H}(3, \mathbb{Z}[i]) := \mathbb{H}(3, \mathbb{C}) \cap \mathrm{GL}(3, \mathbb{Z}[i]).$$

I. Nakamura determined in [7, pp. 94-96] a family of small deformations $\{X_{\mathbf{t}} = (\mathbb{I}, J_{\mathbf{t}})\}_{\mathbf{t} \in \Delta(0, \varepsilon)}$ depending on six parameters

$$\mathbf{t} = (t_{11}, t_{12}, t_{21}, t_{22}, t_{31}, t_{32}) \in \Delta(0, \varepsilon),$$

with

$$\Delta(0, \varepsilon) := \{ \mathbf{s} \in \mathbb{C}^6 \mid |\mathbf{s}| < \varepsilon \},$$

where $\varepsilon > 0$ is small enough.

Furthermore, he provided a co-frame $(\varphi_{\mathbf{t}}^1, \varphi_{\mathbf{t}}^2, \varphi_{\mathbf{t}}^3)$ of left-invariant $(1, 0)$ -forms on $X_{\mathbf{t}}$ satisfying the structure equations given by

$$\begin{cases} d\varphi_{\mathbf{t}}^1 = 0 \\ d\varphi_{\mathbf{t}}^2 = 0 \\ d\varphi_{\mathbf{t}}^3 = \sigma_{12} \varphi_{\mathbf{t}}^1 \wedge \varphi_{\mathbf{t}}^2 + \sigma_{1\bar{1}} \varphi_{\mathbf{t}}^1 \wedge \bar{\varphi}_{\mathbf{t}}^1 + \sigma_{1\bar{2}} \varphi_{\mathbf{t}}^1 \wedge \bar{\varphi}_{\mathbf{t}}^2 + \sigma_{2\bar{1}} \varphi_{\mathbf{t}}^2 \wedge \bar{\varphi}_{\mathbf{t}}^1 + \sigma_{2\bar{2}} \varphi_{\mathbf{t}}^2 \wedge \bar{\varphi}_{\mathbf{t}}^2, \end{cases}$$

where $\sigma_{12}, \sigma_{1\bar{1}}, \sigma_{2\bar{2}}, \sigma_{2\bar{1}}, \sigma_{1\bar{2}} \in \mathbb{C}$ are parameters depending on $\mathbf{t}$. Defining the matrix S as

$$S = \begin{pmatrix} \overline{\sigma_{1\bar{1}}} & \overline{\sigma_{2\bar{2}}} & \overline{\sigma_{1\bar{2}}} & \overline{\sigma_{2\bar{1}}} \\ \sigma_{1\bar{1}} & \sigma_{2\bar{2}} & \sigma_{2\bar{1}} & \sigma_{1\bar{2}} \end{pmatrix}$$

and the complex parameter $D(\mathbf{t})$ as

$$D(\mathbf{t}) := \det \begin{pmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{pmatrix},$$

the small deformations of the Iwasawa manifold can be classified into three classes, based on their Hodge numbers, further subdivided in other subclasses considering their Bott-Chern-numbers:

- (i) $t_{11} = t_{12} = t_{21} = t_{22} = 0$;
- (ii) $D(\mathbf{t}) = 0$ and $(t_{11}, t_{12}, t_{21}, t_{22}) \neq (0, 0, 0, 0)$:
 - (ii.a) $D(\mathbf{t}) = 0$ and $\mathrm{rank} S = 1$;
 - (ii.b) $D(\mathbf{t}) = 0$ and $\mathrm{rank} S = 2$;
- (iii) $D(\mathbf{t}) \neq 0$:
 - (iii.a) $D(\mathbf{t}) \neq 0$ and $\mathrm{rank} S = 1$;
 - (iii.b) $D(\mathbf{t}) \neq 0$ and $\mathrm{rank} S = 2$.

Thanks to some results contained in [10], [1] and the references therein, we can make some considerations about the de Rham, Dolbeault, Aeppli and Bott-Chern cohomologies, in particular for each $\mathbf{t} \in \Delta(0, \varepsilon)$ we get the isomorphisms

$$\begin{aligned} H_{dR}^{\bullet}(\mathfrak{h}, \mathbb{R}) &\simeq H_{dR}^{\bullet}(X_{\mathbf{t}}, \mathbb{R}), & H_{\bar{\partial}}^{\bullet, \bullet}(\mathfrak{h}_{\mathbb{C}}) &\simeq H_{\bar{\partial}}^{\bullet, \bullet}(X_{\mathbf{t}}), \\ H_{BC}^{\bullet, \bullet}(\mathfrak{h}_{\mathbb{C}}) &\simeq H_{BC}^{\bullet, \bullet}(X_{\mathbf{t}}), & H_A^{\bullet, \bullet}(\mathfrak{h}_{\mathbb{C}}) &\simeq H_A^{\bullet, \bullet}(X_{\mathbf{t}}), \end{aligned}$$

where $\mathfrak{h}$ is the Lie Algebra naturally associated to $\mathbb{H}(3, \mathbb{C})$ and $H_{*}^{\bullet, \bullet}(\mathfrak{h}_{\mathbb{C}})$ represents the corresponding cohomology of the subcomplex of left-invariant forms; therefore, to compute dimensions of the cohomologies, it is sufficient to analyze the complex of left-invariant forms. For this reason, in the next section we will restrict our attention on this last complex.

2 - Double complex of left-invariant forms on the Iwasawa manifold and its small deformations

We are now going to analyze some explicit examples of complex of differential forms; in particular we will give an explicit graphic representation, as in the Stelzig process, of the structure of the complex related to the Iwasawa manifold and some of its small deformations. Once we describe the required decomposition as a direct sum of squares and zigzags, it will be easy to obtain the dimensions of the successive pages of the Frölicher spectral sequence.

Given any arbitrary bounded complex, the decomposition described by J. Stelzig is based on the construction of an ascending filtration, which further splits into simpler pieces, thus allowing to identify squares and zigzags appearing in the complex. The procedure works as follows: we determine for each $k \in \mathbb{Z}$ a subcomplex generated by all components of total degree at most k and then a complement of the subcomplex generated by the components of total degree at most $k - 1$ contained in it. This allows us to obtain for every $k \in \mathbb{Z}$ a simpler subcomplex composed by components of degree $k, k + 1$ and $k + 2$, whose direct sum corresponds to complex considered. As a result it is easier to determine the various squares and zigzags contained in every subcomplex to get the desired decomposition.

In the case of the complex of left-invariant differential forms of the Iwasawa manifold and its small deformations the procedure consists of five steps, based respectively on the computation of the images of the left-invariant differential forms of total degree between 2 and 5 by the operators ∂ and $\bar{\partial}$.

In Figure 1, we see the graphs corresponding to some deformations of the Iwasawa manifold obtained in this way. The graph a) corresponds to the complex of differential forms of the Iwasawa manifold itself. Its structure, already

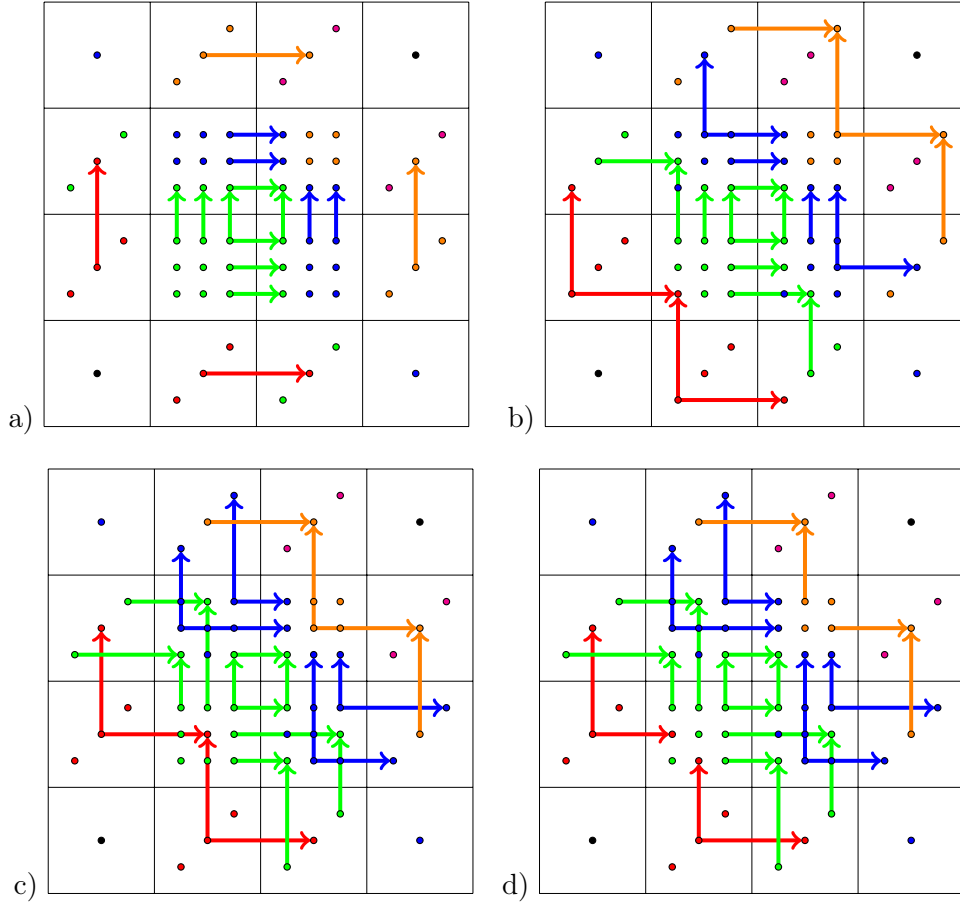


Figure 1: Representation of the structure of the complexes of complex-valued differential forms related to the Iwasawa manifold and some of its small deformations. Horizontal and vertical arrows represent respectively the maps ∂ and $\bar{\partial}$. The different colours correspond to the various steps of the Stelzig process.

determined in [1, Section 4] by D. Angella is quite simple and it can be represented as direct sum of thirtysix zigzags of length 1, twelve zigzags of length 2 and a square. The graph b) instead represents the complex related to a deformation of the Iwasawa manifold in the case where

$$t_{21} \neq 0, \quad t_{11} = t_{12} = t_{22} = t_{31} = t_{32} = 0,$$

that is of class (ii.a) and for which

$$\sigma_{12} = -1 \quad \sigma_{1\bar{1}} = t_{21}, \quad \sigma_{1\bar{2}} = \sigma_{2\bar{1}} = \sigma_{2\bar{2}} = 0.$$

It has a slightly more complicated structure, with twentyeight zigzags of length 1, four zigzags of length 2, four zigzags of length 3, two zigzags of length 5 and a square.

Finally the graphs c) and d) concern a deformation obtained by parameters

$$t_{11} \neq 0, \quad t_{22} \neq 0, \quad t_{12} = t_{21} = t_{31} = t_{32} = 0$$

respectively when $|t_{11}| = |t_{22}|$ and $|t_{11}| \neq |t_{22}|$; the parameters of the structure equations are given in both cases by

$$\sigma_{12} = 1 - \frac{1}{1 - |t_{22}|^2} - \frac{1}{1 - |t_{11}|^2}, \quad \sigma_{1\bar{2}} = \frac{t_{22}}{1 - |t_{22}|^2},$$

$$\sigma_{2\bar{1}} = -\frac{t_{11}}{1 - |t_{11}|^2}, \quad \sigma_{1\bar{1}} = \sigma_{2\bar{2}} = 0.$$

In the first case the deformation represents an example of class (iii.a), while in the second case it belongs to the class (iii.b). The only difference between the two graphs is given by two zigzags of length 5 in the first, which split into four zigzags of length 3 if the two norms are different. In addition to these, we can find also eight more zigzags of length 3 and a square.

Because of the results mentioned before, these representations give another way to compute the dimensions of de Rham, Dolbeault, Bott-Chern and Aeppli cohomologies of the manifolds considered, already determined in [7, p.96] and [1, Section 5]. For example, if we want to check the dimensions of Dolbeault cohomology, we have to remove vertical arrows in the graphs and count the remaining points (see the table in [7, p.96] to check the corresponding values).

3 - Properties of the Frölicher spectral sequence under small deformations

Thanks to the explicit description given in [4], we can easily use the graphs in Figure 1 to determine the dimensions of the successive pages of the Frölicher spectral sequence of the related manifolds, that we record in the table of Figure 2. In particular, we observe that for the Iwasawa Manifold and the deformation of class (ii) analysed the Frölicher spectral sequence degenerates at the second page, while in the case of the deformations of class (iii), in both the examples considered, it degenerates at the first page, according to the Hodge numbers of the Iwasawa manifold and its small deformations already determined by I. Nakamura in [7, p.96]. These facts permit us to make some considerations.

Remark 1. M. Maschio proved in [5, Theorem 1] that, given any compact complex manifold X , if the dimensions of Dolbeault cohomology are constant

a)	1, 0	0, 1	2, 0	1, 1	0, 2	3, 0	2, 1	1, 2	0, 3	3, 1	2, 2	1, 3	3, 2	2, 3
$E_1^{\bullet, \bullet}$	3	2	3	6	2	1	6	6	1	2	6	3	2	3
$E_2^{\bullet, \bullet}$	2	2	2	4	2	1	4	4	1	2	4	2	2	2

b)	1, 0	0, 1	2, 0	1, 1	0, 2	3, 0	2, 1	1, 2	0, 3	3, 1	2, 2	1, 3	3, 2	2, 3
$E_1^{\bullet, \bullet}$	2	2	2	5	2	1	5	5	1	2	5	2	2	2
$E_2^{\bullet, \bullet}$	2	2	2	4	2	1	4	4	1	2	4	2	2	2

c), d)	1, 0	0, 1	2, 0	1, 1	0, 2	3, 0	2, 1	1, 2	0, 3	3, 1	2, 2	1, 3	3, 2	2, 3
$E_1^{\bullet, \bullet}$	2	2	1	5	2	1	4	4	1	2	5	1	2	2

Figure 2: Successive pages of the Frölicher spectral sequence of the Iwasawa manifold (table a)) and some of its small deformations (tables b) and c)).

under deformations and its Frölicher spectral sequence degenerates at most at the second page, then the Frölicher spectral sequence of any small enough deformation of X degenerates at most at the second page as well.

The deformations of the Iwasawa manifold considered above have a behaviour consistent with the result of M. Maschio, despite the dimensions of Dolbeault cohomology are not constant.

However, this property does not hold in general. In [3, Corollary 5.12], M. Ceballos, A. Otal, L. Ugarte and M. Villacampa analysed the behaviour of the Frölicher spectral sequence with respect to a family of small deformations defined on a particular nilmanifold. In particular, they showed that the Frölicher spectral sequence degenerates at the second page for some deformations and at the third for the others.

Another example is given by M. Maschio in an early unpublished version of [5], where he considered the Nakamura manifold, that represents an example of solvmanifold and whose Frölicher spectral sequence degenerates at the second page, determining also a family of small deformations whose Frölicher spectral sequence degenerates at higher pages.

Remark 2. The example considered in [3] permits also to observe ([3, Corollary 4.9]) that, in general, the dimensions of the components $E_r^{p,q}(X_t)$ of the Frölicher spectral sequence do not vary either upper semi-continuously or lower semi-continuously under small deformations, i.e. if $\{X_t = (X, J_t)\}_{t \in \Delta(0, \varepsilon)}$ is a family of small deformations of a compact complex manifold X , then for

every $p, q \in \mathbb{Z}$ and $r \in \mathbb{Z} \setminus \{0, 1\}$ the map

$$\mathbf{t} \mapsto \dim(E_r^{p,q}(X_{\mathbf{t}}))$$

is, in general, neither upper nor lower semi-continuous.

We can deduce it also by analysing values exposed in Figure 2, in particular comparing the numbers corresponding to the components of type $(2, 0)$ and $(1, 1)$ of tables a) and c).

Remark 3. It is evident that the Frölicher spectral sequences of the considered manifolds present also a symmetry property: for each $p, q \in \mathbb{N}$ we have

$$\dim(E_2^{p,q}(X_{\mathbf{t}})) = \dim(E_2^{3-p,3-q}(X_{\mathbf{t}})).$$

This equality represents a numerical generalization of the Serre duality for Dolbeault cohomology, corresponding for each $p, q \in \mathbb{Z}$ to the isomorphism

$$H_{\bar{\partial}}^{p,q}(X_{\mathbf{t}}) \simeq H_{\bar{\partial}}^{n-p,n-q}(X_{\mathbf{t}})^*.$$

It was already noticed by L. Ugarte in [12] for a hypothetical complex structure on S^6 and his proof is due to D. Popovici, that in [8, Prop. 5.1] showed it for any compact complex manifold. The Serre duality for the Frölicher spectral sequence has been instead recently proved in several ways by A. Milojević in [6, Corollary 2.7], by J. Stelzig in [11, Corollary 20] and by D. Popovici, J. Stelzig and L. Ugarte in [9, Corollary 2.9].

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